Monotonicity Paradoxes in Three-Candidate Elections Using Scoring Elimination Rules

Dominique Lepelley (Université de La Réunion, France)
Issofa Moyouwou (Université de Yaoundé, Cameroon)
Hatem Smaoui (Université de La Réunion, France)

Abstract

Scoring Elimination Rules (SER), that give points to candidates according to their rank in voters' preference orders and eliminate the candidate(s) with the lowest number of points, constitute an important class of voting rules. This class of rules, that includes some famous voting methods such as Plurality Runoff or Coombs Rule, suffers from a severe pathology known as *monotonicity paradox* or *monotonicity failure*, that is, getting more points from voters can make a candidate a loser and getting fewer points can make a candidate a winner. In this paper, we study three-candidate elections and we identify, under various conditions, which SER minimizes the probability that a monotonicity paradox occurs. We also analyze some strategic aspects of these monotonicity failures. The probability model on which our results are based is the *Impartial Anonymous Culture* condition, often used in this kind of study.

1. Introduction

In voting theory, a monotonicity paradox occurs each time a voting system reacts in a perverse way to a change in individual preferences. Two forms of monotonicity paradox are usually distinguished and studied in the literature (Brams and Fishburn, 1983; Lepelley et al. 1996; Miller, 2012; Felsenthal and Tideman, 2014); according to Brams and Fishburn:

MORE-IS-LESS PARADOX (or *Upward Monotonicity Failure*) occurs when, if the winner were ranked higher by some voters, all else unchanged, then another candidate might have won.

LESS-IS-MORE PARADOX (or *Downward Monotonicity Failure*) occurs when, if a loser were ranked lower by some voters, all else unchanged, then this loser might have won.

The following example, borrowed from Felsenthal and Tideman (2014), illustrates these two forms of monotonicity failures[1] under the widely used Plurality Runoff system. Under this method, each voter casts one vote for a single candidate and a candidate wins if she obtains an absolute majority of the votes. If no candidate is declared the winner in the first round, a

[1] See also Miller (2012) for a (more or less) real-world example.

second round is organized which confronts the two candidates with the highest number of votes in the first round and the one who obtains the majority of votes wins.

Suppose there are 127 voters whose rankings of the three candidates, *a*, *b* and *c*, are as follows:

Number of voters	Ranking
32	$R_1: a \succ b \succ c$
9	$R_2: a \succ c \succ b$
9	$R_3: b \succ a \succ c$
38	$R_4: b \succ c \succ a$
30	$R_5: c \succ a \succ b$
9	$R_6: c \succ b \succ a$

Under Plurality Runoff, candidate *c* is eliminated in the first round (*a* obtains 41 votes, *b* 47 votes and *c* 39 votes) and candidate *a* beats candidate *b* in the second round (71 to 56): candidate *a* is thus the election winner.

Suppose now that nine out of the 38 voters whose initial ranking is $b > c > a$ change their ranking to $a > b > c$ (thereby increasing *a*'s support). As a result of this change, *b* (rather than *c*) is eliminated in the first round and *c* beats *a* in the second round (68 to 59), illustrating the More-is-Less-Paradox.

Suppose instead that three of the 38 voters whose initial ranking is $b > c > a$ change their ranking to $c > a > b$ (thereby decreasing *b*'s support). As a result of this change, *a* (rather than *c*) is eliminated in the first round and *b* beats *c* in the second round (76 to 51), illustrating the Less-is-More-Paradox.

Plurality Runoff (or Plurality Elimination Rule – PER in what follows) is an example of *Run-off Point Systems* or *Scoring Elimination Rules*. In a remarkable paper, Smith (1973) has shown that the whole class of Scoring Elimination Rules is subject to monotonicity failure. As we only consider in this paper the three-candidate case, we can describe this class of voting systems as follows: in the first round of the choice process, each voter ranks the candidates and the score of each candidate is computed on the basis of a point-system $(1, \lambda, 0)$ that gives 1 point each time a candidate is ranked first in voter's preferences, points for a second position, with $0 \leq \lambda \leq 1$, and 0 point for a third and last position. The candidate with the lowest score is then eliminated and, in a second round, the two remaining candidates are confronted and the one who obtains the majority of votes wins. Plurality Runoff is obtained when $\lambda = 0$ and, in the three-candidate case, is equivalent to the so-called Alternative Vote or Instant Runoff Voting. Taking $\lambda = 1$ gives the Coombs method (or Negative Plurality Elimination Rule - NPER): the candidate who is eliminated in the first round is the one who is ranked last by the largest number of voters. A third well known Scoring Elimination Method is the Borda Elimination Rule (BER), associated to the case where $\lambda = 1/2$; this rule is known to be the only one in the class of Scoring Elimination Rules that always selects the Condorcet winner – i.e. the candidate who beats each other candidate in majority pairwise comparisons – when such a candidate exists (see Smith, 1973). Although Scoring Elimination Rules are not the only voting procedures that exhibit monotonicity paradox[2], we focus in the current study on this particular class which contains a procedure very often used in practice (namely, PER).

As some authors regard a voting method that is susceptible to give rise to monotonicity failure as totally unacceptable (e.g. Doron and Kronick, 1977), it is of interest to investigate the following issues: First, can we consider that monotonicity violations are too rare to be of practical concern? Second, how alternative voting systems compare with respect to their propensity to give rise to such violations? And finally, what is the voting rule that minimizes the probability of monotonicity failure in the class of Scoring Elimination Rules (SER)?

The first attempt to consider some of these questions is Lepelley et al. (1996). Using the Impartial Anonymous Culture (IAC) assumption, which states that all possible voting situations are equally likely to be observed, they compute the probabilities of both More-is-Less and Less-is-More paradoxes for PER (or Plurality Runoff) and for NPER (or Coombs rule) in three-candidate elections. They conclude that "it seems difficult to claim that monotonicity paradoxes are extremely rare and have no practical relevance". Miller (2012) implements some simulations based on the Impartial Culture (IC) assumption[3] and on various special conditions (such as single-peakedness) for evaluating the likelihood of monotonicity failure in PER elections with three candidates. His basic and important finding is that, under PER, monotonicity problems are substantial whenever elections are closely contested by all three alternatives. The results obtained by Plassmann and Tideman (2014) are somewhat more comforting: using a statistical model that simulates ranking profiles that follow the same distribution as ranking profiles in actual elections, they estimate (among other things) the frequency of More-is-Less paradox for PER and NPER in three-candidate elections; it turns out that these frequencies are between 1% and 2%, according to the number of voters.

In this paper, we offer some new and exact results on the likelihood of monotonicity failures for the whole class of SER, completing and extending what has been done before.

Our study is organized as follows. In Section 2, we derive some analytical representations for the probability of Monotonicity Paradox(es) under each of the three most famous SER: PER ($\lambda = 0$), BER ($\lambda = 1/2$) and NPER ($\lambda = 1$). These representations depend on the number of voters and make use of the IAC assumption, which stipulates that every possible "voting situation" is equally likely to occur. The representations for PER and NPER are based on characterizations of voting situations giving rise to monotonicity failure that can be found in previous works (Lepelley et al., 1996; Miller, 2012). The analysis of BER is completely new. Using the same probabilistic assumption as in Section 2, Section 3 studies the case where the number of voters tends to infinity; we obtain some representations giving the desired

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[2] See on this point Fishburn (1982). An example of voting rule which does not belong to SER and fails to satisfy monotonicity is the procedure associated with the name of Dodgson (Lewis Carroll); see Felsenthal and Tideman (2013), (2014).

[3] The IC assumption considers that each voter chooses **independently** and with the same probability (1/6) one of the six possible rankings on the three candidates. By contrast, the IAC assumption introduces some degree of dependence in voters' preferences (see Gehrlein, 2006, for more on these two assumptions).

probabilities as a function of parameter λ and allowing the determination of the optimal value of λ. We consider in Section 4 what happens when preferences are assumed to be single-peaked. We continue in Sections 5 by considering some distinctions in the types of circumstances in which failures of monotonicity can occur; these distinctions have been introduced and analyzed in a recent paper by Felsenthal and Tideman (2013). Section 6 is devoted to the impact of election closeness on the likelihood of monotonicity failure and Section 7 summarizes our results.

