The Probability of Casting a Decisive Vote in a Mixed-Member Electoral System Using Plurality at Large*

Michel Le Breton† Dominique Lepelley‡ Vincent Merlin§

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†Institut Universitaire de France and Toulouse School of Economics, France.

‡CEMOI, Université de La Réunion, France

§CREME, Université de Caen, France.

1 Introduction

The main purpose of this paper is to use the modern analysis of power measurement (Felsentahl and Machover (1998), Laruelle and Valenciano (2008)) to evaluate the probability that a voter casts a decisive vote in a mixed-member electoral system. Any such electoral system combines majority/plurality and proportionality features. The model that we consider can accomodate two interpretations in terms of electoral outcomes.

On one hand, we can consider that the election is a single-seat contest like in the case of the election of the president of a country. Even if the more common electoral system is a one tier system (the district is the nation as a whole) there are countries on top of which the United States where the electoral system is a two-tier sytem. In that sense, we expect our model to help in analysing the U.S. electoral college and potential reforms of this system.

On the other hand, we can consider that the election is milti-seat contest like in the case of the election of the chamber of representatives of some given area. For the sake of concrete illustration, it is useful to think of the chamber of representatives as the chamber of representatives of a country but the same ideas can be transplanted without any modification to any other level of government (region, metropolitan area,...).

In this introduction, we will mostly pay attention to the second interpretation but will also call the attention of the reader on the first interpretation when necessary and interesting.

At one extreme[1], we have single-member districts (SMD). In such systems, the country is divided into N electoral districts : each district elects one representative via a single-seat electoral formula (one round, two rounds,...). In such system, in spite of a reasonable national support, a party may fail to be adequately represented in the chamber.

At the other extreme[2], we have purely proportional systems (PR). In the extreme case, the country has a single electoral district which is the country itself. The parties present closed lists of N names among which the voters must chose. Then a proportional formula is used to transform the votes into seats.

More often, the country is divided into few (but more than one) very large districts and the seats allocated to each district are distributed according to the votes in this district.

A mixed electoral system (M) lies between these two extremes. Of course, the idea of mixing itself can be implemented in many different ways and we are not going to review here the classification of mixed electoral systems done by political scientists[3]. These electoral systems are getting more and more popular and some countries, including the country of residence of

[1]For instance, in France and in the U.K.

[2]For instance in Israel and Netherlands.

[3]We invite the reader to read Biais () as well as Lijphart (1986), Moser and Scheiner (2004) as well as the volume edited by Shugart and Wattenberg (2001).

the authors of this article[4], are considering moving in that direction. While behind SMD and PR, Blais and Massicote (1999) identifiy 29 countries out of 170 which were using a mixed system to elect their national chamber of deputies i.e. approximatively 17% of the total[5].

We should also point out that mixed systems are used in regional and municipal elections, for instance in France, Italy and the United States. In the United States, mixed systems are used by one fifth of the municipalities (or metropolitean areas) to elect their councils. Among the cities whose population is between 5 000 and 1 000 000, it is the second mostly used system (after pure proportional system) and it is used by half of the cities whose population is between 250 000 and 500 000.

Mixed systems display a great heterogeneity. According to Biais and Massicotte: "All mixed systems entail the combination of PR and plurality or majority. It follows that types of mixed systems can be distinguished on the basis of the kind of combination involved. From this perspective, mixed systems can be characterized along three dimensions: which electoral formulas are combined, how they are combined, and how many seats are allocated under each formula. The first and the last of these dimensions are simple and straightforward. One must specify whether PR is blended with plurality or majority, and what percentage of seats are allocated through PR. The manner PR is combined with plurality or majority requires some clarification. We propose below a classification of the different ways in which PR may be combined with majority or plurality..."

In this paper, we will focus on a simple electoral environment with the following features. We will assume that there are two parties that we will label L (left) and R (right). We assume that candidates are identified exclusively by their party affiliation and that voters are either L or R. The size of the parliament (assembly,...) is defined by an exogenous number K of seats. By mixed electoral system, we mean the following. M seats are elected via SMD on the basis of a division of the country into M local electoral districts. A ballot in a local district is a single name. The remaining $K - M$ seats are elected in a national electoral district. A ballot in the

[4]In France, opinions diverge strongly among the political leaders concerning the appropriate fraction of seats in the National Assembly that should be elected in a national district. The current French president has made promises on this issue during his campaign and has included this reform as one of the 60 changes that he would implement during his (first) term. One recent report co-supervided by the head of the chamber of representatives has produced a number of proposals asking for some major changes in political institutions including the electoral system: it advocates that half of the chamber should be elected according to PR and half according to SMD.

[5]Farell (2001) focuses on 59 democratic countries, with a population of at least 2 millions et with a degree of political freedom at least equal to the scaling of the 1999 Freedom House annual survey, leads to a slightly different perspective. Proportional representation is used by 49,2 % of the countries concentrating 18,4 % of the population while one round and two rounds majoritarian systems are used by 23,7 % of the countries concentrating 55,7 % of the population. The mixed systems are used in 27,1 % of these countries concentrating 25,8 % of the population. To summarize, half of these countries used proportional representation while the remaining half is shared equally between majoritarian and mixed.

national district will consist of a blocked/closed list of $K - M$ names. We will detail below the different ways to transform votes into seats. It is important to note that some of these assumptions are truly simplifying assumptions. For instance, we could assume that instead of having SMD districts, we have double-member district (DMD) without changing the analysis. We could also consider the case of more than one large district where each large district is the union of some small districts. The key feature of our model that makes it special is that voters vote twice; they vote a first time in their local district to elect their local representatives and a second time in the national district to elect/support their "list" of national representatives. It is important to know how practically this double vote is implemented. In this paper, we will assume that the ballot is conceived in a way that allows the voter to vote differently in the two districts if he/she wants to do so[6] like in the case of New Zeeland reproduced below. But in some countries, this is not the case. While the vote of any voter is *counted twice* (support of a candidate at the local level and support of the party to which this candidate is affiliated at the national level), the voter may be unable to distangle his two votes.

Insert here an image of a ballot in NZ

In the national district, we have already pointed out that there are several ways to transform seats into votes. Since the lists are blocked, the voters have the choice between two lists: a list of $K - M$ left candidates and a list of $K - M$ right candidates. Note however that since we have assumed that there are only two parties and that voters are exclusively party oriented, as we will see, some systems turn to be in fact to be equivalent.[7]

- *Plurality/Majority At Large:* the national winner (either L or R) gains all national $K - M$ seats. It is important to point out that in this paper it does not make any difference to assume that candidates belong to list or not. In fact, candidates will be identified by a single characteristic: their party affiliation. This means that, in our simplified setting, an electoral system where ballots consist of candidates, or a system where the voters can mix produce the same outcome as a system where the voters have the choice between lists of candidates[8].

[6] Table 2 in Biais and Massicote shows that a large majority of the countries using mixed electoral sytems have two votes. Exceptions are South Korea, Senegal, Seychelles and Taiwan.

[7] The combinatorics of possibilities is very large and we will not explore in this paper this large family of semi proportional systems including cumulative vote, limited vote and SNTV to cite few of the more popular ones.

[8] Sometimes, the electoral system is close to plurality at large up to some qualification. For instance, since 1998, in Equator, the $K = 124$ deputies are elected in 22 local districts electing each 2 deputies plus 1 extra one for each additional 200 000 residents or any population in excess of 133 000 residents. They use a proportional formula: blocked lists in districts which elect 2 deputies: one seat is allocated to the list in first position, the other one going to the second top list under the condition that it has obtained at least 50% of the score of the first list. *If not, the two seats go to the top list.* In districts with more than 2 seats, a double quota proportional formula with blocked lists is used: the lists which do not reach a number of votes equal to 50% of the first electoral quotient are eliminated. A second electoral quotient is then used to allocate the seats among the lists which compete.

- *Proportionality at the Margin:* Proportional Representation : the $K-M$ national seats seats are divided between L and R according to some formula (largest reminders/Hamilton, largest mean/Jefferson,...). Note that the electoral system used in France for regional and munipality elections combine plurality at large (a plurality bonus equal to one quarter of the seats) and proportionality at the margin (three quarters of the seats).[9]

- *Proportionality by Compensation:* [10] : the K-M national seats are allocated in a way to make sure that the allocation of the K seats in the parliament is moving towards to proportionality[11].

In this paper, we will focus exclusively on the first formula. As noted by Eggers and Fouirnaies (2013) : "Multi-member plurality elections are quite common, not just at the local level where they are widely used (in e.g. the U.S., U.K., Canada, Russia, India France and Hungary) but in a sizable number of legislatures as well (in e.g. Mexico, Kuwait, Lebanon, Egypt, Mauritius, Philippines, Bermuda and, historically, both the U.S. and U.K.)". Note that in such case, if the size of the parliament is fixed, the electoral system is defined by a single parameter : how many deputies are elected at the local level and how many deputies are elected at the national level ? In such case, what remains to be determined is the optimal value of M. Note also that if we look at the problem from the perspective of a presidential election as suggested by the first interpretation of our model, this correponds exactly to the U.S. electoral college. Indeed, as already pointed out, if the local district has more than one seat:electoral vote, as soon as the winner in the district gets all of them, the analysis developed in this paper goes through.

The main objective of this paper is to evaluate the probability for a voter to cast a decisive vote in a mixed electoral system when there are two parties, a fixed number of single-member equipopulated[12] districts and a national multi-member district. The computation is conducted

[9]Biais and Massicote use the term *fusion* for such systems.

[10]This is a special case of what Biais and Massicote call a *conditional system* while the two first ones are unconditional in the sense that M and $K-M$ are defined irrespective of the votes the two mechanisms are independent of each other). These compensation mechanisms are used in Germany and Italy. Note however that in such systems, when the proportional side contains a principle leading to a perfect (or almost perfect) correction of the majoritarian formula, it is harder to define such system as mixed. If the correction is partial like in Hungary and Italy (the scorporo) the system is truly mixed while in Germany the electoral system is (in spite of the appearances) truly a proportional system. Compensation systems like the German one are also used in New-Zealand, Albania (since 1997) and Bolivia.

[11]K itself can be variable like in Germany.

[12]Biais and Massicote make a distinction between the case where the territory is divided into two parts, one consisting of majority or plurality and another of PR districts. In such a system some voters vote under one formula and others under another. Thet call such a system *coexistence*. In our paper, there are two kinds of districts but the two formulas apply throughout the territory to all electors. Under such a system, each elector has two sets of representatives, one elected in a majority or plurality district and one elected in a PR district. They call such combination *superposition*.

for two popular probabilistic models and for two set of behavioral assumptions. Precisely, we assume that the local districts This means that the precise objective of this paper is to compute four probabilities. One of them has already been computed by Edelman (2004). All these numbers are going to be small when the total population of voters is going to be large. The precise objective of the paper is therefore to evaluate how small is this probability and how this probability changes when we change the proportion of local and national seats. We know the value of these probabilities in the two extreme cases: a national district but no local district or no national district and exclusively local districts. From that perspective, the added value of this paper is to estimate the dependency of the probability upon the proportion.

The two probabilistic models are the IC and the IAC* models. These models are the two most popular probabilistic models in social choice. The two behavioral assumptions which are considered are as follows. On one hand, we will consider the case where the votes of any voter in the two districts are independent variables. This assumption combined with IC is the setting considered by Edelman in his seminal paper. This paper explore the three other possible configurations. In particular, we consider the polar framework where each voter votes in the same way in both elections and refer to it as the assumption of single/consistent/correlated vote. This model is rather complex. as pointed out by Miller (2014) : *"While an analytic solution to this problem may be possible, the difficulties appear to be formidable"*. We end up getting some anlytical formulas but due to the complexity of some of the computations, we also sometimes offer only asymptotic and/or numerical results.

Why should we be interested in evaluating these probabilities ? There is a long tradition in social choice which has ramified in applied political science under the heading "power measurement". While rarely used as a normative device, the implicit postulate is still that it is good for an electoral system to be responsive to the preferences of the voters. Therefore, acccording to that view, electoral systems where, on average, voters are decisive are considered better. This view is supported by the so-called penrose relationship (stating that there is an affine relationship between utility and decisiveness for the IC probabilistic model. Unfortunately, this relationship does not hold in general if preferences are biaised or correlated as demonstated by several authors (Laruelle and Valenciano (2005), Lasruelle Martinez and Valenciano (2006) and Le Breton and Van Der Straeten (2015)). This means that judging an electoral system from the perspective of decisiveness is not the same thing as judging an electoral system from the perspective of welfare. But in political science, the comparative analysis of electoral systems is rarely based exclusively on welfarist criteria: women representation, fairness, governability,...

