UNIVERSITE DE LA REUNION

DOCUMENTS DE TRAVAIL

CERESUR

2008

SUBSTITUTABILITY AND STABILITY OF A WALRASIAN GENERAL EQUILIBRIUM

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Abstract - *In this paper, I propose to interpret the history of stability analysis of Walrasian exchange economy in the light of a microeconomic concept: the concept of substitutability. A purely mathematical approach of this story does not seem sufficient to account for the way economists have studied the question of stability. I do not want to reduce the importance of the mathematical difficulties the theoreticians have had to cope with. I just mean that substitutability, and all the representations going with it, have partially oriented the path followed by stability analysis from the publication of* Value and Capital *up to the Sonnenschein-Mantel-Debreu theorem and to Smale investigations on the computation of general equilibrium. Thus, I uphold that firstly, the concept of substitutability allows catching the heuristic behind the work on stability, and secondly, it allows replacing the SMD results in a larger context.*

Key words: stability, general equilibrium, gross substitutability, substitutability, complementarity, John Hicks, law of demand, Steve Smale, Sonnenschein-Mantel-Debreu theorem.

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0. Introduction

It is rather a well-trodden path to say that the heart of the general equilibrium research programme has passed away. The hopes for finding general sufficient conditions under which a tâtonnement process is stable, has turned into dark pessimism, and even to disinterest. The story of this research programme is well known. Ingrao and Israel (1990) have identified the steps and boundaries. In return, the methodological appraisal of this story is much less clear. The most common opinion on the subject (Guerrien, 1989; Rivzi, 1997) credits the famous Sonnenschein-Mantel-Debreu results for having withdrawn all serious reference to the « invisible hand ». However, one can find a slightly different position regarding the stability literature in Ingrao and Israel. According to Ingrao and Israel (1990), from the very beginning of the sixties, mathematical knowledge on dynamical systems and the instability results made stability researches a vain task.

The gap between those two positions is not anecdotal. Firstly, according to the position adopted, the place of the SMD result is not the same, both theoretically and from a symbolic point of view. Secondly, there are methodological consequences at stake on the way we can represent the development of general equilibrium theory.

The aim of this paper is to identify some methodological principles at stake in the history of the stability of general equilibrium. More precisely, I would like to identify some criteria, others than analytical rigor, that were in use in order to evaluate and to interpret the results. This methodological look at the stability literature may lead to a more progressive view of the history.

My aim in this article is to provide a first step into the study of stability of a Walrasian exchange economy, from *Value and capital* in 1939. I will put in the foreground a microeconomic concept: the concept of substitutability.

In effect, I think that this concept is a good tool to identify a guiding principle in the general equilibrium programme. It is my contention that the concept of substitutability gives methodological depth to general equilibrium theory. It allows identifying some methodological and heuristic constraints within general equilibrium theory.

It is well known that one of the only sufficient conditions for stability that attracted attention is the gross-substitutability hypothesis. By reconstructing stability analysis through the concept of substitutability, I want to uphold the idea that the representations attached to substitutability were like a “positive heuristic”. It is a way to shed some light on the expansion and decline of stability analysis in general equilibrium theory.

The paper is organised as follows. From *Value and capital* to the middle of the fifties, stability is linked intimately with the search for comparative static results. It is a founding time for the heuristic of substitutability, and more precisely for the idea of a relation between substitutability and stability. And there is no doubt that John Hicks is at the origin of this way if seeing things (**1. Stability and substitutability: a Hicksian tradition**). From the middle of the 50’s, the axiomatisation of general equilibrium enters

the scene. There are hopes for finding relevant conditions of stability. On the one hand, substitutability remains a good guiding principle, it has good descriptive and explicative properties, while on the other hand, the first examples of instability are found (**2. From "Gross Substitutability" to instability examples**). The last time period in this story is much more uneasy and agitated. It is characterised with hidden pessimism, with a spread of work on the subject. One among other results, the SMD theorem comes as a confirmation that the search for stability comes to a dead-lock. But as we will see, it is not the only result that played a role in the neglect of stability analysis (**3. The end of a research programme?**). As a conclusion, I provide an evaluation of the SMD result and of its consequences, within the context of many other results, and I propose to redefine the new lineaments of a research programme on stability.