2. Representations for Monotonicity Paradoxes under three Scoring Elimination Rules

We consider elections with a set of n voters (or individuals) and a set of three candidates (or alternatives), $A = \{a, b, c\}$. We assume that each voter has preference over the alternatives given by one of the six possible strict rankings R_j $(1 \leq j \leq 6)$ defined in the introductory example. We suppose that voters' preferences are accumulated anonymously into groups with common preference rankings, thus we consider *voting situations* (or simply *situations*) represented by six-tuples of the form $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ such that $n_j \geq 0$ $(1 \leq j \leq 6)$, $\sum_{j=1}^{6} n_j = n$, and where n_j $(1 \leq j \leq 6)$ denotes the number of voters with preference ranking R_j. Let $D(n)$ be the set of voting situations with n voters and let D be the set of all voting situations (with any number of voters). A voting rule (or a voting system) is a mapping F from D to X. In this paper, we are interested only in the class of voting rules introduced in the previous section, i.e. the class of Scoring Elimination Rules for three-candidate elections. For a real number $\lambda \in [0, 1]$, we denote by F_λ the SER using the point-system $(1, \lambda, 0)$. For a candidate $w \in A$ and a voting situation $x \in D$, we denote by $S_\lambda(w, x)$ the score obtained by w in the first round, when voters' preferences are described by x and the point-system $(1, \lambda, 0)$ is applied.

Consider x and y in $D(n)$ and w in A. We say that y is *an improvement* of the status of w from x if w is ranked higher in y by some voters, all else unchanged. We say that y is a *deterioration* of the status of w from x if w is ranked lower in y by some voters, all else unchanged. Vulnerability to More-is-Less Paradox (MLP) and Less-is-More Paradox (LMP) can now be formulated as follows. A voting system F is vulnerable to (or exhibits) MLP for situation x if there exists an improvement y of the status of $F(x)$ from x such that $F(y) \neq F(x)$. Similarly, F is vulnerable to (or exhibits) LMP at situation x if there exists a candidate w, $w \neq F(x)$, and a deterioration y of the status of w from x such that $F(y) = w$.

For a given monotonicity paradox M (MLP or LMP) and a voting system F, we define the *vulnerability* of F to M as the probability, $Pr(M, F, n)$, that a situation in $D(n)$ gives rise to M under F. Under the Impartial Anonymous Culture assumption, $Pr(M, F, n)$ is the proportion of voting situations in which F is vulnerable to M: $Pr(M, F, n) = |D(M, F, n)| / |D(n)|$, where $D(M, F, n)$ is the set of situations in $D(n)$ for which F is vulnerable to M. Note that the set $D(MLP, F, n)$ is the disjoint union of the six subsets $D(MLP,F,n)_{\nearrow(w,w')}$ $(w,w' \in A, w \neq w')$ where $D(MLP,F,n)_{\nearrow(w,w')}$ consists of all situations x satisfying the two following conditions: $F(x) = w$ and there exists an improvement y of the status of w from x such that $F(y) = w'$. Similarly, $D(LMP,F,n)$ is the disjoint union of the six subsets $D(LMP,F,n)_{\searrow(w,w')}$ $(w,w' \in A, w \neq w')$ where $D(LMP,F,n)_{\searrow(w,w')}$ consists of all situations x satisfying the two conditions: $F(x) = w$ and there exists a deterioration y of the status of w' from x such that $F(y) = w'$.

We also introduce a global measure for the vulnerability of F to monotonicity paradoxes, denoted by $Pr(GMP,F,n)$ and defined as the probability that a situation gives rise to MLP or LMP under F. If we denote by $Pr(MLP+LMP,F,n)$ the probability that a situation exhibits both MLP and LMP[4], then:

$$Pr(GMP,F,n) = Pr(MLP,F,n) + Pr(LMP,F,n) - Pr(MLP+LMP,F,n) \quad (2.1)$$

It is also easy to see that, by symmetry arguments, we obtain:

$$Pr(MLP,F,n) = 6\left|D(MLP,F,n)_{\nearrow(a,c)}\right|/|D(n)| \quad (2.2)$$

$$Pr(LMP,F,n) = 6\left|D(LMP,F,n)_{\searrow(a,b)}\right|/|D(n)| \quad (2.3)$$

Finally, we can write $Pr(MLP+LMP,F,n)$ in the same way:

$$Pr(MLP+LMP,F,n) = 6\left|D(MLP,F,n)_{\nearrow(a,c)} \cap D(LMP,F,n)_{\searrow(a,b)}\right|/|D(n)| \quad (2.4)$$

Lepelley et al. (1996) provided analytical expressions for $Pr(MLP,F,n)$ and $Pr(LMP,F,n)$ for $F = F_0$ (PER) and $F = F_1$ (NPER). The starting point of our study is to complement their results by extending these representations to the case $F = F_{0.5}$ (BER) and by computing the global vulnerability to monotonicity paradoxes for each of the three classical SER's.

The first step in such calculations is to characterize the situations belonging respectively to $D(MLP,F,n)_{\nearrow(a,c)}$ and $D(LMP,F,n)_{\searrow(,,b)}$ for each M and F under consideration. The characterizations of these sets for F_0 and F_1 are given in Lepelley et al. (1996). The following proposition provides characterizations of the situations belonging to $D(MLP,F_{0.5},n)$ and to $D(LMP,F_{0.5},n)$. Note that, as in Lepelley et al. (1996), in order to simplify our calculations, we ignore the problem of tied elections: we assume that one and only one alternative is eliminated in the first stage as well as in the second (this assumption alters the results only for small values of n).

[4] Miller (2012) refers to this kind of situation as "double monotonicity failure". In a recent paper, Felsenthal and Tideman (2014) have shown that "all prominent voting methods that are vulnerable to monotonicity failure can also display double monotonicity failure".

Proposition 2.1. Let $x = (n_1, n_2, n_3, n_4, n_5, n_6)$ be a voting situation.

1) The situation x belongs to $D(MLP, F_{0.5}, n)_{\nearrow(a,c)}$ if and only if:

$$\begin{cases} -2n_1 - n_2 - n_3 + n_4 + n_5 + 2n_6 < 0 \\ -n_1 + n_2 - 2n_3 + n_4 + 2n_5 + n_6 < 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 < 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 < 0 \\ n_1 - n_2 + n_3 + n_4 - n_5 - n_6 < 0 \end{cases}$$

2) The situation x belongs to $D(LMP, F_{0.5}, n\,)_{\searrow(a,b)}$ if and only if:

$$\begin{cases} -2n_1 - n_2 - n_3 + n_4 + n_5 + 2n_6 < 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 < 0 \\ n_1 + n_2 + n_3 - n_4 - n_6 < 0 \\ n_1 + n_2 + n_3 - n_4 - n_5 - n_6 < 0 \\ 3(n_1 + n_2 - n_6) + n_3 - n_4 + n_5 < 0 \end{cases}$$

Proof. See appendix.

The second step of calculation is now to count the exact number of integer solutions for each of the two systems given by the previous proposition. Note that all (in)equalities in these systems are linear and have integer coefficients on the variables n_j $(1 \leq j \leq 6)$ and on the parameter n. We know from Lepelley et al. (2008) and Wilson and Pritchard (2008) that there is a well-established mathematical theory and efficient algorithms to calculate the number of integer solutions of such systems. Indeed, by Ehrhart's theorem (Ehrhart, 1962), this number is a quasi-polynomial in n, i.e. a polynomial expression $f(n)$ of the form $f(n) = \sum_{k=0}^{d} c_k\,(n)n^k$, where d is the degree of $f(n)$ and where the coefficients $c_k(n)$ are rational periodic numbers in n. A rational periodic number of period q on the integer variable n is a function $u: \mathbb{Z} \to \mathbb{Q}$ such that $u(n) = u(n')$ whenever $n \equiv n'$ (mod q). Each coefficient $c_k(n)$ can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial $f(n)$ and defined as the least common multiple of the periods of all coefficients. To calculate the quasi-polynomials associated with the systems of Proposition 2.1, we use the program proposed by Verdoolaege et al. (2005). This program is based on Barvinok's algorithm (1994), which is known to be one of the most powerful tool that guarantees the polynomial-time counting of integer points inside rational polytopes (for fixed dimension).