This paper is also based on a simple description of the political supply. As already mentioned, candidates will be labelled either L or R. No other dimensions will be considered and we will assume that the political preference of a voter is only based on the party affiliation of the candidate. However, when voter vote both in their local district and in the national district, they can in principle vote differently in the two districts. This is the assumption considered by Edelman (2004). To provide a rational fondation to a possible difference between the two votes, we would need (in principle) a more sophisticated model where local politics and national politics are distinct[13]. The construction of such a general model is beyong the scope of the present paper.

Related Literature

This paper belongs mostly to the literature on power measurement and to a lower extent to electoral engineering as it hopes to contribute to our understanding of mixed-member electoral systems[14]. The closest paper to what we are doing is the seminal paper of Edelman (2004) followed by Edelman (2005). In his paper, Edelman introduces the concept of generalized compound simple game and develop a framework where voters have two independent votes. He defines the power of a voter in that framework as the sum of the powers of the voter via the two channels. His beautiful result than states that if we fix the size of the chamber, then this power is maximized when the number of seats "at large" is equal to the square root of the size of the chamber. The exploration of his framework has been continued by Miller (2014) in the case of various reforms of the U.S. electoral college. Miller studies Edelman's setting in the case where the two preferences are the same and show via simulations how the evaluation of powers differs when we consider that alternative assumption. Is is important to point out that in the case of Miller, the work gets more commplicated as the U.S. electoral college is not a partition of equipoplulated electoral districs. This asymetry is responsible for the complication in the analysis. Both papers limit their exploration to the *IC* probability model.

The other closely related paper is by Le Breton and Lepelley (2014) who study a French electoral law edicted in June 1820 and used for a decade in France to elect the Chamber of Deputies. According to that law, which was called "Loi du Double Vote", some of the voters (in fact one quarter of the voters) had two votes and the chamber was composed with two types of a representatives : three fifth of the deputies were elected by the all electorate while the remaining two fifth were elected by a quarter of the electorate. The model is more complicated than the Edelman's model as here the property of symmetry among the voters is violated: there are two categories of voters and some have more power than others. Another important difference with Edelman is that the large districts are the union of a small number (often three) of local districts. Some of the results make use of some new theoretical results established by

[13]In some occasions politicians serve their constituencies at the expense of national considerations and in some other they proceed otherwise. Some scholars have examined the role of the electoral system in shaping the incentives of politicians and their responsiveness to the to types of concerns.

[14]This paper is a companion to Le Breton, Lepelley and Merlin (2014) who explore the vulnerability of mixed-member electoral systems to election inversions

Le Breton, Lepelley and Smaoui (2012) for a new class of probability models.

2 Theoretical Foundations

2.1 Binary Social Choice

We consider a society N of n voters which must chose among two alternatives: L versus R, status quo versus reform,... Each member i of N is described by his/her preference P_i. There are two possible preferences: L or R. A social choice mechanism (interpreted here as a voting mechanism) is a mapping F defined over $\{L,R\}^n$ with values in $\{L,R\}$: to each profile $P=(P_1,...,P_n)\in\{L,R\}^n$, the mechanism attachs a decision $F(P)\in\{L,R\}$. Any such mechanism F is totally described by the list $F^{-1}(L)$ of coalitions $S\subseteq N$ such that $F(P)=L$. This list $\mathcal{W}$ of coalitions defines a *simple game* as soon as $\varnothing\notin\mathcal{W}$ and $N\in\mathcal{W}$[15] and $\mathcal{W}$ is monotonic ($S\in\mathcal{W}$ and $S\subseteq T\Rightarrow T\in\mathcal{W}$) ; any coalition S in $\mathcal{W}$ is called a winning coalition. A simple game is *proper* if $S,T\in\mathcal{W}\Rightarrow\mathcal{S}\cap\mathcal{T}\neq\varnothing$. It is *strong* if for all $S\subseteq N$: $S\in\mathcal{W}$ or $N\backslash S\in\mathcal{W}$[16].

Intuitively, a voter is powerful (influential,...) in a mechanism F (i.e. $\mathcal{W}$) when its preference P_i plays an important role in the selection of the social alternative. Can we evaluate ex-ante (i.e. before knowing the preferences of the voters) the power of the different voters?

Any such measure relies on a particular probability model π over $\{L,R\}^n$: $\pi(P)$ denotes the probability (frequency,...) of profile P. Let us examine the situation from the perspective of voter i. To evaluate how often i is influential, we have to consider the frequency of profiles P such that $F(L,P_{-i})\neq F(R,P_{-i})$ or equivalently of coalitions $S=S(P)$, such that $S\in\mathcal{W}$ and $S\backslash\{i\}\notin\mathcal{W}$ or $S\notin\mathcal{W}$ et $S\cup\{i\}\in\mathcal{W}$. The probability of such an event is:

$$\sum_{T\notin\mathcal{W}\text{ et }T\cup\{i\}\in\mathcal{W}}\pi(T)+\sum_{T\in\mathcal{W}\text{ et }T\backslash\{i\}\notin\mathcal{W}}\pi(T)$$

This formula makes clear that the evaluation depends upon the probability π which is considered. Two popular specifications have attracted most of the attention and dominate the literature. The first (known under the heading *Impartial Culture (IC))* leads to the Banzhaf's index. It corresponds to the setting where all the preferences P_i proceed from independent

[15] The mechanism is not constant.

[16] A mechanism F treats equally the two alternatives D and G if and only if the game $\mathcal{W}$ is proper and strong. To see it, consider a profile P such that $F(P)=G$. Consequently, if S represents the coalition of players voting G, then $S\in\mathcal{W}$. Consider the profile P' where the voters in S vote now D and those in $N\backslash S$ vote G. Since $S\in\mathcal{W}$, $N\backslash S\notin\mathcal{W}$ i.e.: $F(P')=D$. Reciprocally, one can verify that, if F is neutral, then $\mathcal{W}$ is proper and strong.

Bernoulli draws with parameter $\frac{1}{2}$. In this case, $\pi_{-i}(T) = \frac{1}{2^{n-1}}$ for all $T \subseteq N \backslash \{i\}$. The Banzhaf power B_i of voter i is equal to:

$$\frac{\eta_i(\mathcal{W})}{2^{n-1}}$$

where $\eta_i(\mathcal{W})$ denotes the number of coalitions $T \subseteq N \backslash \{i\}$ such that $T \notin \mathcal{W}$ and $T \cup \{i\} \in \mathcal{W}$ (in the literature, any such coalition T is referred to as a "swing" for voter i). The second model (known under the heading *Impartial Anonymous Culture (IAC) Assumption)* leads to the Shapley-Shubik's index. It is defined as follows. Conditionnally to a draw of the parameter p in the interval $[0, 1]$, according to the uniform distribution, the preferences P_i proceed from independent Bernoulli draws with parameter p. In such a case, we obtain: $\pi_{-i}(T) = p^t(1-p)^{n-1-t}$ where $t \equiv \#T$. The Shapley-Shubik power Sh_i of voter i is equal to:

$$\int_0^1 \left(\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} p^t(1-p)^{n-1-t} \right) dp$$

Hereafter we will mostly focus on two models: the *IC* model and a version of the *IAC* model, called here *IAC**, which is intermediate between *IC* and *IAC*. It was first introduced by May (1949) in his analysis of election inversions and studied more recently by Le Breton, Lepelley and Smaoui (2014). This model is described in appendix 1.

Among simple games, we will pay a specific attention to weighted majority games (including ordinary majority games) and compound simple games. A simple game $(N, \mathcal{W})$ is a *weighted majority game* if there exists a vector of weights $(w_1, w_2, ..., w_n) \in R^n_+$ and a quota $q > 0$ such that :

$$S \in \mathcal{W} \text{ iff } \sum_{i \in S} w_i \geq q$$

The number q denotes the quota needed to validate the choice of alternative L. Depending upon context, the weights may represent the number of representatives of i if i denotes a country in the board of an international organization or the number of deputies affiliated to party i if i denotes a party (with a perfect party discipline) in a parlementary assembly. If $q > \frac{\sum_{i \in N} w_i}{2}$, the game $\mathcal{W}$ is proper. When the weights w_i are integers, the quota $\left\lceil \frac{\sum_{i \in N} w_i}{2} \right\rceil$ where $\lceil x \rceil$ denotes the smallest integer strictly larger than x denotes the majority quota and the corresponding game is called the ordinary majority game. If $\sum_{i \in N} w_i$ is odd, the ordinary majority game $\mathcal{W}$ is strong. When $\sum_{i \in N} w_i$ is even, it is not necessarily the case. When this happens, a second game is used to break the ties; for instance one of the players can be used as a "tie breaker". When all the weights are equal to 1 and $q = \lceil \frac{n}{2} \rceil$, we obtain the symmetric majority game. If n is odd, the symmetric majority game is strong.

2.2 Ordinary and Generalized Compositions: Disjoints Colleges, Nested Colleges and Multiple Votes

Ordinary compound simple games describe two-tier voting systems: the population of voters is partitioned into districts and each district (the bricks of the first tier) elects one or several representatives/delegates. Then, these representatives select the social alternative in the second tier. At this level of generality, the simple games used in each of the districts composing the first tier as well as the simple game describing the decision making (among representatives) in the second tier can be quite arbitrary. The composition operation is very general.

In what follows, we will assume that the second tier is a weighted majority game. Precisely, the population N of voters is partitioned into K electoral districts : $N = \cup_{1 \leq k \leq K} N_k$. The n_k voters of district k elect w_k representatives under majority rule (the w_k representatives of district k are either all left or all right). The total number of representatives in the assembly is therefore $\sum_{1 \leq k \leq K} w_k$. We are going to compute exactly or approximatively the Banzhaf and Shapley-Shubik powers of a voter in such compound simple games.

Calculating the probability of the event "Voter i in district k is pivotal" is rather simple (in principle) when the districts are disjoint. It will happen when the voters (besides i) in district k are divided in two groups of equal size and if the block of the representatives from district k is itself decisive in the assembly. Under IC, this second event is independent from what happens in other districts. Under IAC, things are not so simple. This is why we will consider here a model where IAC is applied separately in each district. This revised IAC model, called $IAC*$ by Le Breton and Lepelley (2014), is a particular case of the general model introduced and investigated in Le Breton, Lepelley and Smaoui (2012).

We note that the block of the w_k representatives of district k is pivotal in the assembly if the subset of districts $G \subseteq \{1, ..., K\} \setminus \{k\}$ voting left is such that :

$$q - w_k \leq \sum_{j \in G} w_j < q$$

Counting the number M_k of sets G verifying the above inequality is not straightforward in general. When q is the majority quota, Penrose's Approximation Theorem asserts that if the number K of blocks is large enough, then the ratio $\frac{M_k}{M_l}$ is close to the ratio $\frac{w_k}{w_l}$ for any pair k, l. This statement does not hold in full generality and its scope of validity has been the subject of several studies[17]. We will see that in our context, Penrose's approximation does not help unless we assume that our mixed electoral system is almost majoritarian.

[17]In the case of an arbitrary quota, Penrose's approximation, in general, does not hold (Lindner and Owen (2007)).

Edelman (2004) has developped a generalization of the notion of composition where the K electoral districts N_k are not necessarily disjoint and where each voter has as many votes as the number of districts to which he belongs. This means that if i belongs to k districts, he will vote k times and that (technically speaking), he may sometimes vote L and sometimes vote R. Strictly speaking, as noted by Edelman, compound settings where electorates are not disjoint are already considered in the abstract literature on simple games (Felsenthal and Machover (1998), Taylor and Zwicker (1999)). For instance, the US Congress is modelled as the union of two simple games with the same set of 537 identical players. Similarly, the EU council of ministers is modeled as the intersection of three simple games with the same set of 27 identical players. Union and Interesection are two particular cases of composition. But in these models, voters have a single vote.

