On the probability to act in the European Union

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Abstract

An important question for the European Union is to know whether its institutions will permit it to escape from political deadlocks each time a question is at stake. Several studies [1, 2, 10] suggest, by using the Impartial Culture assumption to model the voting behavior, that the EU could only take a decision in 2% of the cases with its current voting mechanisms. The Impartial Culture model has been criticized from a theoretical point of view [8], a political one [19] and does not fit with data [15, 20]. The Impartial Anonymous Culture assumption we consider in this paper is an alternative of this first model. First, we study the probability of approval under a generalized version of the Impartial Anonymous Culture assumption for the asymptotic case (reached when the number of countries N goes to infinity). Next with computer enumerations and Monte Carlo simulations, we evaluate the probability to act for a toy example and for the European Union with 27 members. We consider both the Treaty of Nice and some proposals for the European Constitution.

JEL classification: D7

1 Introduction

In the last five years, a considerable body of research on the choice of the best voting rules for federal unions have been inspired by the debates on the Treaty of Nice and the projects for an European Constitution. Without being exhaustive, we can mention the work by Baldwin *et al.* [1], Baldwin and Widgren [2], Barberà and Jackson [3], Bobay [6], Beisbart *et al.* [5], Feix *et al.* [9], Felsenthal and Machover [10, 12], Laruelle and Widgren [17]. All these contributions share a common organization: the authors propose a voting model, and then seek for the voting rule or the constitution that fits better according to some normative criteria.

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In particular, a key parameter for the analysis of voting systems is the *a priori* probability by which a decision is taken against the *status quo*. This probability is called by Coleman [7] 'the power of a collectivity to act', and the 'decision-making efficiency' or the 'probability of passage' by Baldwin and Widgren [2]. There is clearly a trade off between a low and a high value of this probability. If the probability to act is too low, the political system may be inefficient in the sense that no decision, even those supported by a large majority of the voter, may ever be approved. On the other hand, when the protection of minority opinion matters, the probability of passage should stay decently below 50%. This criteria can be used to analyze the different decision making procedures of the European Union with 27 members (EU27 hereafter). Currently, the decision scheme of the council of minister is the one described in the Treaty of Nice. First, each country is endowed with a certain number of mandates, ranging from 3 for Malta to 29 for Germany. A proposal must then receive 255 mandates out of 345[1]. It should also pass two extra conditions: it must be approved by a majority of states, gathering at least 62% of the population. Felsenthal and Machover [10] have shown that there is only a handful of cases out of 2^{27} where the second and third conditions are not met while the first one is met, which justifies the fact that most of time, the analysis only focuses on the first game. This simplification is no longer possible for the decision scheme proposed in the draft constitution proposed by the European Convention in 2003. The convention suggested that a decision would be adopted if it could be supported by 50% of the states gathering 60% of the populations. The constitutional treaty finally proposes a similar procedure: a proposal must receive the support of 55% of the states, representing at least 65% of the population[2].

Recently a welcome and quite useful discussion between a high level Swedish diplomat (Axel Moberg) and scientists (Dan Felsenthal and Moshe Machover) has developed [19, 11]. At the origin, the scientific analysis of the Treaty of Nice [1, 2, 10, 12] claims that the need of 255 (resp. 258) votes on a total of 345 to approve a proposition at the council of ministers of the European Union will result in a serious deadlock with an *a priori* probability of passage of 2%. (resp 1.7%). A. Moberg disagrees strongly, pointing out that the result ignores the "strong consensual culture of the EU". Who is right ? In fact, the scientific analysis given in [1, 2, 10, 12] is only a part of the full story : it is based on the use of the Impartial Culture (IC hereafter) model, which states that each country chooses to vote 'yes' or 'no' independently with equal probability. In other words, each country flips a fair coin to take a decision! But other models to describe the behavior of the states exist! In particular, the Impartial Anonymous Culture (IAC) model asserts that all the distributions of the votes at the Union level are equally likely.[3]

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[1] The Treaty of Nice specified that when all candidate countries have acceded, the blocking minority in a Union of 27 will be raised to 91. Thus, the quota has been lowered to 255 instead of 258, which was first specified elsewhere in the treaty. This strange specification of the treaty explain why the 255 and 258 thresholds have both been studied in the literature. For a detailed analysis of the Treaty of Nice, see Felsenthal and Machover [10]

[2] When the Council of minister is not acting on the basis of a proposal made by the Commission or on the initiative of the Union Minister for Foreign Affairs, this last quota is risen to 72% of the population.

[3] Notice that the widely used Banzhaf power index relies upon the IC probability assumption, which is know as the Independence assumption in the power index literature (Straffin [21]). For its part, the IAC model can be associated to the Shapley–Shubik power index, and is then called the Homogeneity assumption (Straffin [21]). The link between the probability models in social choice theory and power indices literature was first emphasized by Berg [4].

The aim of this note is to show that the use of a model related to the IAC one is able to give answers which are closer to the reality of the European Union with 27 members and, in some way, takes into account the consensual character of the vote. By departing from the common IC assumption, we obtain a theoretical probability of passing a motion that turns out to be higher. Our result concerns not only the Treaty of Nice with its famous 73.9% majority rule (one key vote), but the double key votes decision schemes that have been suggested during the debates for the European Constitution (a motion is passed if it is supported by more than q_A% of the countries gathering more than q_B% of the population).