Result 2.2. (**BER**) For $n \equiv 1\ [12]$ (i.e. n = 13, 25, 37…), we have:

$$Pr(MLP, F_{0.5}, n) = \frac{(n-1)(53n^4+188n^3-1482n^2+9388n-139475)}{1728(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(LMP, F_{0.5}, n) = \frac{(n-1)(n-7)(3n^2-6n-109)}{144(n+1)(n+2)(n+3)(n+4)},$$

$$Pr(MLP + LMP, F_{0.5}, n) = \frac{(n-1)(n-13)(13n^3-163n^2-1801n+8575}{1728(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP, F_{0.5}, n) == \frac{(n-1)(19n^3-n^2-1051n+889)}{432(n+1)(n+2)(n+3)(n+4)}$$

The proof of this result is immediate. Using Barvinok's algorithm, we calculated the quasi polynomials describing the numbers $\left|D(MLP,F,n)_{\nearrow(a,c)}\right|$ et $\left|D(LMP,F,n)_{\searrow(a,b)}\right|$ as functions of n. The number $|D(n)|$ is known and given by $|D(n)| = \binom{n+5}{n}$ for $n \geq 1$; it then suffices to apply formulas (2.2) and (2.3) to obtain the analytical expressions for $Pr(MLP,F,n)$ et $Pr(LMP,F,n)$. To calculate $Pr(GMP,F_{0.5},n)$, we first calculated $Pr(MLP+LMP,F,n)$ and then we applied formula (2.1). The calculation of $Pr(MLP+LMP,F,n)$ is done in three steps: (i) characterization of the situations belonging to $D(MLP,F,n)_{\nearrow(a,b)} \cap D(LMP,F,n)_{\searrow(a,,c)}$ that are simply the situations that jointly satisfy the two systems of Proposition 2.1, (ii) use of Barvinok's algorithm to obtain the quasi-polynomial giving the expression of $\left|D(MLP,F,n)_{\nearrow(a,c)} \cap D(LMP,F,n)_{\searrow(a,b)}\right|$ and (iii) application of formula (2.4).

Note that the obtained quasi-polynomials are of degree 5 and period 12. For simplicity, we have only exhibited here the expression of these quasi-polynomials for integers n that are congruent to 1 modulo 12. However, complete formulas for the probabilities calculated in this proposition for any congruence modulo 12 are available and can be provided on request from the authors. The same remark is true for the two following propositions where we offer similar results for PER and NPER (recall that the results obtained by Lepelley et al. (1996) deal only with the vulnerability to MLP and LMP and only for integers n multiple of 12, i.e. for a congruence 0 modulo 12).

Result 2.3 (PER) For $n \equiv 1\ [12]$ (i.e. *n* = 13, 25, 37…), we have:

$$Pr(MLP,F_0,n) = \frac{(n-1)(n-13)(52n^3+713n^2+2566n+2525)}{1152(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(LMP,F_0,n) = \frac{(n-1)(n+11)(17n^3-45n^2+147n+745)}{864(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(MLP+LMP,PER,n) = \frac{(n-1)(n+11)(n-13)(17n^2+56n-25}{2304(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP,F_0,n) = \frac{(n-1)(397n^4+1292n^3-35298n^2-142228n-142115)}{6912(n+1)(n+2)(n+3)(n+4)(n+5)}$$

Result 2.4 (NPER) For $n \equiv 1\ [12]$ (i.e. *n* = 13, 25, 37…), we have:

$$Pr(MLP,F_1,n) = \frac{(n-1)(4n^3-11n^2+24n+439)}{72(n+1)(n+2)(n+3)(n+4)},$$

$$Pr(LMP,F_1,n) = \frac{(n-1)(7n^3+27n^2-3n-463)}{108(n+1)(n+2)(n+3)(n+4)},$$

$$Pr(MLP+LMP,F_1,n) = \frac{5(n-1)(n-13)(2n^3-15n^2-228n-623)}{2592(n+1)(n+2)(n+3)(n+4)(n+5)},$$

$$Pr(GMP,NPER,n) = \frac{(n-1)(302n^4+2017n^3+2217n^2-3053n-17035)}{2592(n+1)(n+2)(n+3)(n+4)(n+5)}$$

The following Tables display some values of $Pr(M,F_\lambda,n)$ for $M \in \{MLP, LMP, MLP+LMP, GMP\}$, $\lambda \in \{0,\frac{1}{2},1\}$ and *n* = 13 (Table 1), *n* = 109 (Table 2), *n* = ∞ (Table 3).

	MLP	LMP	MLP+LMP	GMP
F_0	0	5/476 = 1.05%	0	5/476 =1.05%
$F_{.5}$	4/357 = 1.12%	1/357 = 0.28%	0	5/ 357= 1.40%
F_1	8/357 = 2.24%	9/238 = 3.78%	0	43/714=6.02%

Table 1: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{.5}$), $n = 13$

	MLP	LMP	MLP+LMP	GMP
F_0	475593/12233606 = 3.89%	63981/3495316 = 1.83%	78021/12233606 = .64%	16143/317756 =5.08%
$F_{.5}$	411/15029 = 2.73%	5559/321937 = 1.73%	30648/6116803 = .50%	12750/312937 = 3.96%
F_1	15789/321937 = 4.90%	3555/58534 = 6.07%	16572/6116803 = .27%	187119/1747658=10.71%

Table 2: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{.5}$), $n = 109$

	MLP	LMP	MLP+LMP	GMP
F_0	13/288 = 4.51%	17/864 = 1.97%	17/2304 = .74%	397/6912 =5.74%
$F_{.5}$	53/1728 = 3.07%	3/144 = 2.08%	13/1728 = .75%	19/432 = 4.40%
F_1	1/18 = 5.56%	7/108 = 6.48%	5/1296 = .39%	151/1296=11.65%

Table 3: Vulnerability for PER (F_0), NPER (F_1) and BER ($F_{.5}$), $n = \infty$

The computed values show that, for the three rules under consideration, the vulnerability to monotonicity paradoxes increases with the number of voters and, with the exception of double monotonicity paradox, this vulnerability reaches values that cannot be considered as negligible. Coombs rule (F_1) clearly exhibits the poorest performance for almost each type of monotonicity failure[5] and each value of n, with a GMP probability close to 12%. when n tends to infinity. BER dominates PER for MLP and GMP but these two rules perform similarly for LMP and MLP+LMP.

3. Limiting Representations for all Scoring Elimination Rules

We suppose in this section that the number n of voters tends to infinity and we derive some general representations for the vulnerability of SER's to monotonicity paradoxes as functions of λ. These representations will allow us to identify, for each form of the paradox, the SER that minimizes the probability monotonicity failure. All the representations in this section (and in the remaining of the paper) are based on the IAC assumption.

In the following, a voting situation $(n_1, n_2, n_3, n_4, n_5, n_6)$ in $D(n)$ will be represented by the 6-tuple $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ where $x_j = n_j/n$ denotes the proportion of individuals with preference R_j. We assume that the number of individuals n is sufficiently large (tends to infinity) and we consider as a voting situation any 6-tuple $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ of non-negative real numbers that sum to 1. We denote by S the set of all such voting situations. We

[5] A noticeable exception if double monotonicity paradox, for which the vulnerability of NPER if lower than the vulnerability of both PER and BER.

keep the same definitions and the same notations as in the previous section, but we adapt them to this new context where $D(n)$ and D are replaced by S. Voting rules are defined as mappings F from S to X; scores $S_\lambda(w,x)$ and SERs F_λ are defined in a similar way as previously. The notions of improvement, deterioration and vulnerability to MLP and LMP are also unchanged. We will just slightly modify some notations and replace $Pr(M,F,n)$, $D(M,F,n)$ (where M is MLP, LMP, GMP or MLP+LMP), $D(MLP,F,n)_{\nearrow(w,w')}$ $D(LMP,F,n)_{\searrow(w,w')}$ respectively with $Pr(M,F,\infty)$, $S(M,F)$, $S(MLP,F)_{\nearrow(w,w')}$ and $S(LMP,F)_{\searrow(w,w')}$.