We continue to denote by N the all population of voters, i.e. $N = \cup_{1 \leq k \leq K} N_k$. A *generalized simple game* is a family $\mathcal{W}$ of K-tuples of sets $(S_1, S_2, ..., S_K)$ where S_k is a subset of N_k for all $k = 1, ..., K$, satisfying:

· $(N_1, N_2, ..., N_K) \in \mathcal{W}$

· $(\varnothing, \varnothing, ..., \varnothing) \notin \mathcal{W}$

· If $(S_1, S_2, ..., S_K) \in \mathcal{W}$ and $S_k \subseteq T_k$ for all $k = 1, ..., K$, then $(T_1, T_2, ..., T_K) \in \mathcal{W}$.

The coalitions $(S_1, S_2, ..., S_K) \in \mathcal{W}$ are called *global winning coalitions* while local components will be called *local coalitions.* In this generalized setting, Edelman defined a new index of power along the lines of Banzhaf as follows. Given a voter i and a global coalition $S = (S_1, S_2, ..., S_K)$, he says that i is pivot in S via component k if $i \in S_k$, $S \in \mathcal{W}$ and $(S_1, ..., S_k \backslash \{i\}, ..., S_K) \notin \mathcal{W}$. Let

$$Piv(i, S, k) = \begin{cases} 1 \text{ if } i \text{ is pivot in } S \text{ via component } k \\ 0 \text{ otherwise} \end{cases}$$

We note $\chi(\mathcal{W}, i)$ the number of times where i is pivot in a global coalition via a component .i.e.:

$$\chi(\mathcal{W}, i) = \sum_{S \in \mathcal{W}} \sum_{k=1}^{K} Piv(i, S, k)$$

Edelman defines the power $\widetilde{B}_i$ of i as follows:

$$\widetilde{B}_i = \frac{\chi(\mathcal{W}, i)}{2^{n_1 + ... + n_K - 1}}$$

We call the attention on the fact that this measure differs from the conventional Banzahf index B_i.

The generalized simple game on which Edelman mostly focuses, that we will call accordingly *Edelman's generalized game*, is the game that we have considered after introducing conventional composition, except for the fact the sets N_k are not anymore disjoint. Given a weighted majority game $[q; w_1, ..., w_K]$ on the set $\{1, 2, ..., K\}$ and K simple games$(N_k, \mathcal{W}_k)$, $(S_1, S_2, ..., S_K) \in \mathcal{W}$ iff:

$$\sum_{\{k:S_k \in \mathcal{W}_k\}} w_k \geq q$$

This generalized composition differs from the ordinary one in that the definition does not force a voter to be in all of them or in none of them: it can be counted in some S_i and not counted in others. Everything is as if any voter had as many twins/clones as the number of votes that he has. A compound simple game is defined with K simple games $((N_k, \mathcal{W}_k))_{1 \leq k \leq K}$ and an additional simple game $\widetilde{\mathcal{W}}$ on the set $\{1, 2, ..., K\}$. The *classical composition* postulates that a coalition $S \subseteq \underset{1 \leq k \leq K}{\cup} N_k$ is winning if $\{k \in \{1, ..., K\} : S \cap N_k \in \mathcal{W}_k\} \in \widetilde{\mathcal{W}}$. The *generalized composition* is defined by a generalized simple game $\mathcal{W}$ where $(S_1, S_2, ..., S_K) \in \mathcal{W}$ iff $\{k \in \{1, ..., K\} : S_k \in \mathcal{W}_k\} \in \widetilde{\mathcal{W}}$.

2.3 The Mixed-Member Electoral System of Edelman

In this paper, we will focus on a symmetric version of the *Edelman's generalized game.* We assume that the total population of voters is partitioned into A regional districts (local electoral districts) of identical size denoted $L = 2r + 1$ where r is a positive integer. In addition to these A regional distícts, we will have a national district which is the union of the regional districts. Its size is therefore $A(2r + 1)$. Hereafter, unless explicitely stated, we will assume that A is odd: this implies that N is also odd.

In our setting, voters will vote two times. Each voter i is member of a single regional district $j(i) \in \{1, 2, ..., A\}$. In this district, the voter choses to vote either for a blocked list of $\underline{D}$ left candidates or for for a blocked list of $\underline{D}$ right candidates. The list getting more votes in regional district j wins the $\underline{D}$ representatives of district j. Herefater, we will refer to these representatives as *local representatives.* As a member of the national district N, voter i votes a second time. Just as at the regional level, he has two options: to vote either for a blocked list of $\overline{D}$ left candidates or for for a blocked list of $\overline{D}$ right candidates. Also as before, the list getting the largest support, wins the totality of the $\overline{D}$ seats. Hereafter, we will refer to these representatives as the *national representatives* (or at-large representatives). The total number of seats and therefore representatives in the national parliament is equal to $A\underline{D} + \overline{D}$. Table 2 describes the values of parameters A, $\underline{D}$ et $\overline{D}$ for a sample[18] of countries using (or having used)

[18]We could also add Croatia and Philippines.

a mixed electoral system of the superposition type. Note that the setting covers the extreme case where $A = \underline{D} = 0$ and $\overline{D} = 0$.

	A	$\underline{D}$	$\overline{D}$
Andorra	7	2	14
Russia (before 2005	9	1 or 4	59
Ukraine	225	1	225
Armenia	41	1	90
Georgia	73		77
Senegal	35	$1 \leq . \leq 5$	60
South Korea (in 1981 and 1985)	92	2	92
Taiwan	125	A lot of Volatility	35
Japan	300	1	11 Very Large districts
Seychelles	25	1	9
Lituania	71	1	70
Macedonia (until 2002)	85	1	35
Albania (until 2008)	100	1	40

Table 2

The study of such mixed electoral sytem will be conducted under different sets of assumptions involving two dimensions. The first dimension concerns preferences while the second refers to the probabilistic model describing the likelihood of the different preference profiles.

Concerning preferences, we will consider in turn two polar assumptions.

A first set of results will be based on the hypothesis (made by Edelman) that the two votes are independent. It is as if each voter has two different preferences that he will use respectively at the local level and at the national level. We will refer to this assumption as the *assumption of double vote*. The second assumption that we will call *assumption of coherent/single/correlated*[19] *vote* supposes that each voter has a unique preference and therefore votes in the same way in both districts.

Concerning the probability model, we will also consider two alternative settings.

Under the assumption of a single vote we will either assume the IC model or the IAC* model which consists in independent application of IAC in each of the local districts. Under

[19] We should say perfect correlation. A model encompassing both assumptions is to assume that the correlation coefficient ρ between the two votes is any number between -1 and 1. Here we limit our investigation to the case where either $\rho = 0$ or $\rho = 1$.

the assumption of a double vote,we will assume that the local preferences are drawn according to IC or to IAC * and that the national preference is independent from the first one but drawn according to the same model.[20] The following table summarizes the road map.

	Double Vote	Single Vote
IC	Section 3	Section 4
$IAC*$	Section 3	Section 4

Table 3

3 The Mixed-Member Electoral System under the Assumption of Double Vote

In this section we compute the Banzhaf and Shapley-Shubik power measures of a generic voter in the case of double vote. This computation raises a difficulty which is not present in Le Breton and Lepelley (2014) analysis of the French electoral law of June 29 1820. In this new setting, it is not appropriate to consider the Penrose approximation. The second tier has a large player and an ocean of small players. The usual conditions of validity of Penrose are not satisfied and it is necessary to proceed to a direct computation. However, in contrast to Le Breton and Lepelley who had two types of voters, here, there is only one type. This is why we can refer to the power of a player without any ambiguity.

3.1 Computation of the Power Indices

Consider in a given local district a voter i. Voter i is pivotal (or decisive) if he is pivotal in his local district $j(i)$ and if the block of $\underline{D}$ representatives elected in district j is decisive in the second tier **or** if he is pivotal in the national district and if the block of $\overline{D}$ representatives elected in the national district is decisive in the second tier (the "or" is inclusive here). Let us introduce the following subsets of $\{L, R\}^n$ attached to the following events :

$\mathbf{P}_i$: "Voter i is decisive" ;

$\mathbf{R}_{d(j)}$: "the block of $\underline{D}$ representatives of district j is decisive in the parliament" ;

$\mathbf{R}_{d(n)}$: "the block of $\overline{D}$ representatives of the national district is decisive in the parliament" ;

$\mathbf{P}_{i,j}$: "Voter i is decisive in his local district $j = j(i)$" ;

[20] In the working paper version of this article, we consider in the case of double vote an additional model: the national vote proceeds from a unique IAC drawing in the national district N, a model that Le Breton and Lepelley (2014) call IAC_{**}.

$\mathbf{P}_{i,n}$: "Voter i is decisive in the national district" ;
We have :

$$\mathbf{P}_i = (\mathbf{P}_{ij} \ \cap \ \mathbf{R}_{d(j)}) \ \cup \ (\mathbf{P}_{i,n} \ \cap \ \mathbf{R}_{d(n)})$$

Let us denote by $\pi_\alpha(\mathbf{X})$ the probability of the event $\mathbf{X}$ when the set of profiles $\{L, R\}^n$ is endowed with the probability α. The voting power of an elector is given by $\pi_\alpha(\mathbf{P}_i)$. The Banzhaf index corresponds to the case $\alpha = IC$ and the revised Shapley-Shubik index correpond to the cases $\alpha = IAC_*$.

The assumption of double vote and the assumptions on the two probability models which are consider lead to the following equality:

$$\begin{aligned}\pi_\alpha(\mathbf{P}_i) &= \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)}) \\ &\quad -\pi_\alpha(\ (\mathbf{P}_{ij} \ \cap \ \mathbf{R}_{d(j)}) \cap (\mathbf{P}_{i,n} \ \cap \ \mathbf{R}_{d(n)})).\end{aligned}$$

In this expression of $\pi_\alpha(\mathbf{P}_i)$, we can neglect the third term as it is second order as compared to the two other ones i.e. we delete the profiles of preferences where a voter is simultaneously decisive in his local district and in the national district. We obtain:

$$\pi_\alpha(\mathbf{P}_i) \approx \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)}) \tag{1}$$

From there we can propose simple formulas for each of the two models which are considered. For Banzhaf, i.e. $\alpha = IC$, we know the power of a voter in a district is inversely proportional to the square root of the population of the district. Precisely, we have: $\pi_{IC}(\mathbf{P}_{i,j}) \approx \sqrt{\frac{2}{\pi L}}$ since there are L voters in any local district and $\pi_{IC}(\mathbf{P}_{i,n})) \approx \sqrt{\frac{2}{\pi LA}}$ since there are LA voters in the nation. Plugging in (1) leads to:

$$\pi_{IC}(\mathbf{P}_i, A, D) \approx \sqrt{\frac{2}{\pi L}}\pi(\mathbf{R}_{d(j)}) \ + \pi(\mathbf{R}_{d(n)})\sqrt{\frac{2}{\pi AL}} \tag{2}$$

For the revised model of Shapley-Shubik i.e. $\alpha = IAC_*$, the power of a voter is inversely proportional to the population size of the district. We obtain: $\pi_{IAC_*}(\mathbf{P}_{i,j}) \approx \frac{1}{L}$ and $\pi_{IAC_*}(\mathbf{P}_{i,n}) \approx \frac{c_A}{LA}$, implying:

$$\pi_{IAC_*}(\mathbf{P}_i; A, D) \approx \frac{1}{L}\pi(\mathbf{R}_{d(j)}) \ + \pi(\mathbf{R}_{d(n)})\frac{c_A}{LA} \tag{3}$$

To compute the power of a voter in any of these three models, we need to calculate $\pi(\mathbf{R}_{d(j)})$ et $\pi(\mathbf{R}_{d(n)})$ and for the second one we also need to calculate c_A. The value of c_A for the first

50 integers is determined in Le Breton, Lepelley et Smaoui (2012) as well as the asymptotic behavior of c_A when A tends to $+\infty$.

A	3	5	7	9	11	13	15	17	...
c_A	2.25	2.995	3.577	4.076	4.521	4.925	5.298	5.647	...

