Nash equilibria in singleton congestion games: Symmetric case

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Abstract.
This paper provides a simple formula describing all Nash equilibria in singleton congestion games, reducing the complexity of computing all these equilibria and giving a simple and short proof (without invoking the FIP or the potential function).

Keywords: Congestion games, FIP, Potential function, Nash equilibrium.

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1 Introduction

In recent years, several economists have shown a growing interest in the study of the existence and the identification of Nash equilibria in congestion and also potential games. Rosenthal [8] introduced the class of symmetric congestion games and proved that they always possess a pure-strategy Nash equilibrium. Monderer and Shapley [6] contributed to the treatment of the class of potential games and showed the relation between potential functions and Nash equilibria: the existence of an exact potential function implies the finite improvement property (FIP). Milchtaich [5] proved Rosenthal's result without invoking the potential function, but by using the FIP property. The aim of this work is to provide a new, short and simple proof establishing the existence of a Nash equilibrium in singleton congestion games and

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to show how to compute this equilibrium using a simple formula that allows to reduce calculation complexity. The rest of this paper is organized as follows: In section 2, we introduce notations, definitions and related work, section 3 provides some main results in singleton symmetric congestion games. We conclude in section 4.

2 Notations, definitions and related work

A game (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the utility function for player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i\in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies. A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a best-reply strategy: for each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy.

In a (standard) congestion game [8] we are given a finite set $R = \{1, \ldots, m\}$ of m resources (also called primary factors). A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \ldots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r\in\sigma_i} d_r(n_r(\sigma))$). Rosenthal [8] shows that every congestion game possesses at least one Nash equilibrium by considering the exact potential function $P : S \to \mathbb{N}$ with $P(\sigma) = \sum_{r\in R}^{m} \sum_{j=1}^{n_r(\sigma)} d_r(j)$ [1], $\forall \sigma \in S$. A consequence of the existence of an exact potential function is the finite improvement property (FIP) (Monderer and Shapley [6]): Any sequence of strategy-profiles in which each strategy-profile differs from the preceding one in only one coordinate and the unique deviator in each step strictly increases his utility (such a sequence is called an improvement path), is finite. Obviously, any maximal improvement path, an improvement path that cannot be extended, is terminated by a Nash equilibrium. A slightly different formulation of congestion games was introduced by Milchtaich [5] under the name of congestion games with player-specific payoff functions [2]. Each player i has individual payoff functions $(d_r^i)_{r\in R}$: The payoff

[1]Rosenthal's potential function shows that congestion games are potential. Monderer and Shapley (1996) proved that every potential game can be represented in a form of a congestion game. Thus, classes of potential games and congestion games coincide. Hence, congestion games are essentially the only class of games for which one can show the existence of pure equilibria with an exact potential function.

[2]This class of games was also investigated, independently and under different names

function associated with each resource is not common but player-specific. In this sense, theses games are more general than Rosenthal's model. However, this generalization is accompanied by two limiting (restrictive) assumptions. The first restriction is that each player is allowed to choose any resource from R but must choose only one. The second restriction is that, for each player i and each resource r, the specific payoff function d_r^i is a monotonically non increasing function along with the number of players selecting r. Milchtaich shows that games in this class do not generally satisfy the FIP (thus they are not potential games anymore) but that they always possess Nash equilibria. In fact, he showed that the best-replay dynamics may cycle, that is improvement paths in which players iteratively shift to the best-replay strategy do not necessarily lead to an equilibrium. Nevertheless, he also proved that there is always at least one best-replay improvement path that connects an arbitrary initial profile to a Nash equilibrium.

In this paper, we are interested in a simple subclass of congestion games which lie in the intersection between Rosenthal's and Milchtaich's model. In the following, we refer to games in this class as monotone singleton congestion games. Such a game can be represented by a tuple $\Gamma(N, R, (d_r)_{r \in R})$ where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a nonincreasing payoff function associated with resource r. The utility of player i for a profile σ is simply given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$. Note that these games are symmetric : players share the same strategy set ($S_i = R$, for all $i \in N$) and the same utility function.

Since this kind of games is a special case of standard congestion games (with the restrictions cited above), the existence of Nash equilibrium is guaranteed by Rosenthal's potential function. This class of games has been initially studied by Milchtaich [5] as the symmetric case of his model. Without invoking a potential function, he showed that, unlike general (nonsymmetric) congestion games with specific-payoff functions, monotone singleton congestion games possess the finite improvement property. It follows from this proof that best-replay dynamics always converge to an equilibrium in at most $O(n^2 \times m)$ steps. Ieong et *al.* [3] generalized this result to the largest class of singleton congestion games (where the payoff functions are not required to be monotone). They also showed that even optimal equilibria (Nash equilibria that maximize the sum of players utilities) can be found in a polynomial time. Holzman and Law-Yone [2] and Voorneveld et *al.* [10] investigated the set of strong Nash equilibria [3] in monotone singleton congestion games. It turns out that this set coincides with the set of Nash equilibria and with the set of profiles which maximize the potential. Variants of (monotone)

by Quint and Shubik [7] and by Konishi et *al.* [4]).