1. Stability and substitutability: a Hicksian tradition

In *Value and Capital* (1939), Hicks made a systematic use of substitutes and complements to express stability conditions. He went as far as to uphold a narrow link between stability and substitutability, giving to the substitutability concept an explicative and descriptive power of the stability of market systems. This idea will imprint the future of the search for stability conditions. I will begin by reminding some technical definitions (**1.1 Some technical definitions**). The, I will present Hicks ideas on stability and substitutability (**1.2 Stability and substitutability according to Hicks**). The end of the section is devoted to the reconstruction of a Hicksian tradition after Samuelson's criticism (**1.3 Hicksian tradition and substitutability**).

1.1 Some technical definitions

Let's remember first some technical definitions. In *Value and Capital*, Hicks provides a definition of substitutes and complements on the basis of the Slutsky (1915) fundamental equation of value. The Hicks-Slutsky decomposition of the derivative of the demand for good *i*, x_i with respect to the price of a good *j*, p_j is:

$$\frac{\partial x_i(p,r)}{\partial p_j} = \frac{\partial h_i(p,u)}{\partial p_j} - x_j \frac{\partial x_i(p,r)}{\partial r} \qquad (1)$$

with $x_i(p,r)$ the Marshallian demand of *i*, $h_i(p,u)$ the compensated demand (or Hicksian demand) of *i* – that is with a constant level of utility – *r* the revenue and $u = v(p,r)$ the (indirect) utility attainable with (p,r).

From equation (1), we say that *i* and *j* are net substitutes, independent or net complementary if the change in the compensated demand of *i* due to a change in p_j is positive, null, or negative:

$$\frac{\partial h_i(p, v(p,r))}{\partial p_j} > 0, = 0, < 0 \qquad (2)$$

From the equation (1) we say that *i* and *j* are gross substitutes, independent or gross complementary if the change in the Marshallian demand of *i* due to a change in the price p_j verifies

$$\frac{\partial x_i(p,r)}{\partial p_j} > 0, = 0, < 0 \qquad (3)$$

At an aggregated level, definitions (2) and (3) are usable for a general description of substitution between different markets, and equation (1) can serve discuss the direction and the strength of revenue effects.

With those definitions in mind, we can enter into the Hicksian analysis of substitutability.

1.2 Stability and substitutability according to Hicks

In *Value and Capital*, Hicks reinstates Walrasian dynamic analysis. This interest for stability and comparative statics, it must be noted, arises precisely from the availability of a new definition of substitutability and from the distinction between revenue and substitution effect.

In Hicks's view, even more certainly than for Walras, there is no doubt that the law of supply and demand leads the economy to equilibrium. Hicks will follow Walras's reasoning on stability, with the aim of providing a precise mathematical account for it and to discuss with much more attention the effect of interdependencies between markets. He makes the well-known distinction between perfect and imperfect stability. The stability will be perfect if the principal minors of the Jacobian matrix, at equilibrium, alternate in sign, the first one being negative. The system is only imperfectly stable if only the last of these determinants respects the sign condition.

Hicks's analysis proceeds from the generalisation of a two goods economy. He thinks that, except for particular cases, revenue effects should tend to compensate between buyers and sellers and then should disappear:

> "Therefore, when dealing with problems of the stability of exchange, it is a reasonable method of approach to begin by assuming that income effects do cancel out, and then to inquire what difference it makes if there is a net income effect in one direction or the other." (Hicks, 1939, 64-65)

Thus, actually, the Jacobian of the system is identical to the matrix of the substitution effects (the Slutsky matrix). And after a rather clumsy discussion, Hicks comes to the following conclusion:

> "To sum up the negative but reassuring conclusions which we have derived from our discussion of stability. There is no doubt that the existence of stable systems of multiple exchanges is entirely consistent with the laws of demand. It cannot, indeed, be proved *a priori* that a system of multiple exchanges is necessarily stable. But the conditions of stability are quite easy conditions, so that it is quite reasonable to

> assume that they will be satisfied in almost any system with which we are likely to be concerned. The only possible ultimate source of instability is strong asymmetry in the income effects. A moderate degree of substitutability among the bulk of commodities will be sufficient to prevent this cause being effective" (Hicks, 1939, 72-73, I underline).

What kind of substitutability is refereed to here? It is referred to the fact that the goods are net substitutes to each other. Consequently, symmetrical revenue effects at the aggregated level will have only a weak effect compared with the substitution effect, so that the Jacobian matrix is approximately symmetric.