3.1. Characterization of MLP situations for F_λ

Given a SER F_λ, we seek a characterization of the voting situations for which F_λ is vulnerable to MLP. Let us consider a situation $x=(x_1,x_2,\dots,x_6)$ in $S(MLP,F_\lambda)_{\nearrow(a,,c)}$. By definition of $S(MLP,F_\lambda)_{\nearrow(a,,c)}$ we know that $F_\lambda(x)=a$, thus:

$$S_\lambda(a,x)>S_\lambda(c,x),\ S_\lambda(b,x)>S_\lambda(c,x) \text{ and } \ aMAJb \text{ in } x, \qquad (3.1)$$

where $aMAJb$ means that a majority of voters prefer a to b. We also know that there is an improvement y of the status of a from x, such that $F(y)=c$. Thus:

$$S_\lambda(a,y)>S_\lambda(b,y),\ S_\lambda(c,y)>S_\lambda(b,y) \text{ and } \ cMAJa \text{ in } y \qquad (3.2)$$

Let $m_{j,k}$ stand for the proportion of individuals with preference R_j who move up candidate a over k candidates ($k=1\,,2$). Since changes from x to y must improve the status of a while being as much as possible at the expense of b (b must be eliminated in the first round) and as little as possible at the expense of c (c must go to the second round and win against a), we can write y as follows:

$$y=(x_1+x_3+m_{4,2},x_2,0,x_4-m_{4,2},x_5+x_6,0)$$

In fact, the progress of a, the elimination of b in the first round and the victory of c against a are only possible under the following conditions:

1. All voters with ranking R_1 or R_2 keep their preferences (no improvement is possible for a). This means that $m_{j,1}=m_{j,2}=0$ for $j=1$ or 2.

2. All voters of type $R_3=bac$ change their preferences to $R_1=abc$. Thus $m_{3,1}=x_3$ and $m_{j,2}=0$.

3. For each voter of type $R_4=bca$, there is two possibilities: move to $R_1=abc$ or maintain R_4. Note that if c can be elected by a passage from R_4 to R_3, he (she) will be also elected by moving from R_4 to R_1. We can therefore take $m_{4,1}=0$ and carefully choose $m_{4,2}$ (with $0\le m_{4,2}\le x_4$).

4. All voters with ranking $R_5=cab$ keep their preferences. Thus $m_{5,1}=m_{5,2}=0$.

5. All voters of type $R_6 = cba$ change their preferences to $R_5 = cab$. Thus $m_{6,1} = x_6$ and $m_{6,2}$=0.

Before continuing, we emphasize that any improvement in favor of a from x is not necessarily of the above form. However, it is not difficult to see that F_λ exhibits MLP at x if and only if there is an improvement of the above form whereby c is elected. The remaining work is therefore to determine a necessary and sufficient condition on $m_{4,2}$ for the election of c when voters' preferences are described by y.

It is immediate that $F_\lambda(y) = c$ if and only if

$$(\lambda - 1)x_1 - x_2 + (\lambda - 1)x_3 + x_4 - \lambda x_5 - \lambda x_6 + (\lambda - 2)m_{4,2} < 0 \quad (3.3)$$

$$\lambda x_1 - \lambda x_2 + \lambda x_3 + (1 - \lambda)x_4 - x_5 - x_6 + (2\lambda - 1)m_{4,2} < 0 \quad (3.4)$$

$$x_1 + x_2 + x_3 - x_4 - x_5 - x_6 + 2m_{4,2} < 0 \quad (3.5)$$

$$0 \leq m_{4,2} \leq x_4 \quad (3.6)$$

Thus we only need to rule out $m_{4,2}$. To do this, we first collect all bounds of $m_{4,2}$ from (3.3)-(3.6). Thereafter, the required characterization is obtained by making sure that those bounds (of $m_{4,2}$) are such that each lower bound is less than each upper bound. In fact, given two real numbers r and s, there exists a real number t such that $t > r$ and $t < s$ if and only if $r < t$. Due to (3.4), we achieve this by distinguishing three cases: $0 \leq \lambda < \frac{1}{2}$, $\lambda = \frac{1}{2}$ and $\frac{1}{2} < \lambda \leq 1$.
After eliminating redundant constraints, we obtain the following:

Proposition 3.1 *A voting situation x belongs to $S(MLP, F_\lambda)_{\nearrow(a,c)}$ if and only if the following conditions are satisfied:*
For $\lambda \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
&-x_1 + (\lambda - 1)x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda)x_5 + x_6 < 0\\
&-\lambda x_1 + \lambda x_2 - x_3 + (\lambda - 1)x_4 + x_5 + (1 - \lambda)x_6 < 0\\
&-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 < 0\\
&x_1 + x_2 + x_3 - x_4 - x_5 - x_6 < 0\\
&x_1 + (1 - 4\lambda)x_2 + x_3 + x_4 + (2\lambda - 3)x_5 + (2\lambda - 3)x_6 < 0
\end{aligned}$$

For $\lambda \in \left[\frac{1}{2}, 1\right]$,

$$\begin{aligned}
&-x_1 + (\lambda - 1)x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda)x_5 + x_6 < 0\\
&-\lambda x_1 + \lambda x_2 - x_3 + (\lambda - 1)x_4 + x_5 + (1 - \lambda)x_6 < 0\\
&-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 < 0\\
&x_1 + x_2 + x_3 - x_4 - x_5 - x_6 < 0\\
&\lambda x_1 - \lambda x_2 + \lambda x_3 + (1 - \lambda)x_4 - x_5 - x_6 < 0
\end{aligned}$$

3.2. Vulnerability to MLP paradox

Recall that S is the unit simplex formed by the set of all voting situations and that $S(MLP, F_\lambda)_{\nearrow(a,,c)}$ is the polytope formed by all voting situations satisfying the conditions of Proposition 3.1. We know that S is of dimension 5, since the components of each voting situation in S sum to 1. Denote by $vol(E)$ the 5-volume of a subset E of S. Similarly to formula 2.2, the probability that F_λ is vulnerable to MLP is given by:

$$Pr(MLP, F_\lambda, \infty) = \frac{6vol(S(MLP, F_\lambda)_{\searrow(a,c)})}{vol(S)} = 720 vol\big(S(MLP, F_\lambda)_{\nearrow(a,c)}\big)$$

Thus, the computation of $vol\big(S(MLP, F_\lambda)_{\nearrow(a,c)}\big)$ provides the probability that F_λ exhibits MLP under the IAC assumption. In this paper, all volumes are computed using a triangulation method derived from the well known Cohen and Hickey algorithm of triangulating a polytope (Cohen and Hickey, 1979). Let P be a given d-dimensional polytope described by some non redundant linear inequalities $E_j : c_j y \leq b_j;\ j = 1,2,\ldots,m$. Each facet F_j of P corresponds to at most one equation $c_j y = bj$ with $j = 1,2,\ldots,m$. Each vertex can then be attached to the subset of facets it belongs to. Choosing a vertex, said v_1, a dissection of P is obtained by considering all pyramids with apex v_1 and bases F_j such that v_1 is out of F_j. This operation is then applied recursively to find a triangulation of P into simplices, each containing $d+1$ points that are affine independent. Finally the volume of P is the sum of the volumes of simplices obtained in its triangulation using the following formula of the d-dimensional volume of a simplex $\Delta(a_0, a_1, \ldots, a_d)$:

$$vol(\Delta(a_0, a_1, \ldots, a_d)) = \frac{|\det(a_1 - a_0, a_2 - a_0, \ldots a_d - a_0)|}{d!} vol_0$$

where each a_j is a vertex of $\Delta(a_0, a_1, \ldots, a_d)$ and the notation *det* stands for the determinant and vol_0 is a constant that depends on the cartesian coordinate system used for vertices. Since each probability in this paper is a ratio, one can simply set $vol_0 = 1$. For illustrations see Gehrlein et al. (2014).