[3] A strong Nash equilibria is a profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies.

singleton congestion games have been studied in terms of the time convergence of the best-reply dynamics to a Nash equilibrium (Even-Dar et *al.* [1]) and in terms of the existence of alternative concept of solution (Rozenfeld and Tennenholtz, [9]).

The purpose of this paper is to present an alternative approach for computing Nash equilibria (and strong Nash equilibria) in monotone singleton congestion game. Our main contribution is to provide a simple exact formula describing all such equilibria for this class of games.

3 Main result

The main drawback of Milchtaich's method is that it gives only one equilibrium. If we want to find all Nash equilibria we have to reiterate the process (FIP). As a result the time complexity becomes exponential. Our purpose is to improve Milchtaich's analysis of the symmetric case by providing a simple formula describing all Nash equilibria, reducing the complexity of computing all these equilibria and giving a more simple and shorter proof (without invoking the FIP or the potential function).

Before introducing our main results, we need the following definition.

Definition 1 *Let $\preceq$ be a (weak) ordering on $R \times N$. An n-sequence derived from $\preceq$ is a subset T of $R \times N$ such that:*

- $|T| = n$.
- $((r,k) \in T$ *and* $(r',k') \notin T) \Rightarrow (r,k) \succsim (r',k')$.
- $(r,k) \in T \Rightarrow ((r,k') \in T, \forall k' < k)$.

To illustrate this definition, let introduce the following example.

Example 1 *In this example, we present two cases of the utility for all players.*

- *Let $N = \{1,2,3,4,5\}$ and $R = \{a,b,c\}$. For simplicity, we will denote the utility of $u(r,k)$ by rk. The utility for all players in **strictly decreasing order** is:*

  $$5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}.$$

  By our definition the unique 5-sequence is $T = \{3c, a, 2c, c, b\}$.

- *Let $N = \{1,2,3,4,5\}$ et $R = \{a,b,c\}$. In non increasing order the utility for all players is:*

  $$5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}.$$

  We have exactly three 8-sequences: $T_1 = \{3d, 2a, b, 2c, c, 2d, a, d\}$; $T_2 = \{2b, 2a, b, 2c, c, 2d, a, d\}$; $T_3 = \{3a, 2a, b, 2c, c, 2d, a, d\}$.

The following theorem is the main result of this paper.

Theorem 1 . *Let $\Gamma(N, R, \precsim)$ be a symmetric singleton congestion game, with $|N| = n$ et $|R| = m$. Then,*

1. *There is a unique Nash equilibrium per n-sequence. Let T be an n-sequence of $\precsim$. The corresponding Nash equilibrium is defined by: $\sigma = ((1, \alpha_1), \ldots, (m, \alpha_m))$, where α_j is the greater integer p satisfying $(r_j, p) \in T$.*

2. *When the preferences of the players are in strictly decreasing order, the game Γ admits one Nash equilibrium.*

3. *The number of Nash equilibria of the game Γ equals the number of all n-sequences extracted from $\precsim$.*

Proof 1 . *We just have to prove the first point. Hence, the second and the third one, are a direct consequence of the latter. Let T be an n-sequence and let $\sigma^* = ((1, \alpha_1), \ldots, (m, \alpha_m))$ where $\alpha_r = max\{p : (r, p) \in T\}$. By definition of T and σ^*, we have $\sum_{r=1}^{m} \alpha_r = n$. Indeed, the sequence T consists exclusively of the following terms: $(1, \alpha_1), \ldots, (1, 1), (2, \alpha_2), \ldots, (2, 1), \ldots, (m, \alpha_m), \ldots, (m, 1)$. Therefore, σ^* is a congestion vector. Furthermore, for all r, r' in R, $(r, \alpha_r) \succsim (r', \alpha_{r'} + 1)$ because $(r, \alpha_r) \in T$ and $(r', \alpha_{r'} + 1) \notin T$. Hence, σ^* is a Nash equilibrium. Reciprocally, let $\sigma^* = ((1, \alpha_1), \ldots, (m, \alpha_m))$ be a Nash equilibrium. It is easy to see that $T = \{\underbrace{(1, \alpha_1), \ldots, (1, 1)}, \ldots, \underbrace{(m, \alpha_m), \ldots, (m, 1)}\}$ is an n-sequence. In fact, as σ^* is a congestion vector, we have $\sum_{r=1}^{m} \alpha_r = n$ and so $|T| = n$. On the other hand, by definition of T, $(r, k) \in T \Rightarrow ((r, k') \in T, \forall k' < k)$. Finally, let $(r, k) \in T$ and $(r', k') \notin T$. We have: $(r, k') \notin T$, $k \preceq \alpha_r$ and $k' \succ \alpha_{r'}$. As σ^* is a Nash equilibrium, we have $(r, \alpha_r) \succeq (r', \alpha_{r'} + 1)$. Thus, $(r, k) \succeq (r, \alpha_r) \succeq (r', \alpha_{r'} + 1) \succeq (r', k')$.*■