The paper is organized as follows. In section 2, we present the voting models, and the different probability assumptions, and we briefly discuss their adequation to the vote at the council. In section 3, we give the theoretical probability of approval under the Generalized Impartial Anonymous Culture assumption in the asymptotic limit, *i.e.* when the number of countries (denoted by N in what follows) goes to infinity. Section 4 checks the relevance of this asymptotic solution for a toy example and for EU27, by providing numerical simulations. We present our conclusions in section 5.

2 The model

2.1 The voting rules

We consider binary issue votes 'yes' or 'no' for the N states (elsewhere voters, MPs, etc...) of a federal union. The decision are made by the use of the conjunction of two weighted quota games. Each state has two mandates a_i and b_i, and his (her) vote ('yes' or 'no') is used in two qualified majority games $\mathcal{A}$ and $\mathcal{B}$, the respective quotas being Q_A and Q_B. Notice that for each state i, it is the same vote ('yes' or 'no') which is used to compute the number of mandates obtained by a motion respectively with keys $\mathcal{A}$ and $\mathcal{B}$. The two quotas must be reached for final approval, each one being related to a certain type of legitimacy. The EU Constitution project proposes for country i to take $a_i = 1$ and b_i equal to the population of state i. Let $A = \sum_{i=1}^{N} a_i$ and $B = \sum_{i=1}^{N} b_i$. We will denote the relative quotas by $q_A = Q_A/A$ and $q_B = Q_B/B$.

2.2 The Impartial Culture

In the IC model, each vote is independent of the others and each voter says 'yes' or 'no' with equal probability $p = 1/2$. IC has serious drawbacks. It describes a vote where everybody is undecided (no exchange of points of view allowing the emergence of a majority has taken place) which leads to the existence of two blocks of equal importance. When we consider one weighted quota game defined by a quota (Q_A) and a vector of weights $(a_i)_{i=1\ldots N}$ with 1) a large number of voters, 2) no dominant player in term of weight, and 3) scattered weights, a natural way to handle the IC case is to notice that the probability that a proposal receives between x and $x + \Delta x$ mandates (with Δx small) can be approximated by a normal distribution (see Feix *et al.* [9] for example) with mean $m = \frac{1}{2}\sum_{i=1}^{N} a_i$ and variance $\sigma^2 = \frac{1}{4}\sum_{i=1}^{N} a_i^2$. Then the vote will be won by a margin in term of mandates going as σ as N grows. This explains the low probability of approval with a quota of 255/345 *i.e.* 73.9% in the Treaty of Nice decision scheme which is characterized by $m = 172,5$ and $\sigma = 39,84$.

2.3 Toward homogeneity

A naturel way to escape from the divided society described by the IC assumption has been, both in the game theory and in social choice literature, to consider that all the partitions with x states in favor of a proposal and $N - x$ against it should be equally likely. Thus, the equiprobability assumption is put on the results of the votes. This leads to the homogeneity assumption and the definition of the Shapley-Shubick index in the power indices literature, and the so called IAC assumption in social choice theory[4]. A classical interpretation of the IAC model is to state that, before the vote, a probability p of voting for the proposal is drawn from the uniform distribution on $[0, 1]$.//

The idea is consequently to introduce a model where a probability p different from 1/2 has emerged. Moreover, our knowledge of p is itself of a probabilistic nature, it is mathematically described by the function $f(p)$ which is the probability distribution of p. The emergence of a probability p different from 1/2 seems rather natural in an assembly where certainly long discussions, explanations, compromises, package deals, etc... precede each vote (the "consensual culture" of A. Moberg). Notice that, all these discussions are resumed in a $p \neq 1/2$ and that the subsequent votes are independent. Then the Generalized Impartial Anonymous Culture (GIAC) model is characterized by a given $f(p)$ with $0 \leq p \leq 1$, $f(p) \geq 0$ and $\int_0^1 f(p)\mathrm{d}p = 1$. The function $f(p) = 1$ for all p gives the IAC model.

3 The probability of approval in the asymptotic limit

Till now, we have discussed the behavior of the voters and the distribution of their votes, without taking into account their number of mandates. However, we need to know the distribution of the mandates in favor of a proposal in order to evaluate the probability to act[5]. Let $F_N(x)$ be the distribution function for x mandates in favor of approval in the single key $\mathcal{A}$ and $F_N(x, y)$ the distribution function of x and y mandates in favor of approval for keys $\mathcal{A}$ and $\mathcal{B}$ respectively. In order to be able to perform analytical computations, to see the role of $f(p)$ and the influence of the different quotas, we suppose that N tends to infinity to use asymptotic calculations. We also assumes that the mandates are scattered enough so that all the configurations with $x \in [0, A]$ are explored. In addition, notice the all the configurations cannot be explored if one state dominates outrageously the union in term of mandates. At the end, we will compare the obtained asymptotic results to numerical simulations and will see that, as already stated in [9], EU27 can be fairly approximated by this limit for the one key vote and two key votes.

Proposition 1 *In the asymptotic limit, for a GIAC model characterized by $f(p)$, the distribution function $F_N(x)$ for x mandates in favor of approval in the single key case reads*

$$F_N(x) = \frac{1}{A} f(x/A), \tag{1}$$

[4]The IAC assumption has been introduced in social choice theory by Fishburn and Gehrlein [13] in order to compute a priori the likelihood of the Condorcet Paradox for three alternatives. Here, there are six possible preference types, and a probability p is now a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ in the unit simplex, where p_i is the probability of picking preference type i for each voter. The IC assumption is based upon the vector $p = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$ while the IAC assumption assumes that p is drawn from an uniform distribution on the unit simplex. For more on the likelihood of the Condorcet paradox and the use of probability models in social choice, see the recent book by Gehrlein [14].