In so doing, Hicks develops a descriptive and explicative point of view on stability, and substitutability is given a prominent role. Substitutability is entrusted to produce a stylised representation of the interdependencies between markets, likely to receive a validation *a priori*. Thus, the idea that substitutes are dominating over the system is regarded as a natural and virtuous property of the economic system.

1.3 Hicksian tradition and substitutability.

Hicks's analysis is first criticised by Samuelson (1941, 1944, and 1947) on a purely technical basis. It will then reappear with Metzler.

According to Samuelson, Hicksian stability is a static approach to stability. It consists mainly "to generalise to any number of markets the stability conditions of a single market" (Samuelson, 1947, 270). Samuelson proposes the first true dynamic expression of Walrasian tâtonnement. It takes the form of a differential equation system, and it is stable if and only if real parts of the eigenvalues associated to the Jacobian are all negative. For this, he shows that Hicksian perfect and imperfect stability conditions are neither necessary nor sufficient. The main lesson to be drawn from Samuelson's analysis is that taking the revenue effect seriously is indispensable to making a serious analysis of stability. But Samuelson does not try to enter into the interpretation of the mathematical conditions of stability.

In the time period following immediately Samuelson, Metzler, Lange, Mosak and Smithies tried to provide interpretable conditions of stability. Metzler established the local stability of a dynamic system under gross substitutability. Gross substitutability is a hypothesis meaning that all the goods are gross substitutes for each other, so that:
$z_{ij}(\mathbf{p}) > 0$, for every $i \neq j$.

Metzler will show first: that Hicksian matrices – matrices with principal minors alternating in sign – are useful in dynamic economics. He also established that under gross substitutability, the conditions of Hicksian stability are the same as the conditions for true dynamic stability.

> "the Hicks conditions of perfect stability are necessary if stability is to be independent of […] price responsivness. Second, and more important, in a certain

> class of market systems Hicksian perfect stability is both necessary and sufficient for true dynamic stability. In particular, if all commodities are gross substitutes, the conditions of true dynamic stability are identical with the Hicks condition of perfect stability. ” (Metzler, 1945, 279-80).

In a few words, the main contribution of the Metzler analysis (together with those mentioned above) is to introduce once more the concept of substitutability into the analysis of stability, and to focus on the interpretative content of the analysis. The Hicksian tradition is reformulated around the gross substitutability hypothesis. And this hypothesis is taken as a fruitful point of departure. It was so much shared by the people at work at the time that Newman could write in 1959:

> “a good deal of the work on the analysis of stability has been directed towards establishing intuitively reasonable – or at least readily comprehensible – conditions on the elements of A, that will ensure stability.” (Newman, 1959, 3)

To make a conclusion on this first group of works on stability, one can say that the concept of substitutability has been worked out so that it can be used to describe and to interpret the main stability properties of an economic system. So, beyond the mathematics of stability, there is an interpretative content and a “positive heuristic”. It is enhanced first by the idea that substitutes are good for stability, and second by the idea that expressing stability conditions as general conditions on the elements of the Jacobian, that is on excess demands, will be enough to find reasonable stability conditions.

Now, the question is: what will be left of this tradition in the arms of the axiomatic turn in general equilibrium theory?

2. From “gross-substitutability” to instability examples

At first sight, the main feature of the axiomatic turn in the general equilibrium literature is to put to the background the interpretative content of the theory. In this section, I would like to put to the foreground the mode of development of general equilibrium analysis after Arrow-Debreu-McKenzie results. I will begin by enhancing the intuitive privilege that was attributed to the stability hypothesis, and as a consequence, the interest for the gross-substitutability assumption (**2.1. “Gross substitutability” as a reference hypothesis**). Then, I will focus on instability examples and the way the results have been received by theoreticians (**2.2 Scarf and Gale counter-examples**).

2.1 “Gross substitutability” as a reference hypothesis

Most of the work on stability in the fifties and sixties is centered on the hypothesis of gross substitutability. It is a sufficient hypothesis for unicity of equilibrium (Arrow and Hurwicz, 1958). It is also the hypothesis with which Arrow, Block and Hurwicz established the global stability of the tâtonnement in 1959. This result was presented as a confirmation of the importance for stability of the substitution among goods.