We obtain:

Result 3.2 For $\in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned} Pr(MLP, F_\lambda, \infty) &= \frac{1152\lambda^{15} - 8232\lambda^{14} + 34154\lambda^{13} - 140310\lambda^{12} + 364162\lambda^{11} - 192564\lambda^{10} - 1235457\lambda^9 + 2958975\lambda^8}{1728(2+\lambda)(1+\lambda)(1-\lambda)^3(2-\lambda)^4(2-3\lambda)^2(2+\lambda-4\lambda^2)} \\ &+ \frac{-2347767\lambda^7 - 422931\lambda^6 + 2009862\lambda^5 - 1213672\lambda^4 - 1032\lambda^3 + 310960\lambda^2 - 137280\lambda + 19968}{1728(2+\lambda)(1+\lambda)(1-\lambda)^3(2-\lambda)^4(2-3\lambda)^2(2+\lambda-4\lambda^2)}. \end{aligned}$$

For $\in \left[\frac{1}{2}, 1\right]$,

$$\begin{aligned} Pr(MLP, F_\lambda, \infty) &= \frac{640\lambda^{13} - 10976\lambda^{12} + 73691\lambda^{11} - 270204\lambda^{10} + 619225\lambda^9 - 949717\lambda^8 + 1006822\lambda^7 - 740377\lambda^6}{432\lambda^3(2-\lambda)^3(5\lambda-2)(4\lambda^2-3\lambda+2)(2\lambda^2-4\lambda+3)} \\ &+ \frac{363979\lambda^5 - 102657\lambda^4 + 3872\lambda^3 + 8470\lambda^2 - 2864\lambda + 312}{432\lambda^3(2-\lambda)^3(5\lambda-2)(4\lambda^2-3\lambda+2)(2\lambda^2-4\lambda+3)}. \end{aligned}$$

3.3. Characterization of LMP situations for F_λ

We now seek to characterize the voting situations for which F_λ is vulnerable to LMP. Let us consider a situation $x = (x_1, x_2, \dots, x_6)$ in $S(LMP, F_\lambda)_{\searrow(a,b)}$. By definition of $S(LMP, F_\lambda)_{\searrow(a,b)}$, candidate c must have the lowest score with x. Indeed, candidate a cannot have the lowest score with x since $F_\lambda(x) = a$, and candidate b cannot have the lowest score with x since she (he) must win after a deterioration y of her (his) status. Note also that $aMAJb$ in x (a and b reach the second round and a wins) and $aMAJb$ in y (the deterioration of the status of b does not change the outcome of the majority duel). Therefore, the only possibility, for b to be the winner with y, is the elimination of a in the first round and the victory against c in the second round. Thus, we must have:

$$S_\lambda(a,x) > S_\lambda(c,x), \qquad S_\lambda(b,x) > S_\lambda(c,x) \text{ and } \ aMAJb \text{ in } x$$

$$S_\lambda(b,y) > S_\lambda(a,y), \qquad S_\lambda(c,y) > S_\lambda(a,y) \text{ and } \ bMAJc \text{ in } y$$

Let $m_{j,k}$ stand for the proportion of individuals with preference R_j who lower the ranking of candidate a by k places ($k = 1\,,2$). The deterioration of the status of b, the elimination of a in the first round and the victory of b against c are only possible under the following conditions:

1. All voters with ranking $R_1 = abc$ keep their preferences or change their preferences to $R_2 = acb$. Hence we take $m_{1,2} = 0$ and carefully choose $m_{1,1}$ (with $0 \leq m_{1,1} \leq x_1$).
2. All voters with ranking R_2 or R_5 keep their preferences (no deterioration is possible for b). This means that $m_{j,1} = m_{j,2} = 0$ pour $j = 2$ ou 5.
3. For each voter of type $R_3 = bac$, there is two possibilities: move to $R_2 = acb$ or maintain R_3. Note that if b can be elected by a passage from R_3 to R_1, he (she) will be also elected by keeping R_3. We can therefore take $m_{3,1} = 0$ and carefully choose $m_{3,2}$ (with $0 \leq m_{3,2} \leq x_3$).
4. For each voter of type $R_4 = bca$, there is two possibilities: move to $R_6 = cba$ or maintain R_4. Note that if b can be elected by a passage from R_4 to $R_5 = cab$, he (she) will be also elected by moving from to R_6. We can therefore take $m_{4,2} = 0$ and carefully choose $m_{4,1}$ (with $0 \leq m_{4,1} \leq x_4$).
5. All voters with ranking $R_6 = cba$ keep their preferences. Thus $m_{6,1} = m_{6,2} = 0$.

So we have a deterioration y (of the status of b) from x that takes the form:

$$y = \left(x_1 - m_{1,1}, x_2 + m_{1,1} + m_{3,2}, x_3 - m_{3,2}, x_4 - m_{4,1}, x_5, x_6 + m_{4,1}\right)$$

Thus the rule F_λ exhibits LMP at x if and only if there is a deterioration y (of the status of b) from x that takes the above form whereby b is elected. The remaining work is therefore to determine a necessary and sufficient condition on $m_{1,1}$, $m_{3,2}$ and $m_{4,1}$ for the election of allowing the election of b when voters' preferences are described by y.

It is immediate that $\mathrm{F}_\lambda(y) = b$ if and only if

$$(1-\lambda)x_1 + x_2 + (1-\lambda)x_3 - x_4 + \lambda x_5 + \lambda x_6 + \lambda m_{1,1} + (2-\lambda)m_{3,2} + (1-\lambda)m_{4,1} < 0$$

$$x_1 + (1-\lambda)x_2 + \lambda x_3 - \lambda x_4 + (\lambda-1)x_5 - x_6 - \lambda m_{1,1} + (1-2\lambda)m_{3,2} + (\lambda-1)m_{4,1} < 0$$

$$-x_1 + x_2 - x_3 - x_4 + x_5 + x_6 + 2m_{1,1} + 2m_{3,2} + 2m_{4,1} < 0$$

$$0 \leq m_{1,1} \leq x_1, \qquad 0 \leq m_{3,2} \leq x_3 \text{ and } 0 \leq m_{4,1} \leq x_4$$

As in the previous section, ruling out step by step each of the three parameters $m_{1,1}$, $m_{3,2}$, $m_{4,1}$ together with redundant constraints leads to the following proposition:

Proposition 3.3 *A voting situation* x *belongs to* $S(LMP, F_\lambda)_{\searrow(a,,b)}$ *if and only if the following conditions are satisfied:*

1. For $\lambda \in \left[0, \frac{1}{2}\right]$,

$$-x_1 + (\lambda-1)x_2 - \lambda x_3 + \lambda x_4 + (1-\lambda)x_5 + x_6 < 0$$
$$-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 < 0$$
$$(2-\lambda)(x_1 + x_2) - (1-2\lambda)(x_3 + x_5) - (1+\lambda)(x_4 + x_6) < 0$$
$$(2-\lambda)(x_1 + x_2 - x_4 - x_6) + \lambda x_3 - (2-3\lambda)x_5 < 0$$
$$(1+\lambda)(x_1 - x_4 - x_6) + (1-\lambda)(3x_2 - x_5) - (1-3\lambda)x_3 < 0$$

2. For $\lambda \in \left[\frac{1}{2}, 1\right]$,

$$-x_1 + (\lambda-1)x_2 - \lambda x_3 + \lambda x_4 + (1-\lambda)x_5 + x_6 < 0$$
$$-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 < 0$$
$$(2-\lambda)(x_1 + x_2) - (1-2\lambda)(x_3 + x_5) - (1+\lambda)(x_4 + x_6) < 0$$
$$(2-\lambda)(x_1 + x_2 - x_6) + \lambda x_3 - 3\lambda x_4 - (2-3\lambda)x_5 < 0$$

3.4. Vulnerability to LMP paradox

Let $S(LMP, F_\lambda)_{\searrow(a,,b}$ be the polytope formed by all voting situations satisfying the conditions of Propositions 2. The probability that F_λ is vulnerable to LMP is given by:

$$Pr(LMP, F_\lambda, \infty) = \frac{6vol(S(LMP, F_\lambda)_{\searrow(a,,b)})}{vol(S)} = 720vol\big(S(LMP, F_\lambda)_{\searrow(a,,b}\big)$$

Thus, the computation of $vol\big(S(LMP, F_\lambda)_{\searrow(a,,b)}\big)$ provides the probability that F_λ exhibits LMP.