[5]The knowledge of this distribution is also useful when one wants to evaluate the influence of a state measured by a power index.

while for the double key case, we obtain

$$F_N(x,y) = \frac{1}{AB}\,\delta(\frac{x}{A} - \frac{y}{B})\, f\left(p = \frac{x}{A} = \frac{y}{B}\right), \tag{2}$$

where δ is the Dirac distribution.

Note that x and y are strictly correlated ($\frac{x}{A} = \frac{y}{B}$) in (2). Recall that in formulae (1) and (2), A and B are defined by

$$A = \sum_{i=1}^{N} a_i, \qquad B = \sum_{i=1}^{N} b_i. \tag{3}$$

The proof of this proposition is given in the appendix.

For the IAC model ($f(p) = 1$) and a single key vote, equation (1) means a flat density from $x = 0$ to $x = A$. Still for the IAC model but for the double key vote, the points are located on the segment joining ($x = 0, y = 0$) and ($x = A, y = B$) with a uniform distribution on this segment. In the approximation used to obtain (1) and (2), votes characterized by p give points located at $x = pA$ and $y = pB$.

Note that these results hold for N going to infinity. It can be shown that the first correction (N large but not infinite) provides a diffusion around these points in $N^{-1/2}$. While this scattering slightly modifies the flatness of $F_N(x)$ for the one key vote, it transforms the segment of the two key vote into a long ellipse with a ratio long over small axes in $N^{1/2}$. A simulation with 100 states will illustrate these facts in the next section.

Coming back to the segment structure, we see that in the double key vote case, with two unequal quotas, it is the one with the highest quota which will set up the frequency of 'yes' votes. Notice also that integration of (2) respectively on y and x gives

$$F_N(x) = \frac{1}{A}\, f(\frac{x}{A}), \quad \text{and} \quad F_N(y) = \frac{1}{B}\, f(\frac{y}{B}), \tag{4}$$

in agreement with the results of the one key vote.

Recall that formulas (1) and (2) are only valid if the weights a_i are sufficiently scattered, for N going to infinity. Thus, from an operational point, their validity may be questioned[6]. We will tackle this question for the IAC model in the next section. First we will observe the convergence to the limit with a toy example where the a_i and the b_i will be drawn randomly and independently from an uniform distribution on the interval $[1, 5]$. Next, we will study whether the discrete distribution of the mandates in the EU27 affects the convergence to the limit.

[6] Precise conditions on the weights that ensures the validity of approximations can seldom be found in the literature. See for example Lindner and Machover [18].

4 Numerical simulations under the IAC assumption

4.1 An example with random weights

In this section, the results of numerical simulations will be shown for the IAC case. Because we want to reach the asymptotic limit which supposes both an important number of elections and a large number of states, Monte Carlo method should be used. Actually, it is not possible, when the number of voters is large, to enumerate, stock and compute the 2^N configurations because of lack of memories and computation time.

In addition, the Monte Carlo technique will illustrate clearly the double probabilistic character of the IAC model. This method consists to make a random sampling among all the vote configurations, but without taking all of them. Then the contribution of all the samples are gathered.

The method has two steps. First a probability p is chosen at random in the distribution function $f(p)$. Second a vote configuration is chosen accordingly this probability p : for each of the N voters, a random number is taken in a uniform distribution, if this number is lower than p, the voter gives its mandates (it is a 'yes' vote) while he doesn't if the number is higher. This is in fact an acceptation-rejection method and if the number of voters is large, the number of 'yes' voters divided by N will tend toward p. This process is repeated for a large number of elections with, at each election, a choice of a new p into $f(p)$ and so on[7].

The mandates of the N states have been taken at random in a uniform distribution between 1 and 5, then the sum has been normalized to 100. Notice that the draws of a_i and b_i are independent. As a consequence, it is possible to find a pair of states (i, j) such that $a_i > a_j$ while $b_i < b_j$. We display the results obtained for $N = 10$, $N = 100$ and $N = 1000$ and $M = 50000$ elections, both for a single key and a double key vote using the Monte Carlo technique.

For the single key case, Figures 1, 3 and 5 show the histogram of the number of configurations, as a function of the related number of mandates. This normalization does not change the ratio 5 between the highest value of the mandates and the smallest one. The histogram becomes flatter as N increases in agreement with equation (1) and the probability of approval is tends to $(1 - q_A)$.

For the double key case, figures 2, 4 and 6 give the distribution of the M elections performed in the plane (x, y), one point representing one election. Because all the points have the same weight, their density gives the value of $F_N(x, y)$ (see equation (2)). As expected, the points are roughly distributed on the segment delimited by the two points $(0, 0)$ and (A, B). In addition to this global behavior, the distribution shows a certain scattering, which is more pronounced for N small. Consequently the results obtained in the asymptotic limit are recovered. We have checked that, for this case, the probability of approval is closely given by $1 - sup(q_A, q_B)$ as shown Table 1. With $N = 100$, for $q_A = q_B = 70\%$, we get 29.31% of approval and for $q_A = 50\%$ and $q_B = 80\%$, we get 20.56%. For Figure 6, the number of voters has been increased up to $N = 1000$. We observed that the scattering of the points decreases as expected.