Table 4

From now on, we will limit our exploration to the case where $\underline{D} = 1$ for all $j = 1, ..., A$ and for the sake of notational simplicity, $\overline{D}$ is denoted D. The computation os $\pi(\mathbf{R}_{d(j)})$ and $\pi(\mathbf{R}_{d(n)})$ is then made easier. Given our assumption of independence across the districts and double vote, this amounts to calculate the Banzhaf index of players in a parlement described as a weighted majority game with A players with a weight of 1 and one player with a weight of D. In Appendix 6, we give the general formula to compute $\pi(\mathbf{R}_{d(j)}) = \pi(\mathbf{R}_j)$ and $\pi(\mathbf{R}_{d(n)})$ as well as some numerical values. We assume that D is an even integer. Therefore, the number $D + A$ of deputies in the parliament is odd and the game is strong. We will also assume that $D < q \equiv \frac{D+A+1}{2}$ otherwise the block of the D national representatives will control the assembly and the game will be trivial. This implies that $D < A + 1$ and therefore $D \leq A$. But since A is odd and D is even this implies in fact that $D \leq A - 1$. We deduce:

$$\pi_{IC}(\mathbf{P}_i, A, D) \approx \sqrt{\frac{2}{\pi L}} \frac{\frac{(A-1)!}{((q-1)!)((A-q)!)}}{2^{A-1}} + \frac{\sum_{i=q-D}^{q-1} \frac{A!}{(i!)((A-i)!)}}{2^A} \sqrt{\frac{2}{\pi A L}} \tag{4}$$

$$\pi_{IAC_*}(\mathbf{P}_i; A, D) \approx \frac{1}{L} \frac{\frac{(A-1)!}{((q-1)!)((A-q)!)}}{2^{A-1}} + \frac{\sum_{i=q-D}^{q-1} \frac{A!}{(i!)((A-i)!)}}{2^A} \frac{c_A}{LA} \tag{5}$$

Formula (4) already appears in Edelman (2004). Edelman maximizes $\pi_{IC}(\mathbf{P}_i, A, D)$ with respect to D and A under the constraint that $D + A$ is constant[21]. Edelman shows that the supremum of $\pi_{IC}(\mathbf{P}_i, A, D)$ is attained when $D \simeq \sqrt{A + D}$. To the best of our knowledge formula (5) is new. The numerical values below illustrate the variations of $\pi_{IC}(\mathbf{P}_i, A, D)$ when $A + D = 101$ (i.e. a chamber of size 101) and a total population of $100001 \times 101 = 1.01 \times 10^7$ i.e. about 10 millions of voters. Up to the multiplicative factor $\sqrt{\frac{2}{\pi \times 1.01 \times 10^7}}$, the Banzhaf index writes:

$$\sqrt{A} \frac{\frac{(A-1)!}{((q-1)!)((A-q)!)}}{2^{A-1}} + \frac{\sum_{i=q-D}^{q-1} \frac{A!}{(i!)((A-i)!)}}{2^A}$$

[21]Note also that the size of the country being given, if we move from A to $A' < A$ (in the same time D moves to D') the population per district move from L to $L' = \frac{A(2r+1)}{A'}$. This means that, strictly speaking, when comparing $\pi_{IC}(\mathbf{P}_i, A, D)$ to $\pi_{IC}(\mathbf{P}_i, A', D')$ we have to change the value of L in formula (4). The best way to proceed is to factorize $\sqrt{\frac{1}{LA}}$ and to multiply by $\sqrt{A}$ the first term in the sum.

Solution 1 (No National District): $A = 101, D = 0$

We obtain $\sqrt{101}\frac{\frac{(100)!}{((50)!)((50)!)}}{2^{100}} = 0.799\,86$

Solution 2 (Almost Half and Half) $A = 53, D = 46$

We obtain $\sqrt{53}\frac{\frac{(52)!}{((50)!)((2)!)}}{2^{52}} + \frac{\sum_{i=5}^{50}\frac{53!}{(i!)((53-i)!)}}{2^{53}} = \frac{663}{2251\,799\,813\,685\,248}\sqrt{53} = 1$

Solution 3 (Edelman) $A = 91, D = 10$

We obtain $\sqrt{91}\frac{\frac{(90)!}{((50)!)((40)!)}}{2^{90}} + \frac{\sum_{i=41}^{50}\frac{91!}{(i!)((91-i)!)}}{2^{91}} = 1.166\,9$

Solution 4 (No Local District) $A = 0, D = 101$

We obtain 1

Note that the Banzhaf so normalized is flat equal to 1 as soon as $D \geq \frac{A+D+1}{2} = 51$

Solution 5 $A = 81, D = 20$

We obtain $\sqrt{81}\frac{\frac{(80)!}{((50)!)((30)!)}}{2^{80}} + \frac{\sum_{i=31}^{50}\frac{81!}{(i!)((81-i)!)}}{2^{81}} = 1.040\,4$

Solution 6 $A = 95, D = 6$

We obtain $\sqrt{95}\frac{\frac{(94)!}{((50)!)((44)!)}}{2^{94}} + \frac{\sum_{i=45}^{50}\frac{95!}{(i!)((95-i)!)}}{2^{95}} = 1.123\,4$

In the IAC*, we get up to a multiplicative factor $\sqrt{\frac{6A}{\pi}}$

$$A\frac{\frac{(A-1)!}{((q-1)!)((A-q)!)}}{2^{A-1}} + c_A\frac{\sum_{i=q-D}^{q-1}\frac{A!}{(i!)((A-i)!)}}{2^A}$$

Solution 1 (No National District): $A = 101, D = 0$

We obtain $101\frac{\frac{(100)!}{((50)!)((50)!)}}{2^{100}} = 8.038\,5$

Solution 2 (Almost Half and Half) $A = 53, D = 46$

We obtain $53\frac{\frac{(52)!}{((50)!)((2)!)}}{2^{52}} + 10\frac{\sum_{i=5}^{50}\frac{53!}{(i!)((53-i)!)}}{2^{53}} = 10.000$

Solution 3 (Edelman) $A = 91, D = 10$

We obtain $91\frac{\frac{(90)!}{((50)!)((40)!)}}{2^{90}} + \sqrt{\frac{6\times 91}{\pi}}\frac{\sum_{i=41}^{50}\frac{91!}{(i!)((91-i)!)}}{2^{91}} = 13.702$

Solution 4 (No Local District) $A = 0, D = 101$

We obtain $\sqrt{\frac{6\times 101}{\pi}} = 13.889$

4 The Mixed-Member Electoral System under the Assumption of Single Vote

In the case where voters have a single preference and vote in the same way in the two elections, the compound mechanism is far more complicated. The complexities arising from this assumption are multiple and even it it looks artificial at first galnce, it is useful to decompose the difficulty into two.

Difficulty 1 : On one hand, if a voter is pivotal in his local district, we learn something about the social preference in his district and this has some impact on the probability that he is also pivotal in the national district.

Difficulty 2 : On the other hand, the computation of the pivot probabilities of the local representatives and the block of national representatives have been based so far on the fact that, under the assumption of a double vote, the political colors of the $A+1$ representatives were independent Bernouilli random variables of parameter $\frac{1}{2}$. This asssumption must be abandoned in the case of a single vote as the $A+1$ electoral outcomes are not anymore independent random variables. The knowledge of the outcomes in local districts biases in one direction or the other the national electoral outcome. To classify the differences between the two settings let us start again from the basic relation:

$$\mathbf{P}_i = (\mathbf{P}_{ij} \ \cap \ \mathbf{R}_{d(j)}) \ \cup \ (\mathbf{P}_{i,n} \ \cap \ \mathbf{R}_{d(n)})$$

In the case of double vote we had written :

$$\pi_\alpha(\mathbf{P}_i) = \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)}) - \pi_\alpha(\ (\mathbf{P}_{ij} \ \cap \ \mathbf{R}_{d(j)}) \cap (\mathbf{P}_{i,n} \ \cap \ \mathbf{R}_{d(n)}))$$

In the case of single vote, we must write:

$$\pi_\alpha(\mathbf{P}_i) = \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)} \mid \mathbf{P}_{i,j}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)} \mid \mathbf{P}_{i,n})$$

$$-\pi_\alpha \left(\mathbf{P}_{ij} \ \cap \mathbf{P}_{i,n}\right) \pi_\alpha(\ (\ \mathbf{R}_{d(j)} \cap \ \mathbf{R}_{d(n)}) \mid (\mathbf{P}_{ij} \ \cap \mathbf{P}_{i,n}) \ )$$

If the term $\pi_\alpha \left(\mathbf{P}_{ij} \ \cap \mathbf{P}_{i,n}\right)$ is of second order as compared to $\pi_\alpha(\mathbf{P}_{i,j})$ and $\pi_\alpha(\mathbf{P}_{i,n})$, we can still use:

$$\pi_\alpha(\mathbf{P}_i) = \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)} \mid \mathbf{P}_{i,j}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)} \mid \mathbf{P}_{i,n})$$

Finally, if we ignore conditioning, we are back to the relation obtained in the case of double vote:

$$\pi(\mathbf{P}_i) = \pi_\alpha(\mathbf{P}_{i,j}) \times \pi_\alpha(\mathbf{R}_{d(j)}) + \pi_\alpha(\mathbf{P}_{i,n}) \times \pi_\alpha(\mathbf{R}_{d(n)})$$

with however a major difference. Indeed, to compute the probabilities $\pi_\alpha(\mathbf{R}_{d(j)})$ we cannot proceed as before as the respective colors of any deputy and the national block of deputies are not (as pointed out above as difficulty 2) independent. In contrast, the computation of $\pi_\alpha(\mathbf{R}_{d(n)})$ can be done as before as the respective colors of any two local deputies are independent.

To do this computation, we must have a good knowlege of the random vector of dimension $A+1$ whose components correspond to the electoral outcomes of the $A+1$ districts. As explained in Appendix 6, given B and C, two disjoints subsets of A, the crucial step is to be able to calculate the following probabilities: $\widetilde{P}_{\alpha,B,C,1}$ is the probability that the local districts in B (respectively C) vote left (respectively right) and the national district vote left and $\widetilde{P}_{\alpha,B,C,0}$ is the probability that the local districts in B (respectively C) vote left (respectively right) and the national district vote right. Note that as soon as $\pi_\alpha$exhibits independence across the local districts:

$$\widetilde{P}_{\alpha,B,C,0} = \frac{1}{2^{|B\cup C|}} - \widetilde{P}_{\alpha,B,C,1}$$

Note also that for the two probability models α considered here, $\widetilde{P}_{\alpha,B,C,0}$ depends only upon the cardinalities b and c of B and C and will be therefore be denoted shorthly $\widetilde{P}_{\alpha,b,c,0}$. In Appendices 4 and 5, we provide formulas to calculate $\widetilde{P}_{\alpha,b,c,0}$ in the cases where $\alpha = IC$ and $\alpha = IAC^{*}$ and in Appendix 6 we provide calculations of $\pi_\alpha(\mathbf{R}_{d(j)})$ et $\pi_\alpha(\mathbf{R}_{d(n)})$. Collecting the terms, we obtain:

$$\pi_\alpha(\mathbf{P}_i) = 2\pi_\alpha(\mathbf{P}_{i,j})\binom{A-1}{q-1}\widetilde{P}_{\alpha,q-1,0} + \pi_\alpha(\mathbf{P}_{i,n})\sum_{i=q-D}^{q-1}\frac{A!}{(i!)\left((A-i)!\right)}\frac{1}{2^A}$$

In the IC case, we obtain:

$$\pi_{IC}(\mathbf{P}_i) \simeq 2\sqrt{\frac{1}{\pi r}}\binom{A-1}{q-1}\widetilde{P}_{IC,q-1,0} + \sqrt{\frac{1}{\pi A r}}\sum_{i=q-D}^{q-1}\frac{A!}{(i!)\left((A-i)!\right)}\frac{1}{2^A} \tag{6}$$

while in the IAC_* case:

$$\pi_{IAC_*}(\mathbf{P}_i) \simeq \frac{1}{r}\binom{A-1}{q-1}\widetilde{P}_{IAC*,q-1,0} + \frac{c_A}{2rA}\sum_{i=q-D}^{q-1}\frac{A!}{(i!)\left((A-i)!\right)}\frac{1}{2^A} \tag{7}$$

It is important to call the attention on the fact that the probabilities $\widetilde{P}_{IC,q-1,0}$ and $\widetilde{P}_{IAC*,q-D-1,1}$ in the above formulas are not the same for the models IC and IAC_*.

Let us now move to the treatment of difficulty 1: how to proceed to a proper evaluation of the effects of conditioning on these two formulas[22] Consider in turn the conditional probabilities: $\pi(\mathbf{R}_{d(j)} \mid \mathbf{P}_{i,j})$ et $\pi(\mathbf{R}_{d(n)} \mid \mathbf{P}_{i,n})$.