Result 3.4 For $\lambda \in \left[0, \frac{1}{2}\right]$,

$$Pr(LMP, F_\lambda, \infty) = \frac{64\lambda^7 + 128\lambda^6 - 92\lambda^5 - 148\lambda^4 - 652\lambda^3 + 1411\lambda^2 - 911\lambda + 204}{5184(2-\lambda)(1-\lambda)^4}.$$

For $\lambda \in \left[\frac{1}{2}, 1\right]$,

$$Pr(LMP, F_\lambda, \infty) = \frac{24\lambda^7 - 180\lambda^6 + 450\lambda^5 - 658\lambda^4 + 748\lambda^3 - 363\lambda^2 + 62\lambda + 1}{1296(2-\lambda)\lambda^3}.$$

3.5. Global vulnerability

Now that the respective vulnerabilities to MLP and LMP are known, it remains to assess the global vulnerability to monotonicity paradoxes. Analogously to the formula 2.1, we have

$$Pr(GMP, F_\lambda, \infty) = Pr(MLP, F_\lambda, \infty) + Pr(LMP, F_\lambda, \infty) - Pr(MLP + LMP, F_\lambda, \infty)$$

So we just need to compute $Pr(MLP + LMP, F_\lambda, \infty)$ which is simply given by the formula:

$$Pr(MLP + LMP, F_\lambda, \infty) = 720vol\left(S(MLP, F_\lambda)_{\nearrow(a,c)} \cap S(LMP, F_\lambda)_{\searrow(a,,b)}\right)$$

Where $S(MLP, F_\lambda)_{\nearrow(a,c)} \cap S(LMP, F_\lambda)_{\searrow(a,,b}$ is the set of voting situations that jointly satisfy the conditions of propositions 3.1 and 3.2. After calculating the volume of this set, we get:

Result 3.5 For $\lambda \in \left[0, \frac{1}{2}\right]$,

$$\Pr(MLP + LMP, F_\lambda, \infty) = \frac{21924\lambda^{17} - 190404\lambda^{16} + 574470\lambda^{15} - 137734\lambda^{14} - 3612736\lambda^{13} + 10010609\lambda^{12} - 9414159\lambda^{11} - 9253752\lambda^{10} + 39905950\lambda^{9}}{5184(2+\lambda)(1+\lambda)(1-\lambda)^4(2-\lambda)^4(2-3\lambda)^3(2-5\lambda+2\lambda^2+2\lambda^3)} + \frac{-58786139\lambda^8 + 50542143\lambda^7 - 25453976\lambda^6 + 4420724\lambda^5 + 3523456\lambda^4 - 3107872\lambda^3 + 1168928\lambda^2 - 231040\lambda + 19584}{5184(2+\lambda)(1+\lambda)(1-\lambda)^4(2-\lambda)^4(2-3\lambda)^3(2-5\lambda+2\lambda^2+2\lambda^3)}.$$

For $\lambda \in \left[\frac{1}{2}, 1\right]$,

$$\Pr(MLP + LMP, F_\lambda, \infty) = \frac{97678\lambda^{15} - 943184\lambda^{14} + 4303838\lambda^{13} - 12343296\lambda^{12} + 24959502\lambda^{11} - 37820113\lambda^{10} + 44503368\lambda^{9}}{2592\lambda^3(2-\lambda)^3(2-5\lambda)(2-3\lambda+4\lambda^2)(3-4\lambda+2\lambda^2)(1-2\lambda+3\lambda^2)^2} + \frac{-41522415\lambda^8 + 31040508\lambda^7 - 18630170\lambda^6 + 8920492\lambda^5 - 3352717\lambda^4 + 959169\lambda^3 - 197544\lambda^2 + 26204\lambda - 1680}{2592\lambda^3(2-\lambda)^3(2-5\lambda)(2-3\lambda+4\lambda^2)(3-4\lambda+2\lambda^2)(1-2\lambda+3\lambda^2)^2}.$$

We can therefore deduce the global vulnerability to monotonicity paradoxes:

Result 3.6 For $\lambda \in \left[0, \frac{1}{2}\right]$,

$$Pr(GMP, F_\lambda, \infty) = \frac{6912\lambda^{19} - 36288\lambda^{18} + 9216\lambda^{17} - 340448\lambda^{16} + 3419456\lambda^{15} - 4955750\lambda^{14} - 19346650\lambda^{13}}{5184(2+\lambda)(1+\lambda)(1-\lambda)^4(2-\lambda)^4(2-3\lambda)^3(2-5\lambda+2\lambda^2+2\lambda^3)} + \frac{66083799\lambda^{12} - 39105246\lambda^{11} - 117106696\lambda^{10} + 232985335\lambda^{9} - 121302386\lambda^{8} - 98661200\lambda^{7} + 175869928\lambda^{6}}{5184(2+\lambda)(1+\lambda)(1-\lambda)^4(2-\lambda)^4(2-3\lambda)^3(2-5\lambda+2\lambda^2+2\lambda^3)} + \frac{-89966904\lambda^5 - 3185872\lambda^4 + 26547904\lambda^3 - 13852928\lambda^2 + 3243392\lambda - 304896}{5184(2+\lambda)(1+\lambda)(1-\lambda)^3(2-\lambda)^4(2-3\lambda)^3(4\lambda^2-\lambda-2)(2-5\lambda+2\lambda^2+2\lambda^3)}.$$

For $\in \left[\frac{1}{2}, 1\right]$,

$$Pr(GMP, F_\lambda, \infty) = \frac{432\lambda^{13} - 4680\lambda^{12} + 16644\lambda^{11} - 16364\lambda^{10} - 34424\lambda^{9} + 109559\lambda^{8} - 132472\lambda^{7} + 92421\lambda^{6} - 36452\lambda^{5}}{2592\lambda^3(2-\lambda)^3(1-2\lambda+3\lambda^2)^2} + \frac{5392\lambda^4 + 2196\lambda^3 - 1279\lambda^2 + 243\lambda - 8}{2592\lambda^3(2-\lambda)^3(1-2\lambda+3\lambda^2)^2}.$$

Observe that we recover the results of Lepelley et al. (1996) for PER ($\lambda = 0$) and NPER ($\lambda = 1$) as well as the results of the previous section for BER ($\lambda = 1/2$). Furthermore, our results allow us, for each type of paradox, to identify the less vulnerable SER:

1. For MLP, the minimum is at $\lambda = 0.4451681$ with a frequency of 0.03040869
2. For LMP, the minimum is at $\lambda = 0.3669500$ with a frequency of 0.01806508
3. For MLP+LMP, the minimum is at $\lambda = 1$ with a frequency of 0.00385802
4. For GMP, the minimum at $\lambda = 0.41877523$ with a frequency of 0.04171907.

$Pr(\text{MLP}, F_\lambda, \infty)$, $Pr(\text{LMP}, F_\lambda, \infty)$, $Pr(\text{MLP+LMP}, F_\lambda, \infty)$ and $Pr(\text{GMP}, F_\lambda, \infty)$ are plotted in Figures 1, 2, 3 and 4.

These figures show that the vulnerability of BER is very close to the optimal value for MLP (we obtain 0.0307 for $= 1/2$) and, to a lesser extent, for GMP. Moreover, it turns out that NPER maximizes the probability of MLP, LMP and GMP.