[7]Notice that the results of the IC model could also be obtained by this technique. The probability p of the N voters is then always equal to 1/2 which corresponds to $f(p) = \delta(p - 1/2)$.

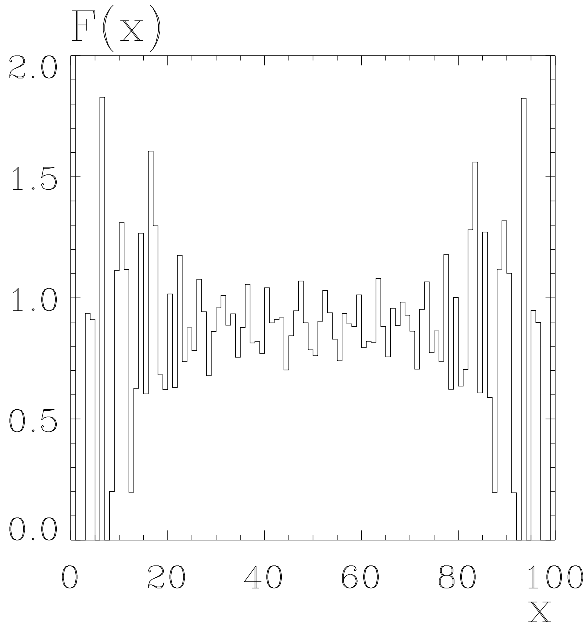

Figure 1: One key vote. Distribution of the results of the votes for $N = 10$ voters and $50,000$ elections using the Monte Carlo technique. The histogram is built with an increment $\Delta x = 1$.

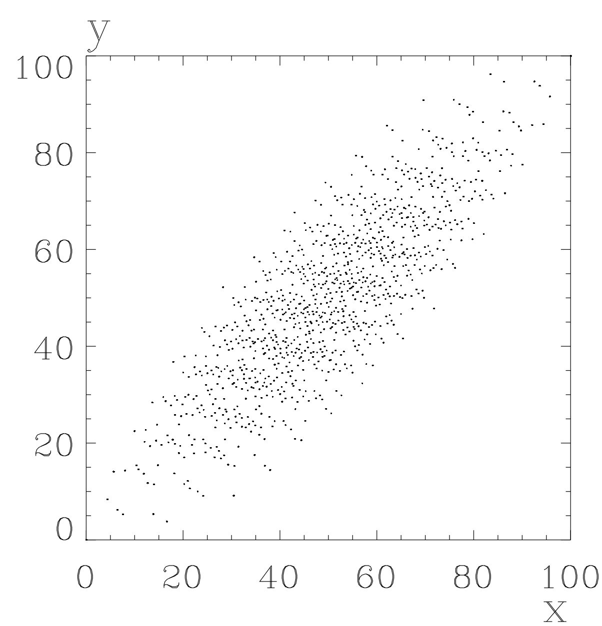

Figure 2: Double key vote. Distribution of the results of the votes for $N = 10$ voters and $50,000$ elections using the Monte Carlo technique. Each point represents the result of an election.

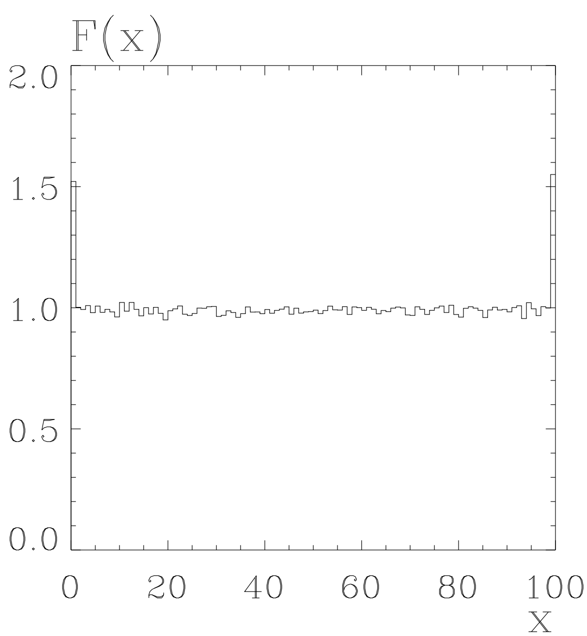

Figure 3: Same as figure 1 but for $N = 100$ voters.

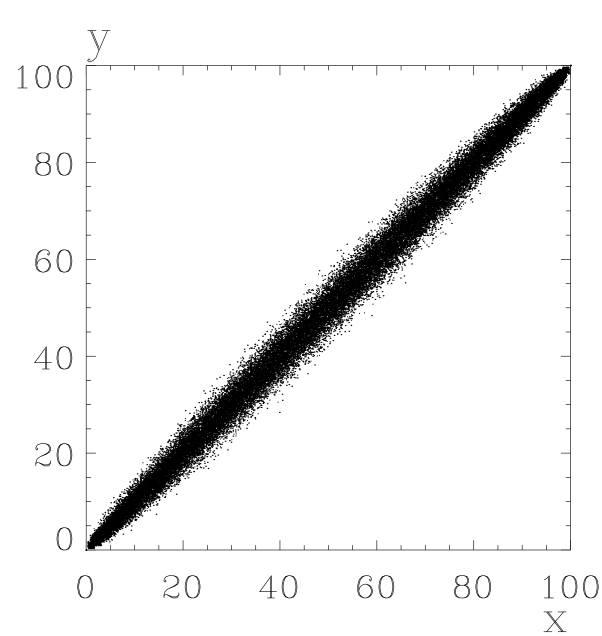

Figure 4: Same as figure 2 but for $N = 100$ voters.