Remark 1 *The proof we have suggested implicitly contains an algorithm to compute all Nash equilibria. Thanks to this algorithm, the computation time for a single Nash equilibrium becomes linear $O(n)$ and independent of m. What constitutes an improvement of the mechanism proposed by Milchtaich, where a Nash equilibrium can be found in polynomial time $O(n^2 \times m)$.*

To illustrate the above theorem, we continue to the previous examples to show how to easily determine all Nash equilibria.

Example 2 *By using the same example as before and applying our theorem, we can easily find the Nash equilibrium for each n-sequence.*

- *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $R = \{a, b, c\}$. *Considering the utility:* $5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}$, *we obtain* $T = \{b, c, 2c, a, 3c\}$ *and we choose the greatest integer of each resource. The corresponding Nash equilibrium is* $\sigma^* = (a, b, 3c)$.

- *Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *and* $R = \{a, b, c, d\}$. *We consider the following utility:* $8c \sim 8b \prec 8a \sim 8d \prec 7c \sim 7b \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5d \sim 5b \sim 4b \sim 7a \prec 5a \sim 4d \sim 6a \prec 4a \sim 3a \sim 3b \sim 2b \sim 3d \prec 2a \sim b \sim 2c \prec c \sim 2d \prec a \sim d$.

  Here we find one Nash equilibrium per n-sequence:

$$T_1 = \{d, a, 2d, c, 2c, b, 2a, 3d\} \rightarrow \sigma_1^* = (2a, b, 2c, 3d);$$
$$T_2 = \{d, a, 2d, c, 2c, b, 2a, 2b\} \rightarrow \sigma_2^* = (2a, 2b, 2c, 2d);$$
$$T_3 = \{d, a, 2d, c, 2c, b, 2a, 3a\} \rightarrow \sigma_3^* = (3a, b, 2c, 2d).$$

  Indeed, the number of Nash equilibria equals the number of all n-sequences.

Note that for the non-symmetric case, Theorem 1 does not work. The following example illustrates this fact.

Example 3 *Let* $N = \{1, 2, 3\}$ *and* $R = \{a, b, c\}$. *We consider the following utility of players:*

$$3a \prec_1 3b \prec_1 2 \prec_1 3c \prec_1 2b \prec_1 a \prec_1 b \prec_1 2c \prec_1 c.$$
$$3c \prec_2 2c \prec_2 3b \prec_2 c \prec_2 2b \prec_2 3a \prec_2 b \prec_2 2a \prec_2 a.$$
$$3c \prec_3 3a \prec_3 2a \prec_3 2c \prec_3 3b \prec_3 c \prec_3 2b \prec_3 b \prec_3 a.$$

The concept of sequence of n previous terms may not apply in this case because we have three different utility functions. For player 1, we have the sequence $b \prec_1 2c \prec_1 c$, for player 2: $b \prec_2 2a \prec_2 a$ and for player 3: $2b \prec_3 b \prec_3 a$. Applying the above theorem to these sequences, we obtain the following three vectors of congestion: $(b, 2c)$, $(b, 2a)$ et $(2b, a)$. None of these three vectors of congestion is appropriate to the three players simultaneously. We can then think about taking the last term of each of the three utility functions. In this way, the strategy profile is (c, a, a). But one can easily check that does not correspond to a Nash equilibrium. Nevertheless, there is a Nash equilibrium (c, a, b).

4 Discussion

In this paper we have first presented a new method which allows to find all Nash equilibria and we provided all the necessary analysis. As a future work, we will present our result in the non symmetric case with two resources ($m = 2$) and try to extend our approach in the general case (n players and $m \succ 2$ resources). However, our aim will be to calculate all Nash equilibria and to reduce the complexity of their computation, improving Milchtaich's mechanism for the nonsymmetric case in congestion games.

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