Concerning $\pi(\mathbf{R}_{d(j)} \mid \mathbf{P}_{i,j})$, things are pretty simple. Indeed, the information: "Without i, district $j(i)$ is divided equally between left and right" does not convey any information on other

[22]We know from Le Breton and Lepelley (2014) that, sometimes, they are not of second order.

local districts. Regarding the national district, it conveys a pretty clear and simple information: everything is as if the national district without i was now composed of $(A-1)L$ voters. The calculations are those of Appendices 4 and 5 after substituting $A-1$ to A in the formulas.

Concerning $\pi(\mathbf{R}_{d(n)} \mid \mathbf{P}_{i,n})$, things seem (at first sight) less simple. Given that the national district is balanced, what are the probabilities of the different political patterns for the A local representatives? So far, we have assumed that all configurations had the same probability $\frac{1}{2^A}$ to occur. How to alter this computation to take into account that information is not straightforward, as will be shown through specific cases in Appendices 2 and 3.

What do we get for instance when $A=3$ and $D=2$? From the results in Appendix 5, when $\pi = IAC_*$, we deduce:

$$\pi_{IAC_*}(\mathbf{P}_i) \simeq \frac{2.25}{6r}\left(\sum_{i=1}^{2} \frac{3!}{(i!)\left((3-i)!\right)} \frac{1}{2^3}\right) \simeq \frac{0.28125}{r}$$

We will see in Appendix 2 that the exact value of the probability $\pi_{IAC_*}(\mathbf{P}_i)$ when r is large is $\frac{3}{8r} = \frac{0.375}{r}$. Why do we get such a gap ? As explained above, the calculation of the probability that the block of two national deputies is pivotal was ignoring the fact that the national district without i was divided. The block of 2 deputies will be pivot in these parlement of 5 deputies if the local deputies are distributed as follows: two on the left and one on the right or two on the right and one on the left. For the sake of illustration, consider the event GGD = "The deputies of the first two local districts are on the left and the deputy of the the third district is on the right". In the formula to compute $\pi_{IAC_*}(\mathbf{P}_i)$, we have used $\pi\left(GGD\right) = \frac{1}{8}$ instead of $\pi(GGD$ |The national district without i is balanced). From Bayes's formula:

$$\pi(GGD \mid \text{The national district without } i \text{ is balanced}) =$$

$$\frac{\pi(GGD \text{ and the national district without } i \text{ is balanced})}{\pi(\text{The national district without } i \text{ is balanced})}$$

$$= \frac{\pi(\text{The national district without } i \text{ is balanced} \mid GGD)}{\pi(\text{The national district without } i \text{ is balanced})}\pi\left(GGD\right)$$

While replacing $\pi(GGD$ |The national district without i is balanced) by $\pi\left(GGD\right)$, we have biased the computation of $\pi_{IAC_*}(\mathbf{P}_i)$ by the factor:

$$\frac{\pi(\text{The national district without } i \text{ is balanced} \mid GGD)}{\pi(\text{The national district without } i \text{ is balanced})}$$

Intuition suggests that this factor is larger than 1. Indeed, knowing that the local districts are themselves balanced between L and R, we should expect at the national level a probility of tie larger than the average one. Consequently, $\pi\left(GGD\right) = \frac{1}{8}$ under-estimates the value to be taken into consideration to compute $\pi_{IAC_*}(\mathbf{P}_i)$.

We have not been able to derive in full generality the effect of conditioning on the probability of being pivotal. We hen $\pi = IC$, we suggest in Appendix 7 a revision of formula (6) which could be used when A is large enough. We obtain then the exact formula:

$$\pi_{IC}(\mathbf{P}_i) = 2\sqrt{\frac{1}{\pi r}}\binom{A-1}{q-1}P_{q-1,0} + \sqrt{\frac{1}{\pi A r}}\sum_{i=q-D}^{q-1}\xi(A,i)\frac{A!}{(i!)\,((A-i)!)}\frac{1}{2^A} \tag{8}$$

where the exact value of the correction term $\xi(A,i)$ is conjectured in Appendix 7.

We have not been able to derive in full generality the effect of conditioning on the probability of being pivotal. We hen $\pi = IAC*$, we suggest in Appendix 8 a revision of formula (7) which could be used when A is large enough. We obtain then the exact formula:

$$\pi_{IC}(\mathbf{P}_i) = \frac{1}{r}\binom{A-1}{q-1}\widetilde{P}_{IAC*,q-1,0} + \frac{c_A}{2rA}\sum_{i=q-D}^{q-1}\psi(A,i)\frac{A!}{(i!)\,((A-i)!)}\frac{1}{2^A} \tag{9}$$

where the exact value of the correction term $\psi(A,i)$ is conjectured in Appendix 8.

5 Revisiting Miller's Simulations

To the best of our knowledge, Miller (2008) is the unique paper who has pursued the investigation of Edelman. Like Edelman, the analysis is conducted in the IC framework. There are no analytical results and instead, Miller conducts a number of very interesting simulations. He focuses mostly on the following symmetric version of the US electoral college:

$$A = 45,\ n = 100035 \text{ (and therefore } 2r+1 = \frac{100035}{45} = 2223\text{) and } D = 6$$

The chamber has 51 representatives. To evaluate the impact of A and D (keeping the size of the chamber constant), he considers also the case:

$$A = 51, 2r+1 = 1961, D = 0 \text{ and } A = 27, D = 24$$

We plan to revisit the Miller's simulations by using our exact formulas in the IC case under the assumption of a single vote.

We could also consider his introductory example where $A = 3$ and $n = 9$. He examines successively the situations : $D = 0, D = 1, D = 2$ and $D \geq 4$. In this case, since $A = 3$ and $r = 1$ are small numbers, we can proceed to a direct computation. Truly, the cases $D = 0$ and $D \geq 4$ are trivial. For IC we obtain respectively $\frac{1}{2} \times \frac{1}{2} = 0.25$ and $\binom{8}{4} \times \frac{1}{2^8} = 0.273\,44$. For $D = 2$, we also obtain $0.273\,44$.

For IAC_*, in Appendix 2, we demonstrate that when $D = 2$, the probability is equal to $\frac{3r+2}{4(r+1)(2r+1)}$. When $r = 1$, we obtain: $\frac{5}{24} = 0.208\,33$. When $D \geq 4$, we obtain of course $\frac{1}{9} = 0.111\,11$ and when $D = 0$, we deduce from Le Breton, Lepelley et Smaoui (2012) that the probability is equal to $\frac{9(n+1)}{4n(n+3)}$ which is equal here to $\frac{90}{432} = 0.208\,33$.

When $A = 3$ and large values of r, the orthant probabilities of Appendix 4 are appropriate to conduct the analysis in the IC case. From formula (6), if we ignore conditioning, we deduce that the probability of casting a decisive vote is equal to $\sqrt{\frac{1}{3\pi}} \sum_{i=1}^{2} \frac{3!}{(i!)((3-i)!)} \frac{1}{2^3} \simeq \frac{0.2443}{\sqrt{r}}$.

6 Concluding Remarks

In this paper, we have evaluated the the power of a voter in a mixed-member electoral system for four diferent configurations. All the results have been established under the common assumption that the "at large" component was allocated to the party with the largest countrywide majority. In some ongoing research, we are exploring the same question for the two other alternatives ways of allocating the seats in the national district : proportionality at the margin and proportionality with compensation. In some future research, we would like also to deal with the case where the country districting is not symmetric in contrast to what has been assumed here? How to deal with districts unequal in sizes and in number of local representatives? How the possibility of malapportionment interfers with the values derived in this paper? Such extension would be particularly relevant to deal with the US electoral college or situations where the districts result from natural boundaries.

7 Appendixes

7.1 Appendix 1: The Revised Shapley-Shubik Model IAC*

To define the Revised Shapley-Shubik Model IAC*, we assume that the total population of voters N is partitioned into K groups: $N = \cup_{1 \leq k \leq K} N_k$. Conditionnally on K independent and identically distributed draws $p_1, ..., p_K$ in the interval $[0, 1]$, according to the uniform distribution, the preferences in group N_k proceed from independent Bernoulli draws with parameter p_k. In such a case, the probability of profile P is equal to:

$$\prod_{1 \leq k \leq K} \binom{n_k}{l_k} \frac{1}{(n_k + 1)}$$

where l_k denotes the number of voters voting left in group N_k. This model is very general but in the paper we use it in the case where the groups coincide with the local districts. In the case where district k uses simple game $\mathcal{W}_k$ to elect its representative and the second tier uses the weighted majoriy game to decide, we obtain that conditionaly on the draw $(p_1, ..., p_A)$, the probability that $i \in N_k$ is pivotal is equal to :

$$\left(\sum_{T \notin \mathcal{W}_k \text{ and } T \cup \{i\} \in \mathcal{W}_k} p_k^t (1-p_k)^{n_k - 1 - t}\right) \left(\sum_{T \subseteq \{1,...,A\} \setminus \{k\} : q - w_k \leq \sum_{j \in T} w_j < q} \prod_{j \in T} \pi_j \prod_{j \notin T} (1-\pi_j)\right)$$

where:

$$\pi_j \equiv \sum_{T \in \mathcal{W}_j} p_j^t (1-p_j)^{n_j - 1 - t}$$

The average probability that $i \in N_k$ is pivotal is then :

$$\left[\int_0^1 \left(\sum_{T \notin \mathcal{W}_k \text{ and } T \cup \{i\} \in \mathcal{W}_k} p_k^t (1-p_k)^{n_k - 1 - t}\right) dp_k\right]$$
$$\times \sum_{T \subseteq \{1,...,A\} \setminus \{k\} : q - w_k \leq \sum_{j \in T} w_j < q} \prod_{j \in T} \overline{\pi_j} \prod_{j \notin T} (1-\overline{\pi_j})$$

where:

$$\overline{\pi_j} \equiv \int_0^1 \left(\sum_{T \in \mathcal{W}_j} p_j^t (1-p_j)^{n_j - t}\right) dp_j$$

or more compactly:

$$\left[\int_0^1 \left(\sum_{T \notin \mathcal{W}_k \text{ and } T \cup \{i\} \in \mathcal{W}_k} p_k^t (1-p_k)^{n_k - 1 - t}\right) dp_k\right] \times P\left(q - w_k \leq Y_k < q\right)$$

where Y_k is the random variable $\sum_{j \in \{1,...,A\} \setminus \{k\}} Z_j$ and Z_j is the random variable equal to w_j with probability $\overline{\pi_j}$ and 0 otherwise. When $\mathcal{W}_k$ is the ordinary majority game, $\overline{\pi_k} = \frac{1}{2}$ for all $k = 1, ..., A$. If the Penrose'a approximation is valid, and if A is large, we deduce that the power of any voter $i \in N_k$ is approximatively equal to:

$$\frac{w_k}{n_k} \sqrt{\frac{2}{\pi \sum_j w_j^2}}.$$

When $n_k = n$ and $w_k = 1$ for all $k = 1, .., A$, this probability is equal to:

$$\frac{1}{n} \sqrt{\frac{2}{\pi \sqrt{A}}}.$$

Finally, in this framework, we can also determine the probability that a voter i is pivotal within a superdistrict that would consist of the union of the above A districts. The computation of this probability is more complicated and is investigated in Le Breton, Lepelley et Smaoui (2012). They demonstrate that this probability behaves as $\frac{c_A}{nA}$ when n is large where c_A is a constant which is tabulated in their article.

7.2 Appendix 2 : A Country with Three Districts

7.2.1 The Electoral Outcome under IC

We consider a country with $N = 6r + 3$ voters, where r is an integer. The country is divided into 3 local districts having each $2r+1$ voters. We suppose that that the votes are independent and equidistributed: each voter vote with a probability $\frac{1}{2}$ for the left candidate or the right candidate. There are therefore 8 possible electoral configurations (16 if we pay attention to district identity): (3G;G), (2G,D;G), (G,2D;G) et (3D;G), (3G;D), (2G,D;D), (1G,2D;D), (3D;D) where the first two letters correspond to the local outcomes. The final outcome is described by the realization of the random vector (X^1, X^2, X^3, X^4) where X^4 denotes the electoral outcome in the national district. For instance, the probability that simultaneously $X^1 = G$ and $X^4 = G$ is equal to:

$$P_{GG}^{IC}(r) = \frac{\sum_{i=r+1}^{2r+1} \sum_{j=3r+2-i}^{4r+2} \binom{2r+1}{i} \binom{4r+2}{j}}{2^{6r+3}}$$

It is interesting to know by how much this value deviates from $\frac{1}{4}$. Table 6 below reproduces $P_{GG}^{IC}(r)$ for few values of r:

r	3	5	7	9	21
$P_{GG}^{IC}(r)$	0.35201	0.35053	0.34984	0.34944	0.34861

Table 6 : $P_{GG}^{IC}(r)$

We will show with the tools in Appendix 4 that $\underset{r\to\infty}{Lim} P_{GG}^{IC}(r) = 0.347\,96$.