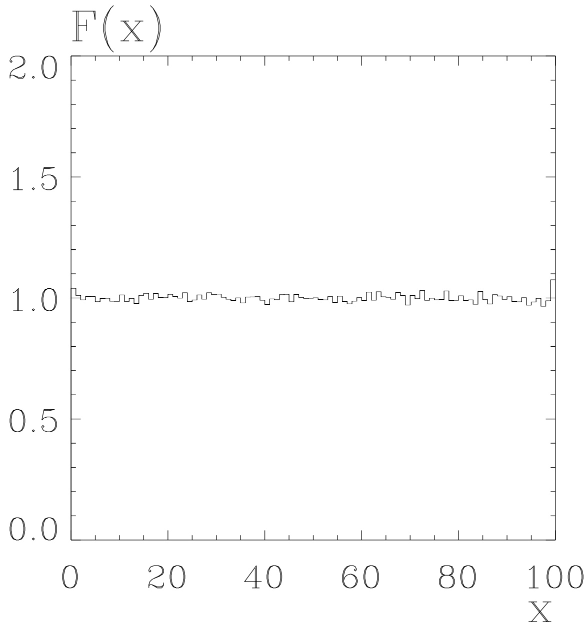

Figure 5: Same as figure 1 but for $N = 1000$ voters.

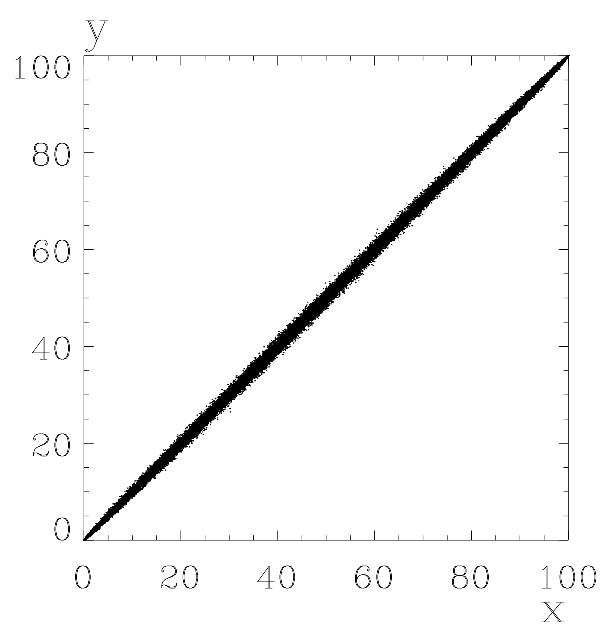

Figure 6: Same as figure 2 but for $N = 1000$ voters.

		Key $\mathcal{A}$				
		50%	60%	70%	80%	90%
	50%	49,04	40.25	30.43	20.58	10.54
	60%	40.19	39.01	30.43	20.58	10.54
Key $\mathcal{B}$	70%	30.27	30.27	29.31	20.58	10.54
	80%	20.56	20.56	20.56	19.66	10.54
	90%	10.61	10.61	10.61	10.61	9.86

Table 1: Double key vote. Probability of passage for the simulation presented figure 3 as a function of the two quotas q_A and q_B.

4.2 The probability to act for EU27

Now, the question is to know whether or not the asymptotic limit is a good approximation for the EU27, where the mandates are integers ranging from 3 to 29[8]. It is here possible to enumerate the 2^{27} vote configurations (taking care of their different weights).

For the single key case, Figure 7 shows the histogram of the number of configurations as a function of the related number of mandates. The central part of the curve is flat, in accordance with equation (1) but we cannot avoid the effect of a finite number of states on the edges. For $Q_A = 255$, the probability of approval is 27.50%, rather close to the result predicted by the asymptotic limit, $(1 - q_A) = (1 - 255/245) = 26.08\%$. Also notice that we are far above the 2% level of approval predicted by the IC model!

Figure 8 (respectively 9) shows the histogram when the number of mandates have been taken proportional to the state populations (respectively to the square root of the state populations). In figure 8, the weights are more scattered compared to Figure 7, but the ratio between the smallest and the biggest state has increased; thus, the flatness of the curve is less pronounced. The case of mandates proportional to the square root of the population constitutes a good compromise between the state legitimacy and the citizen legitimacy (see [6]). Moreover, the weights are scattered and the ratio between the mandates of the states remains relatively low. This explain why the curve is rather flat, at least for q between 0.2 and 0.8, indicating that the asymptotic limit could be used for this single key vote.

For the double key case, the probability to act is given for different values of the keys Q_A and Q_B in Table 2 which proves that the rule $1 - sup(Q_A, Q_B)$ for the approval is fairly satisfied. In particular, we observe that for 15 states gathering 65% of the population, the probability of passage is 35.24%, far above the probabilities obtained by Baldwin and Widgren [2] for different two keys decision method under the IC assumption.

We turn back to Monte Carlo simulations to analyze the dispersion of the votes for the two keys (although complete enumeration is possible) because each point has the same weight. Then, it is easier to interpret Figure 10 which gives the distribution of 2,700 vote configurations in the plane (x, y) (one point represents result for an election). For key $\mathcal{A}$, all the mandates are equal to 1 (state legitimacy) while for key $\mathcal{B}$, the number of mandates of a state is proportional to its population. The sum of the mandates of key $\mathcal{B}$ has been normalized to 100. Because of the discrete nature of the key $\mathcal{A}$ mandates, the points are aligned on vertical lines distant of 1. The scattering of the points, not negligible, is compatible with the $N^{-1/2}$ law as stated before.