Let's make a short digression on the status of concept and hypothesis within the axiomatic phase of general equilibrium theory. Axiomatisation is usually at odds with the interpretative content of the concepts and hypothesis (Debreu, 1986; Ingrao and Israel, 1990). The question is thus whether the heuristic properties of substitutability should remain relevant in this context. The answer is yes. Leaving aside the relevance of the tâtonnement as a descriptive tool, the fact is that most of the theoreticians, I mean those who were interested in the work on stability, tend to consider that the concept and hypothesis used should have some heuristic properties. This aspect of the work on stability, compared with other fields of general equilibrium theory, has never been underlined. In any case, it is certainly a key to study the development of stability analysis and to understand the reactions of the main protagonists. Otherwise stated, everything happens as if the descriptive content of general equilibrium was not only at the level of the dynamic process but also at the level of the properties of the excess demand functions giving stability. In this sense, substitutability plays a heuristic role in the stability analysis, in conformity with Hicks's ideas. It is also to be mentioned that some theoreticians have always privileged a rather soft axiomatic. This is exemplified in Arrow and Hahn (1971) and it can be traced back to Abram Wald (1936).

Lets come back to Arrow and Hurwicz (1958) article. It is a good example of the optimism of the time. In this article, they show that certain kinds of complementarity relations are logically impossible within the framework of a Walrasian economy. This is taken as reducing the conditions of instability:

> "[The] theorem (…) suggests the possibility that complementarity situations which might upset stability may be incompatible with the basic assumptions of the competitive process" (Arrow and Hurwicz, 1958, 550)

In the same time, the gross substitutability assumption is seen as not realistic. But gross substitutability is after all nothing more than a sufficient condition for stability, and the field of investigation seems to be open for less stringent hypothesis, introducing complementarity.

So, during the axiomatic turn, there is a slight epistemological shift in stability analysis. On the one hand, there is still the Hicksian idea that substitutes are good for stability, but it is quite clear that substitutes, as opposed to complementary goods, will not do all the work, and that the task will not be so easy to achieve. The fact is that generalisation of the gross substitutability assumption (the weak gross substitutability) was not so easy to obtain. On the other hand, it is clear also that substitutability is still regarded as the most important concept in order to express stability conditions and to describe the structural properties of an economy. Neither the diagonal dominance, nor the weak axiom of revealed preferences in the aggregate caught that much interest.

As a consequence of this optimism and of the heuristic content of substitutability, one can understand the situation of the work on stability at the end of the fifties. There is

a kind of benevolence to the gross substitutability assumption. It is the task for unsatisfied people to prove that gross substitutability is not so appealing, and that the concept of substitutability may not be enough to study stability. What makes the interest of this story is that counter-examples of unstable economies will arrive a few months later.

2.2 Scarf and Gale's counter-examples

The two important contributions of Scarf (1960) and Gale (1963) will shift the debate on stability. I will not enter precisely into their construction here. Just to go to the point of my analysis, they construct a general equilibrium model with three goods, based on individual rational agents, so that the tâtonnement process of the economy does not converge to the unique equilibrium. Scarf's example implies complementarities between two goods, and asymmetrical income effects. Scarf comments on his results, underlying that instability comes from *pathological* excess demand functions. Scarf's attitude towards this result is ambiguous. On the one hand, he asserts that "Though it is difficult to characterise precisely those markets which are unstable, it seems clear that instability is a relatively common phenomenon" (Scarf, 1960, 160). On the other hand, he gives some possible objections to the empirical relevance of his model:

> "As a final interpretation, it might be argued that the types and diversities of complementarities exhibited in this paper do not appear in reality, and that only relatively similar utility functions should be postulated, and also that some restrictions should be placed on the distribution of initial holdings. This view may be substantiated by the known fact that if all the individuals are identical in their utility functions and initial holdings, then global stability obtains" (Scarf, 1960, 160-161)

As for Gale (1960, 8), he will insist on Giffen goods to explain the instability examples obtained. The same kind of interpretation can be found in Negishi (1962) and also in Quirk and Saposnik (1968, 191), who are of opinion that the stability of a tâtonnement "is closely tied up with the absence of strongly inferior goods".