7.2.2 The Electoral Outcome under IAC*

Let us now determine the probabilities of the various electoral outcomes under $IAC*$. A configuration is defined as a vector (x_1, x_2, x_3), where x_i denotes the number of left voters in the local district i, $i = 1, 2, 3$.

As in the paragraph above, we first compute the probability that the first district votes left and that the national district votes left. This event happens when $x_1 \geq r+1$ and $x_1+x_2+x_3 \geq$

$3r+2$ while the number of configurations is defined by the number $(2r+2)^3$ of integer solutions of the inequalities $0 \leq x_i \leq 2r+1,\ i=1,2,3.$ The number of cases attached to the event that we are considering is given by:

$$\sum_{i=r+1}^{2r+1} \sum_{j=0}^{2r+1} \sum_{\max(0,3r+2-i-j)}^{2r+1} = \frac{17}{6}r^3 + \frac{17}{2}r^2 + \frac{26}{3}r + 3.$$

We deduce then that:

$$P_{GG}^{IAC^*}(r) = \frac{17r^2+34r+18}{48(r+1)^2}.$$

When r tends to infinity, $P_{GG}^{IAC^*}(r)$ tends to $\frac{17}{48} = 0.354$.

Along the same lines, we can calculate the probability of the eight possible elctoral outcomes. We obtain:

$$P_{(3G;G)}^{IAC^*}(r) = P_{(3D;D)}^{IAC^*}(r) = \frac{1}{8};$$

$$P_{(2G,D;G)}^{IAC^*}(r) = P_{(2D,G;D)}^{IAC^*}(r) = \frac{5r^2+10r+6}{16(r+1)^2};$$

$$P_{(G,2D;G)}^{IAC^*}(r) = P_{(D,2G;D)}^{IAC^*}(r) = \frac{r(r+2)}{16(r+1)^2};$$

$$P_{(3D;G)}^{IAC^*}(r) = P_{(3G;D)}^{IAC^*}(r) = 0.$$

When r tends to infinity, these values tend to the values of Table 8 in Appendix 5.

7.2.3 Probability of Casting a Decisive Vote under IAC*

We assume in that section that $D=2$. Consider, without loss of generality, the case of a voter from local district 1. The set of possible patterns is defined by the inequalities :

$$0 \leq x_1 \leq 2r \ \text{ and } \ 0 \leq x_i \leq 2r+1 \text{ for } i=2,3.$$

The total number of configurations is now eqaul to :

$$(2r+1)(2r+2)^2.$$

To evaluate the number of configurations where the considered voter is decisive, we distinguish three cases.

Case 1: The voter is pivotal in the national district (and only in that district) and the local districts dont send three left deputies or three right deputies. It will happen if :

$$x_1+x_2+x_3 = 3r+1 \text{ and } [(x_1 \geq r+1,\ x_2 \leq r \text{ and } x_3 \leq r) \text{ or}$$

$$(x_1 < r,\ x_2 \geq r+1 \text{ et } x_3 \leq r) \text{ or } (x_1 < r,\ x_2 \leq r$$

$$\text{and } x_3 \geq r+1)]$$

or

$$x_1+x_2+x_3 = 3r+1 \ \text{ and } [(x_1 \geq r+1,\ x_2 \geq r+1 \text{ and } x_3 \leq r) \text{ or }$$

$$(x_1 < r,\ x_2 \geq r+1 \text{ and } x_3 \geq r+1) \text{ or } (x_1 \geq r+1,\ x_2 \leq r$$

$$\text{and } x_3 \geq r+1)].$$

We obtain that the number of patterns attached to case 1 is equal to:

$$6\frac{r(r+1)}{2}$$

Case 2: The voter is pivotal in his local district (and only in that district) and the national district vote L (respectively R) and the local districts 2 and 3 vote R (respectively L). For this to hold true, the pattern must satisfy the inequalities:

$$x_1 = r,\ x_1+x_2+x_3 \geq 3r+2\ ,\ x_2 \leq r,\ x_3 \leq r$$

or

$$x_1 = r,\ x_1+x_2+x_3 < 3r+1,\ x_2 \geq r+1,\ x_3 \geq r+1.$$

These inequalities being incompatible, the number of configurations is equal to 0.

Case 3: the voter is pivotal in his local district and in the national district. In such a cse:

$$x_1+x_2+x_3 = 3r+1 \quad \text{and} \quad x_1 = r.$$

The number of attached configurations is equal to:

$$2r+2.$$

Summing the three types of patterns and dividing by the total number of configurations, we obtain that the probability that any voter casts a decisive vote is equal to

$$\frac{3r^2+5r+2}{(2r+1)(2r+2)^2} = \frac{3r+2}{4(r+1)(2r+1)}$$

which behaves as $\frac{3}{8r}$ when r tends to $+\infty$.

7.3 Appendix 3 : A Country with Five Districts

7.3.1 The electoral Outcome under IC

We consider a country with $N = 10r+5$ voters, where r is an integer. The country is divided into 3 local districts having each $2r+1$ voters. We suppose that that the votes are independent and equidistributed: each voter votes with a probability $\frac{1}{2}$ for the left candidate or the right candidate. There are therefore 12 possible electoral configurations : (5G;G), (4G,D;G), (3G,2D;G) et (2G,3D;G), (G,4D;G), (5D;G), (5G;D), (4G,D;D), (3G,2D;D), (2G,3D;D), (G,4D;D) et (5D;D) where the first two letters correspond to the local outcomes. The final outcome is described by the realization of the random vector $(X^1, X^2, X^3, X^4, X^5, X^6)$ where X^6 denotes the electoral outcome in the national district. For instance, the probability that simultaneously $X^1 = G$ and $X^6 = G$ is equal to:

$$P_{GG}^{IC}(r) = \frac{\sum_{i=r+1}^{2r+1} \sum_{j=5r+3-i}^{8r+4} \binom{2r+1}{i} \binom{8r+4}{j}}{2^{10r+5}}$$

It is interesting to know by how much this value deviates from $\frac{1}{4}$. Table 7 below reproduces $P_{GG}^{IC}(r)$ for few values of r:

r	3	5	7	9	21
$P_{GG}^{IC}(r)$	0.32667	0.32562	0.32513	0.32484	0.32426

Table 7 : $P_{GG}^{IC}(r)$

We will show with the tools in Appendix 4 that $\underset{r\to\infty}{Lim} P_{GG}^{IC}(r) = 0.32379$

7.3.2 The Electoral Outcome under IAC*

A configuration is defined as a vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i denotes the number of left voters in the local district i, $i = 1, 2, 3, 4, 5$.

As in the paragraph above, we first compute the probability that the first district vote left and that the national district vote left. This event happens iff

$$r+1 \leq x_1 \leq 2r+1, \ \ 0 \leq x_2, \ x_3, \ x_4, \ x_5 \leq 2r+1, \ x_1+x_2+x_3+x_4+x_5 \geq 5r+3.$$

The number of solutions to these inequalities is equal to:

$$\frac{421}{40}r^5 + \frac{421}{8}r^4 + \frac{845}{8}r^3 + \frac{851}{8}r^2 + \frac{1077}{20}r + 11,$$

Since the total number of patterns is equal to $(2r+2)^5 = 2^5(r+1)^5$, we obtain:

$$P_{GG}^{IAC^*} = \frac{421r^4 + 1684r^3 + 2541r^2 + 1714r + 440}{1280(r+1)^4}$$

which tends to $\frac{421}{1280} = 0.3289$ when r tends to $+\infty$.

Proceeding similarly, we derive the robabilities of the twelve possible electoral outcomes:

$$P^{IAC^*}_{(5G;G)}(r) = P^{IAC^*}_{(5D;D)}(r) = \frac{1}{32} = \frac{1}{2^5};$$

$$P^{IAC^*}_{(4G,D;G)}(r) = P^{IAC^*}_{(4D,G;D)}(r) = \frac{119r^4 + 476r^3 + 719r^2 + 486r + 120}{768(r+1)^4};$$

$$P^{IAC^*}_{(3G,2D;G)}(r) = P^{IAC^*}_{(3D,2G;D)}(r) = \frac{31r^4 + 124r^3 + 191r^2 + 134r + 40}{128(r+1)^4};$$

$$P^{IAC^*}_{(2G,3D;G)}(r) = P^{IAC^*}_{(2D,3G;D)}(r) = \frac{r(9r^3 + 36r^2 + 719r^2 + 49r + 26}{128(r+1)^4};$$

$$P^{IAC^*}_{(G,4D;G)}(r) = P^{IAC^*}_{(D,4G;D)}(r) = \frac{r(r^3 + 4r^2 + r - 6}{768(r+1)^4};$$

$$P^{IAC^*}_{(5D;G)}(r) = P^{IAC^*}_{(5G;D)}(r) = 0.$$

We note that the limit values when r tends to ∞ correspond to the values reported in Table 9.

7.3.3 Probability of Casting a Decisive Vote under IAC*

Consider the case of a voter from district 1. The total number of configurations is the number of integer solutions of the following inequalities:

$$0 \leq x_1 \leq 2r \text{ et } 0 \leq x_i \leq 2r + 1 \text{ pour } i = 2, 3, 4, 5.$$

which is equal to :

$$(2r + 1)(2r + 2)^4.$$

We will evaluate the power of the voter for two values of D : $D = 2$ (a parliament of 7 deputies) and $D = 4$ (a parliament of 9 deputies).

Consider first the case $D = 2$. To evaluate the number of configurations for which the voter is decisive, we distinguish between three cases.

Case 1.1: The voter is pivotal in the national district (and only in that district) and the local districts send two left deputies and three right deputies or 2 right deputies and three left deputies. This will happen when for instance :

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5r + 2 \text{ and } (x_1 \geq r + 1, x_2 \geq r + 1, x_3 \leq r, x_4 \leq r, x_5 \leq r.$$

Since there are 10 ways to chose 2 districts out of 5 and also 10 ways to chose 3 districts out of 5, we deduce that the number of solutions of case 1 is equal to:

$$\frac{20r(r+1)(11r^2 + 23r + 14)}{24}.$$

Case 1.2: The voter is pivotal in his local district (and only in that district) and the 4 other local districts send 3 left (right) deputies and one right (left) deputy to the chamber and the national district is on the right (left). Counting the number of such patterns leads to:

$$\frac{8r(r+1)(r-1)(r+2)}{24}.$$

Case 1.3: The voter is pivotal in his local district and in the national district and the four other local districts send either two left deputies and two right deputies, or one right deputy and three left deputies or one left deputy and three right deputies. The total number of patterns is equal to

$$\frac{2(r+1)(8r^2+16r+9)}{3}.$$

The probability that a voter casts a decisive vote follows by summation of the above numbers and division by the total number of configurations. We obtain:

$$\frac{57r^3+149r^2+130r+36}{96(r+1)^3(2r+1)}$$

which behaves as $\frac{19}{64r} = \frac{0.297}{r}$ when r tends to $+\infty$.

Let us consider now the case where $D = 4$. We also consider in turn three cases.

Case 2.1: The voter is pivotal in the national district (and only in that district) and the local districts send either 1, 2, 3 or 4 left deputies to the chamber. The number of configurations with one left deputy and four right deputies or four left deputies and one deputy from the right is equal to:

$$\frac{10r(r+1)(r-1)(r+2)}{24}.$$

The number of configurations with 2 deputies from the left and three deputies from the right or 2 deputies from the left and 3 deputies from the right is equal to:

$$\frac{20r(r+1)(11r^2+23r+14)}{24}.$$

Case 2.2: The voter is pivotal in his local district (and only in that district) and the 4 other local districts send 4 deputies from the left (right) to the chamber and the national district vote right (left). This pattern cannot happen.