5 Conclusion

In most of the applications of statistical models to voting theory, like the studies computing the Condorcet effect probability or evaluating the Condorcet efficiency of scoring rules, it was often found that the IC and IAC models were giving very similar results in terms of the magnitude of the paradoxes. It is with the study of binary votes that the fundamental differences between the two models become apparent. The issue is of great importance if we remember that the

[8]The number of mandates and the population data for 2003 can be found in Moberg [19].

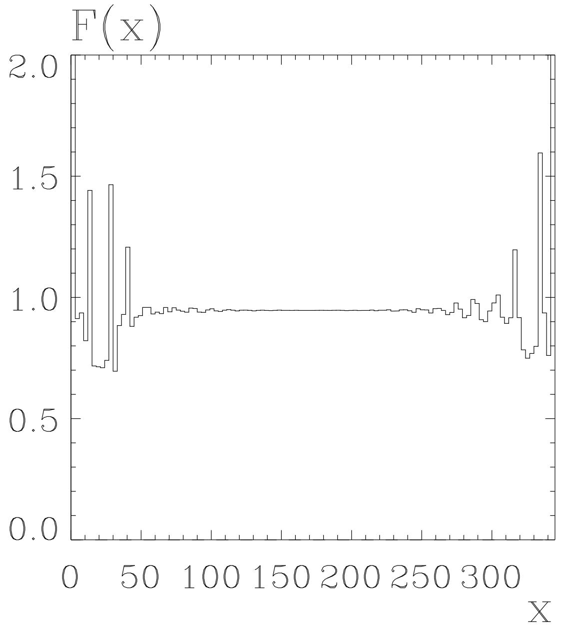

Figure 7: One key vote. Distribution of the results of the votes for the EU27 for the Treaty of Nice. The histogram is built with a bin size equal to 3.

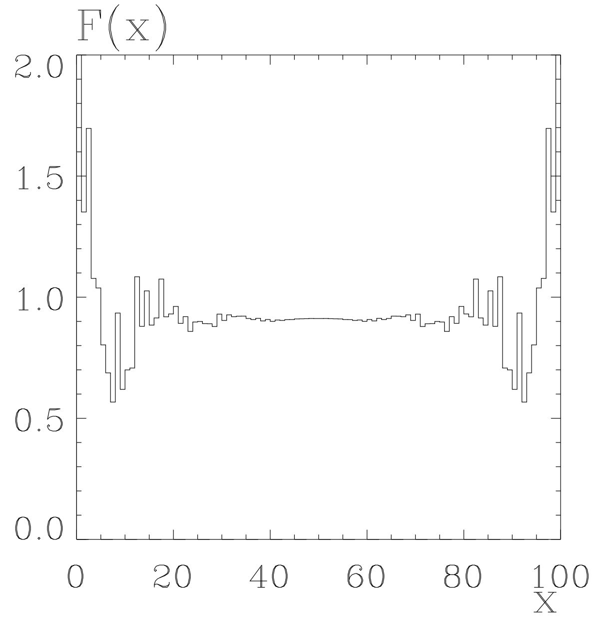

Figure 8: One key vote. Distribution of the results of the votes for the EU27. The mandates are proportional to the populations of the states and the sum is normalized to 100.

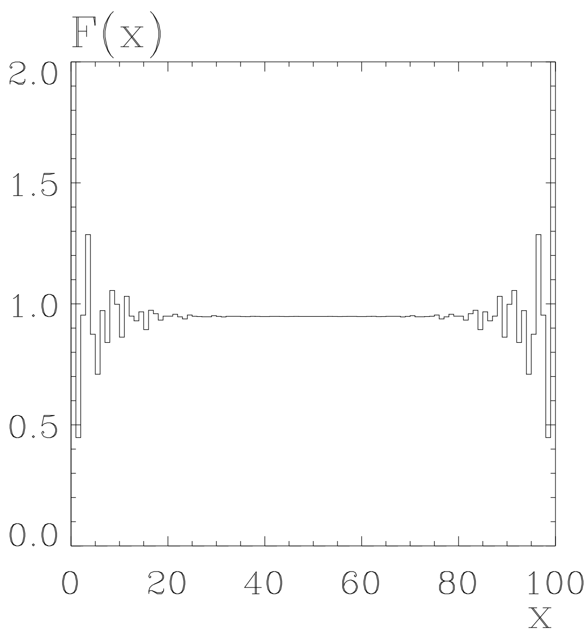

Figure 9: One key vote. Distribution of the results of the votes for the EU27. The mandates are proportional to the square root of the populations of the states and the sum is normalized to 100.