Nevertheless, the Scarf and Gale examples are received with a kind of perplexity. Everything happens as if their models where singular models, and thus as if they were not affecting the general idea that systems including enough substitutability may be stable. Anyway, it is now urgent to find less stringent conditions including complementarity, guaranteeing stability. At this moment, the interpretative content of substitutability is at stake.

With Scarf's and Gale's examples, the situation is reversed. The suspicion is now clearly on stable systems, and it is now the task for all those who have a positive *a priori* in favour of stability to produce examples of stable systems including complementarity relations. In fact, Scarf's results make it possible to question the heuristic content of substitutability. Actually, by identifying many possible sources of instability, relating to the spread of initial endowments and to the variety of preferences, and to their

implications on demand, the interpretative and descriptive content of substitutability tend to vanish. It does not seem possible to express *only* with substitutes and complements the characteristics of an economy, and its properties for stability. Nevertheless, substitutability remains the main concept with which it is thinkable to search for stability conditions. As a proof for this, it is remarkable that neither the diagonal dominance hypothesis, nor the weak axiom of revealed preference caught the same attention. Diagonal dominance, for example, is less stringent than gross substitutability, because gross substitutability implies diagonal dominance. But practically, no utility function has been found that may imply diagonal dominance without implying also gross substitutability. Moreover, only certain forms of diagonal dominance do guarantee stability. Then, it seems easier to provide less stringent conditions by taking gross substitutability as a yardstick than by taking diagonal dominance as a reference. One reason for this choice is that diagonal dominance has a "Marshallian flavor" (Arrow and Hahn, 1971, 242). But it is clear that it will not be so easy to introduce complementarity in the system. The question now is: How to introduce complementarity in a stable system. An eventful period is bound to take place.

3. The end of a research programme?

So far, I have indicated how a general framework of interpretation of the work on stability of an exchange economy was constructed. As was seen in the first section, the idea to search for sufficient conditions introducing complementarity pre-existed to the Arrow Block Hurwicz result and to the Scarf and Gale counter examples. In this section, I want to focus on two different kinds of obstacles that were put on the road. Firstly, from an internal point of view, all the attempts that were made to generalise the gross substitutability assumption did not give many results. What is clear from Scarf and Gale is that it was no more possible to introduce complementarity arbitrarily (**3.1 The impossible generalisation)**. From an external point of view, then, some work in the seventies and eighties questioned radically the research programme, as it had been formulated by Walras (**3.2 Through the looking-glass, and what Sonnenschein, Mantel, Debreu, Smale and others found there**).

3.1 The impossible generalisation

This is an important point for my thesis. The time period following immediately the Scarf results shows a progressive renunciation of the hope to obtain much better than the Arrow, Block and Hurwicz 1959 paper. It is the true moment when the heuristic of substitutability failed.

Actually, the hope to find stability results with complementarity relations was on the agenda from 1945 on (Metzler), and it had just been confirmed by the results on global stability. From the beginning of the fifties on, Morishima spend a lot of time working on this agenda (Morishima, 1952, 1954, 1960 and 1970). Morishima's idea was

to introduce complementarity relations between certain goods. He thus proposed an economy whose excess demands would verify that goods were grouped together so that all the substitutes of substitutes are substitutes to each other and so that all complementary goods of complementary goods where complementary to each other. In the same spirit, McKenzie (1960) established the dynamic stability of an exchange economy for which certain sum of the partial derivatives of the excess demand with respect to the prices is positive, which allows introducing a certain amount of complementarity into the system.[1] Finally, Nikaido (1964) proposed the generalised gross substitutability assumption, i.e. that the sum of the symmetrical terms relative to the diagonal of the Jacobian be positive.[2] In such a system, if tea is a gross complement to sugar, the sugar must be a gross substitute to tea.

Remarks on all these developments are in order:

Firstly, all the results obtained have strong limitations relatively to the programme of general equilibrium theory. They are not independent of the choice of the numéraire good and they are valid only locally. For example, the Morishima case was showed to be incompatible with a Walrasian economy, because of the properties of the numéraire commodity. Ohyama (1972) added a condition on the properties of substitution of the numéraire to ensure stability.