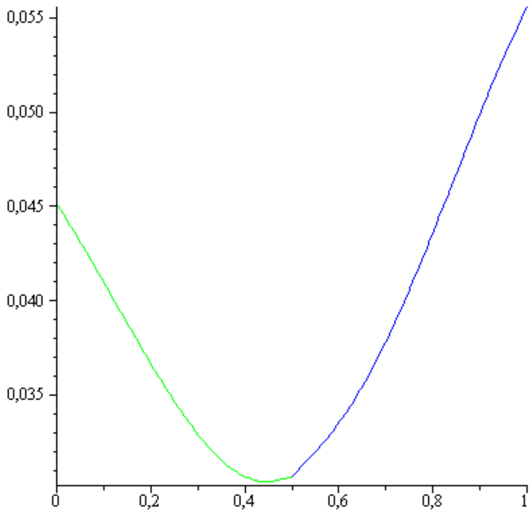

Figure 1. MLP

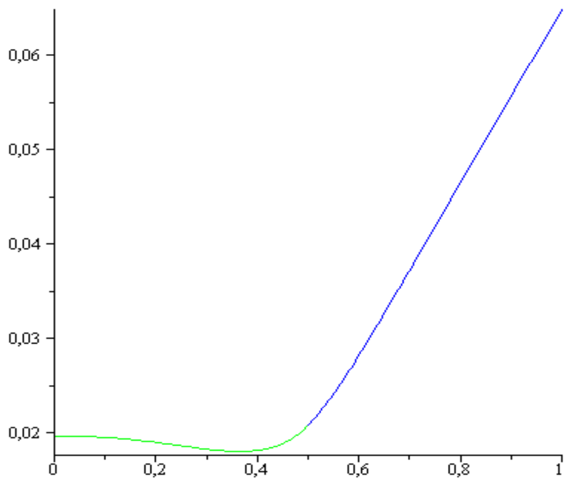

Figure 2. LMP

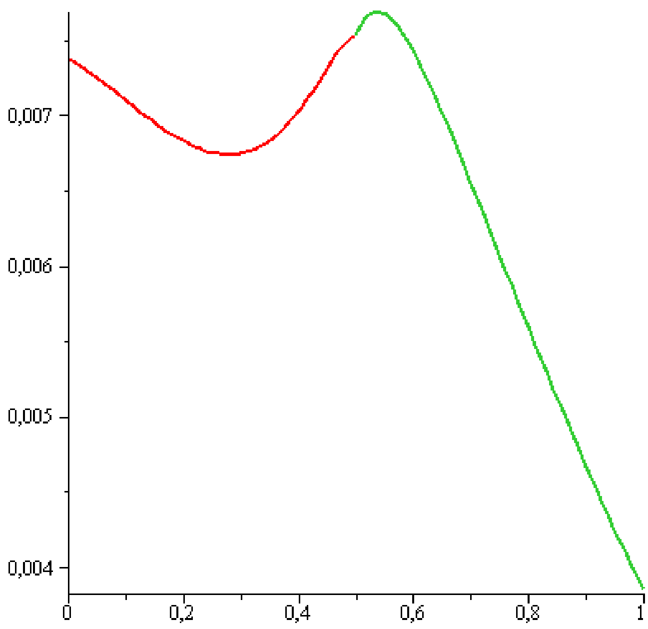

Figure 3. MLP+LMP

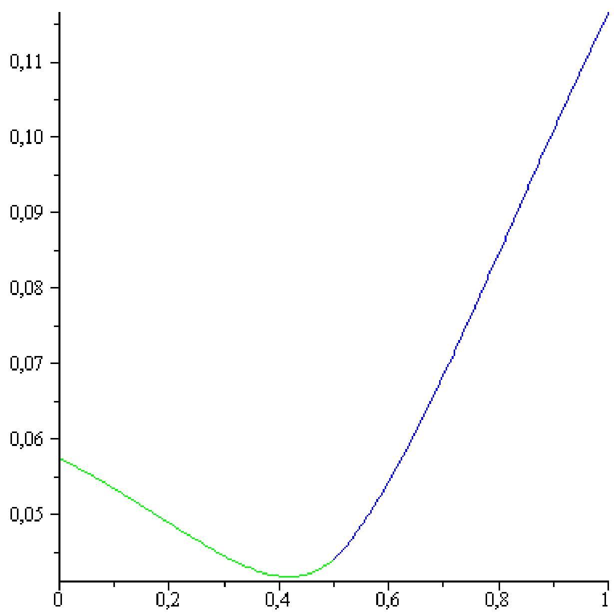

Figure 4. GMP

4. Monotonicity Failure under Single-peakedness

In some political or economic contexts, some preference rankings appear to be very unlikely. One common way to take this observation into account is to assume that preferences are *single-peaked*. When preferences are single-peaked and three candidates are in contention, every voter agrees to consider that (at least) one of the candidates is not the worst. Under this assumption, the number of possible preference rankings is reduced from six to four.

4.1. More-is-Less Paradox under single-peakedness: [MLP+SP]

As shown in previous sections, scoring elimination rule F_λ behave differently depending on whether the weight λ is less than or greater than $\frac{1}{2}$. For three-candidate elections, we are going to show that F_λ exhibits MLP under single-peakedness only for $\lambda < \frac{1}{2}$. This generalizes earlier results by Lepelley et al. (1996) showing that, under single-peakedness, F_0 exhibits MLP while F_1 does not. Without loss of generality, we assume that individual preferences are single-peaked with respect to ideological axis abc (a is the leftist candidate, b the centrist candidate and c the rightist candidate). This amounts to assuming that only abc, bac, bca and cba are admissible while acb and cab are not. Analytically, only voting situations of the form $x = (x_1, 0, x_3, x_4, 0, x_6)$ are considered ($x_2 = x_5 = 0$), with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$ and $x_1 + x_3 + x_4 + x_6 = 1$, and using IAC in this context is tantamount to consider that every single-peaked situations is equally likely to occur.

Proposition 4.1 [MLP+SP] *Assume that individual preferences are single-peaked. Then*

1. F_λ *exhibits MLP at a voting situation x only if the center-alternative b is not qualified for the second round but wins after the winner at x is moved up by some voters;*

2. *MLP never occurs under F_λ for $\lambda \geq \frac{1}{2}$.*

Proof. Assume that individual preferences are single-peaked with respect to the axis abc.

1. Suppose that b is qualified for the second round and for example is opposed to a. There are two possible cases: (i) a defeats b. Then $x_1 > x_3 + x_4 + x_6$ and therefore a is the Condorcet winner. Then a up still wins after being moving up by some voters. (ii) b defeats . Suppose b loses against c after being moving up by some voters. Recalling that individual preferences are single-peaked with respect to abc, it follows that $x_1 > x_3 + x_4 + x_1$. But c receives less points than a. Thus is $x_6 \leq x_6 + \lambda x_4 < x_1 + \lambda x_3 \leq x_3 + x_4 + x_1$ which yields a contradiction. Thus b still wins. In both cases, MLP does not occur. Clearly, F_λ exhibits MLP only if b is not qualified for the second round.

Now assume that b is not qualified for the second round and for example that a wins against c in the runoff. Since c is still losing against a in a majority voting, then MLP occurs only when a is losing against b after being moved up by some voters.

2. Assume that $\lambda \geq \frac{1}{2}$ and suppose that F_λ exhibits MLP at a voting situation x. As shown above, b is ruled out at the first round. That is $x_1 + \lambda x_3 > x_3 + x_4 + \lambda x_1 + \lambda x_6$ and $x_6 + \lambda x_4 > x_3 + x_4 + \lambda x_1 + \lambda x_6$. Or equivalently $(1-\lambda)x_1 > (1-\lambda)x_3 + x_4 + \lambda x_6$ and $(1-\lambda)x_6 > x_3 + \lambda x_1 + (1-\lambda)x_4$. If $\lambda = 1$ then $x_4 + x_6 < 0$ which is contradictory. Now assume that $\lambda < 1$, then $x_1 > x_3 + \frac{1}{1-\lambda}x_4 + \frac{\lambda}{1-\lambda}x_6$ and $x_6 > x_4 + \frac{1}{1-\lambda}x_3 + \frac{\lambda}{1-\lambda}x_1$. Since

$1 > \lambda \geq \frac{1}{2}$ then $\frac{1}{1-\lambda} > 1$ and $\frac{\lambda}{1-\lambda} \geq 1$. Thus $x_1 > x_3 + x_4 + x_6 \geq x_6$ and $x_6 > x_4 + x_3 + x_1 \geq x_1$. Clearly these relations are contradictory. Thus for $\geq \frac{1}{2}$, the center-alternative is always qualified for the runoff and MLP never occurs.■

For $\lambda < \frac{1}{2}$, the inequalities that characterize MLP occurrences are derived from Proposition 3.1. For example when a wins against b but loses against c after being moved up by some voters, corresponding voting situations are characterized by the new set of inequalities obtained from Proposition 3.1 by observing that b and c hold symmetric roles; this amounts to interchanging x_1 and x_2 ; x_3 and x_5; and x_4 and x_6. For the single-peakedness assumption, we set $x_2 = x_5 = 0$. We compute the corresponding 3-dimensional volume to obtain the MLP probability under single-peakedness as follows:

Result 4.2 [MLP+SP]

For $0 \leq \lambda \leq \frac{1}{2}, Pr(MLP + SP, F_\lambda) = \frac{(4\lambda^5 - 25\lambda^3 - 50\lambda^2 + 18\lambda + 20)((1-2\lambda)^3}{72(2+\lambda)(1-\lambda)^2(2-\lambda)^2(2+\lambda-4\lambda^2)}$

For $\lambda \leq \frac{1}{2}, \; Pr(MLP + SP, F_\lambda) = 0$

4.2. Less-is-More Paradox under single-peakedness: [LMP+SP]

As in the case of MLP, LMP still occurs under single-peakedness but for some specific circumstances. Nevertheless a LMP seems to be widespread as all SER F_λ still exhibit LMP except for $\lambda = 0$ and for $\lambda = 0.5$. In fact, this was already known from Lepelley et al. (1996) for $\lambda = 0$. Moreover, under the single-peakedness assumption with three candidates, there always exists a Condorcet winner. Since for $\lambda = 0.5$, F_λ always chooses the Condorcet winner (see Smith, 1973), LMP never occurs under BER. In general, we observe the followings:

Proposition 4.3 [LMP+SP] *Assume that individual preferences are single-peaked. Then*

1. *For* $0 < \lambda < \frac{1}{2}$, F_λ *exhibits LMP at a voting situation only if a (or c) loses in the runoff against b but wins after being moved down by some voters;*

2. *For* $\frac{1}{2} < \lambda < 1$, F_λ *exhibits LMP at a voting situation only if a (or c) wins in the runoff against b but loses after being moved down by some voters.*

Proof. Very similar to the proof of Proposition 4.1 [LMP+SP].