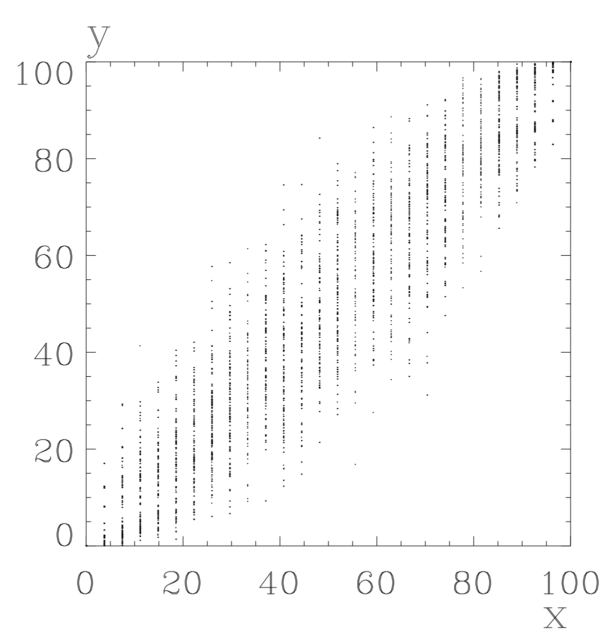

Figure 10: Double key vote. Distribution of the results of the votes for the EU27. For key $\mathcal{A}$ (x variable) all the mandates are equal to 1, for key $\mathcal{B}$ (y variable) the mandates are proportional to the populations of the states and the sum is normalized to 100.

two main power indices (Banhzaf and Shapley–Shubik) are respectively based on IC and IAC assumptions. In particular any recommendation on the number of mandates to attribute to a state in a federal union based upon a power index is contingent to the underlying probability model that we use.//
Similarly, in our study on the probability to act, except if we take quotas closed to 1/2, IC and IAC give results which differ by a large factor. While many studies [1, 2] suggest a fast decline of the probability of passage under IC when the quota rises, the use of the IAC assumption to model a more consensual behavior of the states gives a different picture. We have shown, both for the IC case (see Feix *et al.* [9]) and for IAC (this paper) that the density function of probability to get x yes votes in the EU27 is already quite close to the results one would obtain in the asymptotic case. Thus, in the asymptotic limit, the probability of passage tends to $(1-q_A)$ for one key vote and $1-max(q_A, q_B)$ for two key vote under IAC. As a consequence, quotas of 60% to 70% can still be considered as acceptable with the IAC model, while a similar study done with IC would lead to the opposite conclusion.To some extent, we have shown that the critics of A. Moberg were directed against the IC model but can be easily answered through the use of the IAC model or another GIAC model characterized by the adequate $f(p)$.//
Thus, can we decide which model is the more appropriate ? At this point, after years of studies of the voting rules with a priori models, to which Peter Fishburn greatly contributed, it is worth noticing that scientists are starting to look more closely at the data or stylized fact. For example, the fact that most of the decision are taken at the unanimity in the European Union have inspired Laruelle and Valenciano [16] to design their model of bargaining in committees. In voting theory, a recent study by Gelman *et al.* [15] gives first insights on the nature of the relevant probability models for two candidates. The chief merit of this study is that it analyzes data from American and European elections. It is shown that, for elections with a large number of voters N, the $N^{1/2}$ scale for the differences between two issues is not correct and must be replaced by an N^{α} scale [9]. This confirms that the search for the adequate $f(p)$ (which must be reasonably stable from one election to the other) is of crucial importance. Similarly, Regenwetter *et al.* [20] have started to analyze the repartition of the preferences among three or more candidates, and revised the common wisdom on the probability of voting paradoxes. Thus, after a first age, where the a priori assumption played a crucial role, is seems that the probabilistic analysis of voting rule is entering a new age, where the probability model must, in some way, be related to the observed behavior of the voters. Our results are a modest contribution to this approach, as they clearly state that the conclusions on the probability of passage of different decision scheme could be wrongly evaluated if one does not consider the right a priori probability model.

References

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Appendix

We provide in this appendix the details of the calculations leading to relations (3) and (4). We suppose that N is large enough to use asymptotic calculations.

1. One key vote

Let us suppose that after the i first votes, the number x of "yes" mandates has a repartition $F_i(x)$. The next vote brings x at $x + a_{i+1}$ ("yes" vote) with probability p and x stays at its value ("no" vote) with probability $q = 1 - p$. Consequently $F_{i+1}(x)$ reads

$$F_{i+1}(x) = \int F_i(x-a)\, P_{i+1}(a) \mathrm{d}a \tag{5}$$

where $P_{i+1}(a)$ is the probability that x increases of a. Here

$$P_{i+1}(a) = p\, \delta(a - a_{i+1}) + q\, \delta(a) \tag{6}$$

where δ is the Dirac distribution defined by the following properties

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \mathrm{d}x = f(x_0) \quad ; \quad \delta(\alpha x) = \frac{1}{|\alpha|} \delta(x)$$

The distribution $\delta(x)$ can be considered as the limit when $\sigma \to 0$ of the Gaussian

$$(2\pi)^{-1/2} \sigma^{-1} \exp(-x^2/2\sigma^2)$$

with $\sigma > 0$.