Secondly, most of the results I have mentioned suppose that they refer to quantitative constraints on excess demand functions, in the sense that they suppose comparing the relative strength of partial derivatives. From this point of view, the diagonal dominance hypothesis goes in the same direction. All these limitations illustrate the doubts that arose regarding the hopes for finding a true generalisation of the gross substitutability hypothesis. The heuristic of substitutability is shrinking. The question now is to discuss the relevance of quantitative and structural restrictions on excess demand functions. The change in the spirit and in the state of mind of the theoreticians can be clearly felt. Just to give a quotation by Quirk (1970):

> "In contrast to the Arrow-Hurwicz results, here we do not prove instability but instead show that stability cannot be proved from the qualitative properties of the competitive model alone, (…), except in the gross substitute case" (Quirk, 1970, 358).

It is a very clear way to renounce establishing general properties compatible with stability. This reduces quite naturally the analytical appeal of substitutability (see also Ohyama, 1972, 202). Finally, at this moment, theoreticians have realised to what degree

[1] For any partition of the set of goods J = (1, ...,n) into two subsets J_1 et J_2, there is the relation : $\sum_{i \in J_1} z_{i\ j_2} + \sum_{i \in J_2} z_{i\ j_1} > 0$ for all $j_1 \in J_1$, $j_2 \in J_2$.

[2] I.e. that $z_{ij}+z_{ji}$ is positive or null for every *i*, *j*

the gross substitutability assumption was specific, as the only qualitative hypothesis on excess demand functions guaranteeing stability of a tâtonnement process.

To sum up, in the beginning of the seventies, the work on stability gives a very pessimistic, and even negative, answer to the agenda originally formulated by Metzler and then by Arrow, Block and Hurwicz. Two kinds of results will come and evacuate a bit more any interest with this kind of work: the well-known Sonnenschein-Mantel-Debreu theorem on the one hand, and the Smale-Saari-Simon results on the other hand.

3.2 through the looking glass and what Sonnenschein, Mantel, Debreu, Smale and others found there.

Already in the forties and fifties, it was known that some properties of individual demand behaviour would not be preserved at the aggregated level in general (see for example Schultz, 1938 and Hicks, 1956). There was a gap between weak restrictions on the demand side and stringent sufficient conditions for stability. So, while the work on stability was progressing only very slowly, and not with the results that were expected, a group of theoreticians was engaged in taking the problem from the other side, that is from the hypothesis of individual maximising behaviour:

> "Beyond Walras' Identity and Continuity, that literature makes no use of the fact that community demand is derived by summing the maximizing actions of agents" (Sonnenschein, 1973, 353)

If it is not possible to demonstrate that an economic system comprising complementarity is stable, is it not possible to show that any general equilibrium system exhibits some properties regarding the excess demand functions. This would be at least a way to measure the gap. The answer to this question is well known. It is a series of negative results known as Sonnenschein-Mantel-Debreu theorem. Market excess demand generated by an arbitrary spread of preferences and initial endowments will exhibit no other properties than Walras law and the homogeneity of degree zero relative to prices. Otherwise stated, given an arbitrary set of excess demands, one can always construct an economy that will produce those excess demands.

The question raised by Sonnenschein, Mantel, Debreu and others goes against the usual stream of investigation concerning stability. But it is the most natural stream in terms of the individualistic methodological foundations of the general equilibrium program. Nevertheless, this result raised some dubitative reactions. After all, the distribution of endowments and of preferences allowing for such arbitrary excess demand in the aggregate may well be no more realistic than the conditions generating a representative agent (Deaton, 1975). An article by Kirman and Koch (1986) showed that the class of excess demands would not be restrained even if the agents had the same preference relations and co-linear endowments. To improve further on the constraints would mean to construct a representative agent. So, the S-M-D result would imply that

Giffen goods are quite "normal" goods in a general equilibrium framework, and following Scarf and Gale conclusions, "instability" would be a common feature of economic systems. Then, the S-M-D theorem reduces still a bit more the relevance of quantitative restrictions that would yield stability. The change in the spirit of the economists has been portrayed by Mantel:

> "Another field in which new answers are obtained is that of stability of multimarket equilibrium. It is not so long ago that the optimistic view that the usual price adjustment process for competitive economies is, as a rule, stable, could be found – an outstanding representative is that of Arrow, Block, and Hurwicz (1959). Counterexamples with economies with a single unstable equilibrium by Scarf (1960) and Gale (1963) had a sobering effect, without destroying the impression that the competitive pricing processes show some kind of inherent stability. Here the question arises whether such counterexamples are likely, or whether they are just unlikely exceptions" (Mantel, 1977, 112)

Well, after the S-M-D result, Scarf and Gale counterexamples could no longer be regarded as improbable, if the excess demand should have arbitrary properties.