Case 2.3: The voter is pivotal in his local district and in the national district i.e:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5r + 2 \quad \text{and} \quad x_1 = r.$$

In such case, the voter is decisive irrespective of the electoral outcome in the other local districts. The number of such configurations is equal to:

$$\frac{4(r+1)(2r+1)(2r+3)}{3}.$$

Adding these numbers, we obtain that the probability that a voter casts a decisive vote is equal to::

$$\frac{115r^3 + 299r^2 + 258r + 48}{192(r+1)^3(2r+1)}$$

which behaves as $\frac{115}{384r} = \frac{0.299}{r}$ when r tends to $+\infty$. The increase in the number of national deputies leads to a slight increase of the power of the voters.

7.4 Appendix 4 : Computation of $\widetilde{P}^B_{\alpha,K,0}$ when $\alpha = IC$ and Gaussian Orthant Probabilities

In this Appendix, we present an asymptotic approximation of $\widetilde{P}^B_{\alpha,K,1}$ in the case where $\alpha = IC$. For every district $j = 1, ..., A$, we consider the random variable:

$$S^r_j = \sum_{i=1}^{2r+1} X_{ij}$$

where the random variaable X_{ij} codes the vote of voter i in district $j = j(i)$ as follows:

$$X_{ij} = \begin{cases} 1 \text{ if voter } i \text{ has voted } L \\ 0 \text{ if voter } i \text{ has voted } R \end{cases}$$

Therefore S^r_j denotes the number of voters from district j who voted left. On the other hand let S^r_d denote the random variable:

$$S^r_d = \sum_{j=1}^{A} \sum_{i=1}^{2r+1} X_{ij}$$

S^r_d denotes the number of voters in the national district who voted left. We consider finally the random vector $S^r = (S^r_1, S^r_2, ..., S^r_A, S^r_d)$ of dimension $A + 1$. To study the behavior of this vector, consider first the vector $\widehat{S^r}$ of dimension A :

$$\widehat{S^r} = \left(\sum_{i=1}^{2r+1} X_{i1}, \sum_{i=1}^{2r+1} X_{i2}, ..., \sum_{i=r+1}^{2r+1} X_{iA} \right)$$

From the central limit theorem, we deduce that when r is large this vector is approximatively a gaussian $A-$dimensional vector $N\left(\widehat{\mu}, \widehat{\Omega}\right)$ where:

$$\widehat{\mu} = \left(\frac{2r+1}{2}, \frac{2r+1}{2}, ..., \frac{2r+1}{2} \right)$$

$$\widehat{\Omega} = \begin{pmatrix} \frac{\sqrt{2r+1}}{2} & 0 & 0 & . & 0 & 0 \\ 0 & \frac{\sqrt{2r+1}}{2} & 0 & . & 0 & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & 0 & . & 0 & 0 \\ 0 & 0 & . & . & 0 & \frac{\sqrt{2r+1}}{2} \end{pmatrix}$$

Since $S^r = \Delta \widehat{S^r}$ where Δ is the $(A+1) \times A$ matrix defined as follows:

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & . & 0 & 1 \\ 1 & 1 & 1 & . & 1 & 1 \end{pmatrix}$$

we deduce that S^r is itself approximatively a $A+1$-dimensional-gaussian vector $N(\mu, \Omega)$ where:

$$\mu = \left(\frac{2r+1}{2}, \frac{2r+1}{2}, ..., \frac{2r+1}{2}, \frac{A(2r+1)}{2} \right)$$

Concerning the variance-covariance matrix Ω, it is enough (as they are all identical) to compute the covariance between say S_1^r et S_d^r. We obtain:

$$Cov\left(S_1^r, S_d^r\right) = \sum_{i=1}^{2r+1} \left(X_{i1} - \frac{1}{2} \right)^2 = \frac{2r+1}{4}$$

Since otherwise:

$$\begin{aligned} Var\left(S_1^r\right) &= \frac{2r+1}{4} \\ Var\left(S_d^r\right) &= \frac{A(2r+1)}{4} \end{aligned}$$

we obtain:

$$\sqrt{\Omega} = \begin{pmatrix} \frac{\sqrt{2r+1}}{2} & 0 & 0 & . & 0 & \frac{\sqrt{2r+1}}{2} \\ 0 & \frac{\sqrt{2r+1}}{2} & 0 & . & 0 & \frac{\sqrt{2r+1}}{2} \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & 0 & . & \frac{\sqrt{2r+1}}{2} & \frac{\sqrt{2r+1}}{2} \\ \frac{\sqrt{2r+1}}{2} & \frac{\sqrt{2r+1}}{2} & . & . & \frac{\sqrt{2r+1}}{2} & \frac{\sqrt{A(2r+1)}}{2} \end{pmatrix}$$

The matrix is almost diagonal. The covariances are located on the last row and last column as the votes between local districts are uncorrelated. We observe that the correlation coefficient ρ on this last row is equal to:

$$\rho = \frac{\frac{2r+1}{4}}{\frac{\sqrt{2r+1}}{2} \times \frac{\sqrt{A(2r+1)}}{2}} \simeq \frac{1}{\sqrt{A}}$$

After normalization, we can write the matrix in the following compact form:[23]

$$\sqrt{\Omega} = \begin{pmatrix} 1 & 0 & 0 & . & 0 & \rho \\ 0 & 1 & 0 & . & 0 & \rho \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & 0 & . & 1 & \rho \\ \rho & \rho & . & . & \rho & 1 \end{pmatrix}$$

Let B and C be two disjoint subsets of $\{1, 2, ..., A\}$. What is the probability $\widetilde{P}_{B,C,1}$(respectively $\widetilde{P}_{B,C,0}$) of the event "The districts whose index belongs to B have voted left, the districts whose number belongs to C have voted right and the national district has voted left (respectively right)"? Considering the centered vector $\widetilde{S}^r = S^r - \mu$, it can be seen that:

$$\widetilde{P}_{B,C,1} = Prob\left(\widetilde{S}_j^{\,r} > 0 \text{ for all } j \in B,\ \widetilde{S}_j^{\,r} < 0 \text{ for all } j \in C \text{ and } \widehat{S_d}^{\,r} > 0\right)$$

Under the gaussian approximation, this amounts to compute the probability that the centered gaussian random vector of dimension $|B|+|C|+1$ and with matrix of variances-covariances $\Omega_{|B|+|C|+1}$ (where $\Omega_{|B|+|C|+1}$ is the sub-matrix of Ω corresponding to indices in $B \cup C$ and to the last index) takes values in the orthant $R_+^{|B|+1} \times \mathbb{R}_-^{|\mathbb{C}|}$

For instance, when $A = 3$, we have a quadrivariate Gaussian vector with matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 \end{pmatrix}.$$

The probability $\widetilde{P}_{\{3\},\varnothing,1}$ that the last local district votes left and that the national district votes left is the probability that the two-dimensional vector composed of the last two coordinates belongs to the orthant $\mathbb{R}_+ \times \mathbb{R}_+$. An important literature in statistics is dedicated to the exact or

[23] The matrix Ω has a rank equal to A which means that the gaussian vector takes its values in a subspace of dimension A.

approximate computation of these probabilities (Gupta (1963)). In particular it is well known that $\widetilde{P}_{\{A\},\varnothing,1} = \frac{1}{4} + \frac{\arcsin\rho}{2\pi}$.[24]

The tabulated sequence of the probabilities $\widetilde{P}_{k,0}$ can be obtained through simulations $\widetilde{P}_{k,0}$The R software uses a Monte-Carlo method to produce a sequence of independent draws from the specified gaussian random vector. If we take the random vector composed of the first A coordinates (wich has a diagonal variance-covariance matrix), we record for each raw how many coordinates were positive (and so how many were negative) and also the sign of the sum. We then obtain empirical frequencies of all the $2^{A+1}-2$ possible events. We deduce from Glivenko-Cantelli's theorem that these empirical frequencies converge almost surely to the true values when the size of the sample tends to infinity. The tabulated values appear below.

Insert here the tabulated values that were in the word document sent some time ago.

7.5 Appendix 5 : Computation of $\widetilde{P}_{\alpha,k,0}$ when $\alpha = IAC^*$ and Uniform Orthant Probabilities

In this Appendix, we offer an asymptotic argument in defense of the computation of the probabilities of the electoral outcomes which appear in Appendices 2 and 3 when the probability model α is $IAC*$ Once again, we reduce the problem to the computation that a centered random vector takes values in an orthant. The unique difference with the IC setting is that now the marginal laws are uniform instead of being gaussian. We consider A local districts of size $2r+1$ where A is an odd integer. Using the notations of Appendix 4, we can replicate the argument of Chamberlain and Rothschild (1981)'s Proposition 4 to conclude that, if r is large enough, then for all $j = 1, ..., A$, the law of $\frac{\sum_{i=1}^{2r+1} X_{ij}}{2r+1}$ is approximatively uniform on $[0,1]$. As in the preceding section, for all $1 \leq k \leq A$, we denote by $\widetilde{P}_{k,0}$ the probability of the event "the first k local districts vote left, the following $A-k$ local districts vote right and the national district vote right ". May (1948) has proved that:

$$\widetilde{P}_{k,0} = \frac{1}{A!2^A} \sum_{i=0}^{A-k-1} (-1)^i \binom{A}{i} (A-k-i)^A$$

The tabulated values of $\widetilde{P}_{k,0}$ appear below for all odd integers until 19. It provides a lot of useful information on the distribution of districts won by the left when the left loses the national district. For the application, we will need this probability in the case where $k = \frac{A+D+1}{2}$

[24]It is interesting to note that $6\widetilde{P}_{\{1,2\},\{3\},0}$ defines the probability of the referendum paradox in the case when $\alpha = IC$ and $A = 3$ (see Feix, Lepelley, Merlin and Rouet (2004)).

A,k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1
3	0.125	0.10417	0.02083	0													
5	0.031	0.031	0.02422	7.0313×10^{-3}	2.6042×10^{-4}	0											
7																	
9																	
11																	
13																	
15																	
17																	
19																	

Table ?

7.6 Appendix 6 : Computation of $\pi(\mathbf{R}_{j(i)})$ and $\pi(\mathbf{R}_{d(n)})$

We consider a parliament with A deputies with a weight of 1 and a (single) deputy with a weight of D. We assume that $A+D$ is odd and $D \leq \frac{A+D-1}{2}$. This implies that the weighted majority game $\mathcal{W}$ attached to the quota $q \equiv \frac{A+D+1}{2}$is strong but not dictatorial.

Consider a deputy with a weight of 1. He is pivotal when the coalition of deputies voting left (or voting right, since the game $\mathcal{W}$ is strong) has a weight equal to $\frac{A+D-1}{2}$. Counting the number of profiles where a deputy with a weight of 1 is pivot is equivalent to count the number of combinations of deputies such that their cumulative weight is equal to $q-1 = \frac{A+D-1}{2}$ (of course all the rearrangements of this combinations must be counted). It is straightforward to show that:

$$\underline{B}(A,D) = \binom{A-1}{q-1} \widetilde{P}^{A-1}_{q-1,0} + \binom{A-1}{q-1-D} \widetilde{P}^{A-1}_{q-D,1}$$

and

$$\overline{B}(A,D) = \frac{\sum_{i=q-D}^{q-1} \binom{A}{i}}{2^A} = \sum_{i=q-D}^{q-1} \frac{A!}{(i!)\left((A-i)!\right)} \frac{1}{2^A}$$

Since $\binom{A-1}{q-1} = \binom{A-1}{q-1-D}$ and $\widetilde{P}_{q-1,0} = \widetilde{P}_{q-D,1}$, the first formula simplifies to:

$$\underline{B}(A,D) = 2\binom{A-1}{q-1} P_{q-1,0}$$

Under the assumption of double vote, we get $\widetilde{P}_{k,0} = \widetilde{P}_{k,1} = \frac{1}{2^A}$ and therefore:[25]

[25]These formulas have been first demonstrated by Edelman (2004).

$$\underline{B}(A,D)=\frac{\binom{A-1}{q-1}}{2^{A-1}}=\frac{\frac{(A-1)!}{((q-1)!)((A-q)!)}}{2^{A-1}}$$

We are quite far from the Penrose's approximation[26] as illustrated by the following numerical values:

$$\overline{B}(A,D)=\begin{cases}0.89063 \text{ when } (A,D)=(10,5)\\ 0.97435 \text{ when } (A,D)=(80,19)\end{cases}$$

et

$$\underline{B}(A,D)=\begin{cases}7.0313\times 10^{-2} \text{ when } (A,D)=(10,5)\\ 9.1728\times 10^{-3} \text{ when } (A,D)=(80,19)\end{cases}$$

We observe that the ratio $\frac{\overline{B}(A,D)}{\underline{B}(A,D)}$ takes successively the values 12.667 and 106.22 while the ratio of the weights takes the values 5 and 19. With the Leech's software at Warwick, we have calculated $\overline{B}(A,D)$ and $\underline{B}(A,D)$ for a sample of values of the parameters A and D. These values are reported in the following tables.