We recognize in equation (5) a convolution product which is changed into an ordinary product using a Fourier transform. Calling $\hat{F}_{i+1}(k)$, $\hat{F}_i(k)$ and $\hat{P}(k)$ the Fourier transforms of $F_{i+1}(x)$, $F_i(x)$ and $P(x)$ respectively, equation (6) reads

$$\hat{P}_{i+1}(k) = p\, \exp(ika_{i+1}) + q, \tag{7}$$

while the Fourier transform of the final $F_N(x,p)$ reads

$$\hat{F}_N(k,p) = \prod_{i=1}^{N} \left(p\, \exp(ika_i) + q\right). \tag{8}$$

The inverse Fourier transform gives the expression of $F_N(x,p)$:

$$F_N(x,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(p\, \exp(ika_i) + q\right) \exp(-ikx) \mathrm{d}k. \tag{9}$$

Notice that equation (9) is an exact formula valid for any N and describes exactly the 2^N possible configurations associated to the N votes. In the asymptotic limit, the term given by equation (8) only contributes to the integral (9) around $k = 0$. In this operation, we break the true nature of $F_N(x,p)$ which is a sum of 2^N Dirac and turn $F_N(x,p)$ into a continuum. Consequently to obtain the asymptotic solution, we expand this term to first order in k :

$$p\,\exp(ika_i) + q = p(1 + i\,ka_i) + q = 1 + i\,p\,ka_i. \tag{10}$$

Taking the logarithm to compute the product we get

$$\hat{F}_N(k,p) = \exp(ikp\sum_{i=1}^{N} a_i). \tag{11}$$

Calling $A = \sum_{i=1}^{N} a_i$, $F_N(x,p)$ reads

$$F_N(x,p) = \delta(x - pA). \tag{12}$$

But equation (12) is again a quite understandable result. It states that after N votes with probability p of "yes", the number of mandates obtained is pA.

Taking into account in equation (10) the k^2 terms would bring a diffusion around this value. The average displacement varies as A (*i.e.* as N) when the ignored diffusion has a scale of $\sqrt{A}$ (*i.e.* varies as $N^{-1/2}$). The last step is the p integration with

$$F_N(x) = \int_0^1 f(p)\,\delta(x - pA)\mathrm{d}p = \frac{1}{A} f(x/A). \tag{13}$$

Notice that if all the voters have the same number of mandate, it is possible to compute F_N which is, in this case, given by the number of n "yes" votes. It reads

$$C_N^n \int_0^1 f(p) p^n\,(1-p)^{N-n}\mathrm{d}p. \tag{14}$$

If both n and N go to infinity while the ratio n/N is kept constant, $p^n\,(1-p)^{N-n}$ is more and more picked around n/N, then to first order we get

$$F_N(n) \sim \frac{1}{N}\,f(\frac{n}{N}). \tag{15}$$

Equation (13) is strictly identical to (15) where all voters have the same number of mandates (one, for example). We simply have to replace N by A. In all cases, keeping only the leading term of the expansion, the distribution of x/A reproduces the distribution $f(p)$ which characterizes the GIAC.

2. Double key vote

For the double key vote, the computation is quite similar to the preceding one. The probability $P_{i+1}(a,b)$ that voter $i+1$ moves the score (x,y) to $(x+a, y+b)$ is

$$P_{i+1}(a,b) = p\,\delta(a - a_{i+1})\,\delta(b - b_{i+1}) + q\,\delta(a)\,\delta(b) \tag{16}$$

with, as usual, $q = 1 - p$. The double Fourier transform of equation (16) reads

$$\hat{P}_{i+1}(k,l) = p\,\exp(i\,k\,a_{i+1} + i\,l\,b_{i+1}) + q \tag{17}$$

and the double Fourier transform $\hat{F}_N(k,l,p)$ of $F_N(x,y,p)$ is

$$\hat{F}_N(k,l,p) = \prod_{i=1}^{N} \left[p\,\exp\left(i(k\,a_i + l\,b_i)\right) + q\right]. \tag{18}$$

Again equation (18) is an exact formula. Now to compute its Fourier transform, we expand up to first order in k and l and obtain in this approximation

$$\hat{F}_N(k,l,p) = \exp\left(ip(k\sum_{i=1}^{N} a_i + l\sum_{i=1}^{N} b_i)\right), \tag{19}$$

which is a straightforward generalization of equation (11). Introducing $A = \sum_{i=1}^{N} a_i$ and $B = \sum_{i=1}^{N} b_i$, taking the inverse Fourier transform, we get for these votes characterized by p

$$F_N(x,y,p) = \delta(x - pA)\,\delta(y - pB) \tag{20}$$

which is a generalization of equation (12). For the last step, the p integration, we use the relation

$$\int_{-\infty}^{\infty} \delta(p-\alpha)\,\delta(p-\beta)\,f(p)\mathrm{d}p = \delta(\alpha-\beta)\,f(\xi)$$

where $\alpha = \beta = \xi$. Finally

$$F_N(x,y) = \frac{1}{AB}\,\delta(\frac{x}{A} - \frac{y}{B})\,f\left(p = \frac{x}{A} = \frac{y}{B}\right). \tag{21}$$

	Key $\mathcal{A}$					
	14/27	15/27	17/27	19/27	22/27	25/27
	51.85%	55.56%	62.96%	70.37%	81.48%	92.59%
50%	45.42	43.42	38.24	31.91	21.42	10.71
60%	39.54	38.60	35.57	30.84	21.33	10.71
65%	35.77	**35.24**	33.24	29.62	21.17	10.71
Key $\mathcal{B}$ 70%	31.55	31.29	30.16	27.68	20.67	10.71
72%	29.84	**29.66**	28.79	26.74	20.37	10.68
80%	22.80	22.77	22.56	21.86	18.46	10.56
90%	13.86	13.86	13.86	13.82	12.97	9.10

Table 2: Double key vote. Percentage of approval for two key votes in the EU27 under IAC as a function of the two quotas Q_A and Q_B. Results for the rules suggested by the constitutional treaty are highlighted in bold. The results have been obtained by complete enumeration of all the vote configurations.