But from a historical point of view, one must keep in mind that there was a twelve year gap between the reception of the SMD result and the strengthening of the result by Kirman and Koch. What happened during that time is also very fruitful for our subject. For all those who were discouraged by the negative turn of the events, for those who had only a poor faith in the possibility to find results, the Scarf counterexamples were a starting point for something else.

We have seen that Scarf felt uncomfortable with the instability result. This was the starting point for an inquiry into dynamic systems and algorithmic computation of equilibrium (Scarf, 1973). In this field of research, Steve Smale endeavoured to cope with the question of stability. His purely mathematical look at the subject kept the interpretative content outside, and he readily understood that in general equilibrium "complexity keeps us from analysing very far ahead" (Smale, 1976c, 290). Rather than concentrating his reproaches on the descriptive content of the tâtonnement process and on the stability conditions that were found, Smale tackles another question, quite different from that of Sonnenschein, Mantel and Debreu. If equilibrium exists, how is this equilibrium reached? After Scarf (1973), Smale will found a dynamic process much more complex than Walrasian tâtonnement which allows finding the equilibrium, for any arbitrary structure of the excess demand functions. This process is called the Global Newton method", and it is a generalisation of a classical algorithm of computation of equilibrium. In this process, the variations in the prices on each market dp_i/dt will not depend solely on the sign of the excess demand $z_i(\mathbf{p})$ on this market, but also on the excess demands on other markets. This dynamic process is

$$Dz(\mathbf{p})\frac{dp}{dt} = -\lambda\, z(\mathbf{p}) \qquad (4)$$

with λ having the same sign as the determinant of the Jacobian.

Hahn's (1982) reaction to this kind of process is embarrassed. Indeed, Smale's process is shifting the general equilibrium programme. What can be the meaning of a dynamic process in which the behaviour of the prices on each market depends of the situation on every other market? Hahn does not have any answer to give. The fundamental problem is that this process is very demanding in terms of information. While the Walrasian auctioneer does not have anything else to know than excess demands at a given price vector, the fictional auctioneer of the Global Newton method will have to know about the qualitative properties of each excess demands. In 1978, Saari and Simon established that this amount of information was the price to be paid to find a computational method independent of the sign of the excess demand. This is precisely this kind of information that general equilibrium theory wanted to avoid. With the Sonnenschein-Mantel-Debreu and Kirman and Koch's theorems on the one hand and with Smale, Saari and Simon's results on the other, the stability research programme, in its original form, has collapsed.

4. Final remarks

In this paper, my aim was to put to the foreground the uses of the concept of substitutability in general equilibrium theory. Substitutability, as the main concept used to describe the qualitative properties of an economic system, was expected to provide also good interpretative properties i.e. it was hoped that substitutability would be a sufficient way to express general conditions under which the stability of the tâtonnement would be guaranteed. I have interpreted this very general idea as a guiding principle for the researches on stability. It was thought that substitutes and complements should represent enough information to formulate "reasonable" or "hardly credible" stability conditions. The point was then to see how this guiding principle, this positive heuristic, has been affected by the mathematical results that were found, and how it came to be deprived of interpretative content. Thus, progressively, one can see the vanishing of the heuristic of substitutability.

Of course, I do not pretend that substitutability was the only concept implied in the elaboration of the research programme. It is quite clear from my presentation that the formalisation of the Walrasian tâtonnement, the reflection on quantitative constraints, have also played a role in this story. But in the final analysis, the concept of substitutability has served as a criterion in order to evaluate the relevance of most of the results and to appraise the theoretical consequences of those results on the research programme. It has been a tool for rationalising the path followed by stability analysis.

From a methodological point of view, a conclusion that can be drawn from this study of stability is that the weakening of a research programme and its reformulation within the framework of a purely mathematical theory, do not depend on a unique result,

be it a negative result. The matter depends more pragmatically on the accumulation of many negative or weak results that come to be interpreted as a bundle of results indicating that something else must be done and that the programme must be amended. And it might be that the Sonnenschein-Mantel-Debreu result was not the most important result with regard to this amendment.

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