	$\overline{B}(A,D)$	$\underline{B}(A,D)$
$A=1$	1	0
$A=3$	0.75	0.25
$A=5$	0.625	0.25
$A=7$	0.546875	0.234375
$A=9$	0.492188	0.218750

Table 10. $\overline{B}(A,D)$ and $\underline{B}(A,D)$ **as a function of A** $(D=2)$

	$\frac{\overline{B}(A,D)}{\underline{B}(A,D)}$
11	6.6
21	5.1334
31	4.7238
41	4.5316
51	4.4200
61	4.3471
81	4.2577
121	4.1701
141	4.1453
161	4.1269

[26] The central limit theorem does not apply here as the variance of the random vote of the deputy of weight D is a non negligible fraction of the variance of the total vote.

Table 11. $\overline{B}(A,D)/\underline{B}(A,D)$ **as a function of A** ($D=4$)

	$\overline{B}(A,D)$	$\underline{B}(A,D)$
$D=2$	0.392761	0.183289
$D=4$	0.698242	0.122192
$D=6$	0.881531	0.061096
$D=8$	0.964844	0.022217
$D=10$	0.992615	0.005554

Table 12. $\overline{B}(A,D)$ and $\underline{B}(A,D)$ **as a function of D** ($A=15$)

7.7 Appendix 7: A Conjecture in the IC Case where the Number A of Local Districts is Large

In the case where the number A of local districts is large, we can proceed differently for computing the probabilities $\widetilde{P}^{A}_{k,1}$ along the lines suggested by Feix, Lepelley, Merlin and Rouet (2007). The random variable $Z^r_{k,1} \equiv S^r_d$ |the districts $1,2,...,k$ have voted left while the districts $k+1,...,A$ have voted right can be expressed as[27] :

$$\sum_{j=1}^{k}\left[S^r_j \mid S^r_j > \frac{2r+1}{2}\right] + \sum_{j=k+1}^{A}\left[S^r_j \mid S^r_j < \frac{2r+1}{2}\right]$$

Consider the normalized random variable $W^r_{k,1} \equiv \frac{Z^r_{k,1} - \frac{A(2r+1)}{2}}{\frac{\sqrt{A(2r+1)}}{2}}$. We have:

$$W^r_{k,1} = \frac{1}{\sqrt{A}}\left[\sum_{j=1}^{k}\frac{S^r_j - \frac{2r+1}{2}}{\frac{\sqrt{(2r+1}}{2}} \mid \frac{S^r_j - \frac{2r+1}{2}}{\frac{\sqrt{2r+1}}{2}} > 0 + \sum_{j=k+1}^{A}\frac{S^r_j - \frac{2r+1}{2}}{\frac{\sqrt{2r+1}}{2}} \mid \frac{S^r_j - \frac{2r+1}{2}}{\frac{\sqrt{2r+1}}{2}}\right] < 0$$

To summarize, we have obtained that:

$$W^r_{k,1} = \frac{1}{\sqrt{A}}\left[\sum_{j=1}^{k} W^+_j + \sum_{j=k+1}^{A} W^-_j\right]$$

where the random variables W^+_j and W^-_j for $j = 1,2,...,A$ are independent and with respective probability laws gaussian truncated on the right and gaussian truncated on the left. Hereafter, we will denote by μ^+ and σ^+ the mean and the standard deviation of W^+_j and by μ^- and σ^- the mean and the standard deviation of W^-_j. It is well known that:

$$\mu^+ = \sqrt{\frac{2}{\pi}}, \mu^- = -\sqrt{\frac{2}{\pi}}\text{and } \sigma^{2+} = \sigma^{2-} = 1 - \frac{2}{\pi}$$

[27] Since the random variables $S^r_j \mid S^r_j > \frac{2r+1}{2}$ and $S^r_j \mid S^r_j < \frac{2r+1}{2}$, $j = 1,...,A$ are independent.

From the Lindeberg-Lyapounov's version of the central limit theorem, we deduce that if A is large, the law of $\sqrt{A}W_{k,1}^{r}$ is approximatively a gaussian law with mean $\mu = k\sqrt{\frac{2}{\pi}} - (A-k)\sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}}(2k - A)$ and variance $\sigma^2 = A\left(1 - \frac{2}{\pi}\right)$. We deduce from that:

$$\widetilde{P}_{k,1}^{A} = \frac{1}{2^A}\binom{A}{k}\text{Prob }\left[W_{k,1}^{r} > 0\right] \simeq \frac{1}{2^A}\binom{A}{k}\text{Prob }\left[Z > -\frac{\sqrt{\frac{2}{\pi}}(2k-A)}{\left(1 - \frac{2}{\pi}\right)}\right]$$

where Z is the standard Gaussian law $N(0,1)$. From the tabulated values of the cdf of Z, we compute the probabilities $\widetilde{P}_{k,1}^{A}$. Since $Z_{k,1}^{r}$ is approximatively a gaussian:

$$\frac{A(2r+1)}{2} + \frac{\sqrt{2r+1}}{2}N\left(\sqrt{\frac{2}{\pi}}(2k-A), \sqrt{A\left(1 - \frac{2}{\pi}\right)}\right)$$

we conjecture[28] that the probability that the national district is balanced without the vote of voter i, given that the the districts $1, 2, ..., k$ have voted left and the districts $k+1, ..., A$ have voted right, is equal to:

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{4\pi}{(\pi-2)}}\frac{1}{\sqrt{A(2r+1)}}e^{-\frac{(2k-A)^2}{A(\pi-2)}}$$

Under the presumption that this formula holds true, we obtain the exact value of the probability of casting a decisive vote under the assumption of a single vote when the probability model is IC. We have to replace $\sqrt{\frac{1}{\pi A r}}$ in formula (6) by $\sqrt{\frac{1}{\pi A r}}\xi(A,k)$ where:

$$\xi(A,k) \equiv \sqrt{\frac{\pi}{(\pi-2)}}e^{-\frac{(2k-A)^2}{A(\pi-2)}}$$

For the sake of illustration, we find for instance that when $A = 51$, $\xi(51, 25) = \sqrt{\frac{\pi}{(\pi-2)}}e^{-\frac{1}{51(\pi-2)}} = 1.6306$ and $\xi(51, 24) = \sqrt{\frac{\pi}{(\pi-2)}}e^{-\frac{9}{51(\pi-2)}} = 1.4213$ while $\xi(51, 10) = \sqrt{\frac{\pi}{(\pi-2)}}e^{-\frac{(31)^2}{51(\pi-2)}} = 1.1255 \times 10^{-7}$.

7.8 Appendix 8: A Conjecture in the IAC* Case where the Number A of Local Districts is Large

In the case where the number A of local districts is large, we can proceed along the lines of appendix 7 for computing the probabilities $\widetilde{P}_{k,1}^{A}$. As before, the random variable $Z_{k,1}^{r} \equiv S_d^r$ |the districts $1, 2, ..., k$ have voted left while the districts $k+1, ..., A$ have voted right can be expressed as[29] :

[28] The conjecture is based on a "local" version of the central limit theorem that we have not proved.

[29] Since the random variables $X_j^r \mid X_j^r > \frac{2r+1}{2}$ and $X_j^r \mid X_j^r < \frac{2r+1}{2}$, $j = 1, ..., A$ are independent.

$$\sum_{j=1}^{k}\left[S_j^r \mid S_j^r > \frac{2r+1}{2}\right] + \sum_{j=k+1}^{A}\left[S_j^r \mid S_j^r < \frac{2r+1}{2}\right]$$

Consider the normalized random variable $W_{k,1}^r \equiv \frac{Z_{k,1}^r}{\frac{A(2r+1)}{2}}$. We have:

$$W_{k,1}^r = \frac{1}{A}\left[\sum_{j=1}^{k}\frac{S_j^r}{2r+1} \mid \frac{S_j^r - \frac{2r+1}{2}}{2r+1} > 0 + \sum_{j=k+1}^{A}\frac{S_j^r}{2r+1} \mid \frac{S_j^r - \frac{2r+1}{2}}{2r+1}\right] < 0$$

Since for all $j = 1, ..., A$, $\frac{S_j^r}{2r+1}$ behaves as a uniform random variable $U(0,1)$ when r gets large, the random variables $\frac{S_j^r}{2r+1} \mid \frac{S_j^r - \frac{2r+1}{2}}{2r+1} > 0$ and $\frac{S_j^r}{2r+1} \mid \frac{S_j^r - \frac{2r+1}{2}}{2r+1} < 0$ behave respectively as uniform random variables $U(\frac{1}{2}, 1)$ and $U(0, \frac{1}{2})$. This implies:

$$W_{k,1}^r = \frac{1}{A}\left[\sum_{j=1}^{k} W_j^+ + \sum_{j=k+1}^{A} W_j^-\right]$$

where the random variables W_j^+ and W_j^- for $j = 1, 2, ..., A$ are independent and with respective probability laws $U(\frac{1}{2}, 1)$ and $U(0, \frac{1}{2})$. Let us denote by μ^+ and σ^+ the mean and the standard deviation of W_j^+ and by μ^- and σ^- the mean and the standard deviation of W_j^-. Straightforward computations lead to: $\mu^+ = \frac{3}{4}, \mu^- = \frac{1}{4}$and $\sigma^{2+} = \sigma^{2-} = \frac{1}{48}$. From the Lindeberg-Lyapounov's version of the central limit theorem, we deduce that if A is large, the law of $\sqrt{A}W_{k,1}^r$ is approximatively a gaussian law with mean $\mu = \frac{3k}{4} + \frac{(A-k)}{4} = \frac{k}{2} + \frac{A}{4}$ and variance $\sigma^2 = \frac{A}{48}$. We deduce from that:

$$\widetilde{P}_{k,1}^A = \frac{1}{2^A}\binom{A}{k}\text{Prob }\left[W_{k,1}^r > 0\right] \simeq \frac{1}{2^A}\binom{A}{k}\text{Prob }\left[Z > -\frac{\frac{k}{2} + \frac{A-k}{4}}{\sqrt{\frac{A}{48}}}\right]$$

where Z is the standard Gaussian law $N(0,1)$. From the tabulated values of the cdf of Z, we compute the probabilities $\widetilde{P}_{k,1}^A$. Since $Z_{k,1}^r$ is approximatively a gaussian:

$$(2r+1)\,N(\frac{k}{2} + \frac{A}{4}, \sqrt{\frac{A}{48}}) = N(\frac{k(2r+1)}{2} + \frac{A(2r+1)}{4}, \sqrt{\frac{A(2r+1)^2}{48}})$$

we conjecture[30] that the probability that the national district is balanced without the vote of voter i, given that the districts $1, 2, ..., k$ have voted left and the districts $k+1, ..., A$ have voted right, is equal to:

[30] Again as before, the conjecture is based on a "local" version of the central limit theorem that we have not proved.

$$\frac{1}{\sqrt{2\pi}}\frac{4\sqrt{3}}{(2r+1)}\frac{1}{\sqrt{A}}e^{-\frac{\left(\frac{k(2r+1)}{2}-\frac{A(2r+1)}{4}\right)^2}{\frac{A(2r+1)}{24}}}$$

Under the presumption that this formula holds true, we obtain the exact value of the probability of casting a decisive vote under the assumption of a single vote when the probability model is $IAC*$. We have to replace $\frac{c_A}{2rA}$ in formula (7) by $\frac{c_A}{2rA}\psi(A,k)$ where:

$$\psi(A,k) \equiv \frac{1}{c_A}\sqrt{\frac{24A}{\pi}}e^{-\frac{\left(\frac{k(2r+1)}{2}-\frac{A(2r+1)}{4}\right)^2}{\frac{A(2r+1)}{24}}} \simeq 2e^{-\frac{\left(\frac{k(2r+1)}{2}-\frac{A(2r+1)}{4}\right)^2}{\frac{A(2r+1)}{24}}} \text{ since } c_A \simeq \sqrt{\frac{6A}{\pi}} \text{ when } A \text{ gets large}$$

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