For each possible case of LMP in Proposition 4.3, we derive from Proposition 3.3 the corresponding set of inequalities and set $x_2 = x_5 = 0$ for single-peakedness. The LMP probability under single-peakedness is then giving by:

.

Result 4.4 [LMP+SP]

$For\ 0 \leq \lambda \leq \frac{1}{2}, Pr([LMP + SP], F_\lambda) = \frac{\lambda(1-2\lambda)^3(19+4\lambda)}{432(1-\lambda)^4(2-\lambda)}$

$For\ \frac{1}{2} \leq \lambda \leq 1,\ Pr([LMP + SP], F_\lambda\ ) = \frac{(1-2\lambda)^2(6-\lambda)}{108\lambda}$

4.3. Monotonicity failure under single-peakedness [M+SP]

Due to the existence of a Condorcet winner, there is no possible double monotonicity failure under single-peakedness. We then deduce the probability that F_λ fails to satisfy monotonicity as $Pr(GMP + SP, F_\lambda\ ) = Pr(MLP + SP, F_\lambda) + Pr(LMP + SP, F_\lambda)$, given by:

Result 4.5 [GMP+SP]

$For\ 0 \leq \lambda \leq \frac{1}{2}, Pr(M + SP, F_\lambda) = \frac{(1-2\lambda)^3(8\lambda^6-96\lambda^5-59\lambda^4+176\lambda^3+300\lambda^2-140\lambda-120)}{432(2+\lambda)(4\lambda^2-\lambda-2)(2-\lambda)^2(1-\lambda)^3}$

$For\ \frac{1}{2} \leq \lambda \leq 1,\ Pr(M + SP, F_\lambda) = \frac{(1-2\lambda)^2(6-\lambda)}{108\lambda}$

Observe that a direct consequence of our results is the following:

Corollary 4.6 *When preferences are single-peaked, there is one and only one SER for which Monotonivcity failure never occurs in three-candidate elections, namely the Borda Elimination Rule.*[6]

Table 4 displays some computed values of $Pr(MLP + SP, F_\lambda)$, $Pr(LMP + SP, F_\lambda)$ and $Pr(GMP + SP, F_\lambda)$ in terms of percentage.

λ	$Pr(MLP + SP, F_\lambda)$	$Pr(LMP + SP,\ F_\lambda^*)$	$Pr(GMP + SP, F_\lambda)$
0	1.74	0	1.74
0.1	1.20	0.17	1.36
0.2	0.68	0.21	0.90
0.3	0.28	0.15	0.44
0.4	0.05	0.04	0.09
0.5	0	0	0
0.6	0	0.33	0.33
0.7	0	1.12	1.12
0.8	0	2.17	2.17
0.9	0	3.36	3.36
1	0	4.63	4.63

Table 4. Vulnerability to monotonicity paradoxes with single-peaked preferences

[6] We could give a direct proof of the more general assertion that, when there is a Condorcet winner, there is one and only one SER for which Monotonicity holds, namely BER.

The main conclusions of this section are summarized in Table 5 and illustrated by Figures 4, 5 and 6.

Type	MLP+SP	LMP+SP	GMP+SP
λ_{min}	$[1/2,1]$	$\{0,1/2\}$	½
$Prob_{min}$	0	0	0
λ_{max}	0	1	1
$Prob_{max}$	$\frac{5}{288} \approx 1.74\%$	$\frac{5}{108} \approx 4.63\%$	$\frac{5}{108} \approx 4.63\%$

Table 5. Maximal and minimal vulnerability under single-peakedness

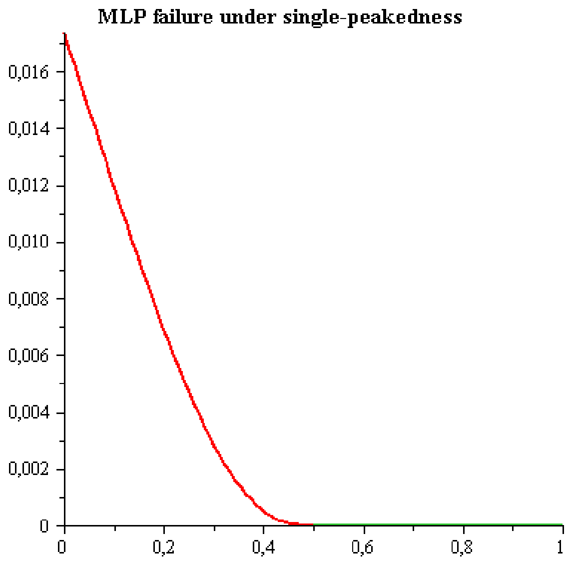

Figure 3.

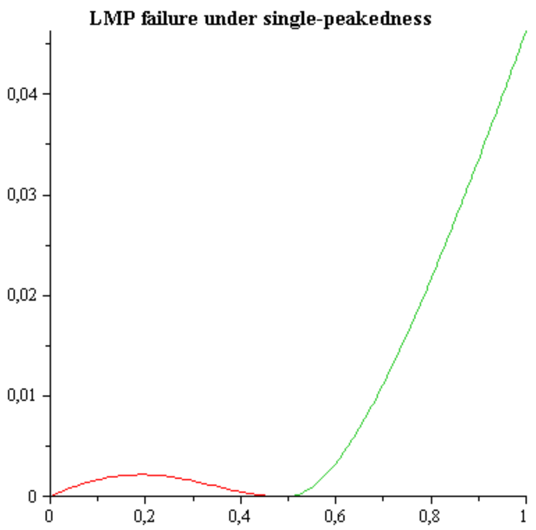

Figure 4.

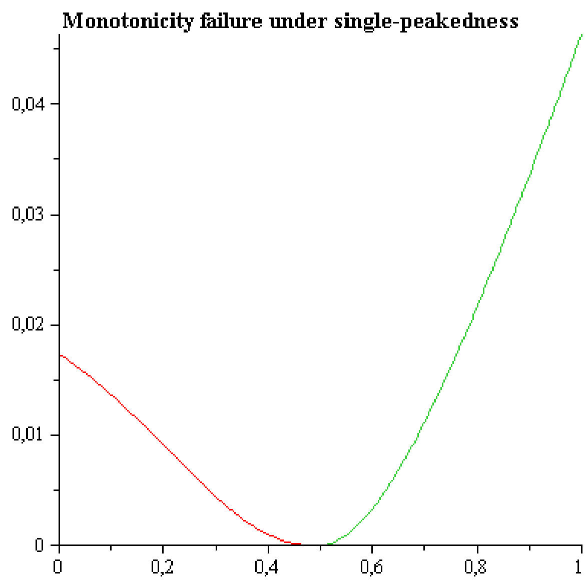

Figure 5.

5. Monotonicity of Scoring Elimination Rules on Specific Subdomains

Following Felsenthal and Tideman (2013), we now examine several sub-types of monotonicity failure by considering alternately only voting situations at which the rule exhibits a monotonicity paradox while the following assumptions successively hold: [CW] a Condorcet winner exists; [CYC] a cycle exists; [B] dynamic voters[7] are better off from changing their rankings; or [W] dynamic voters are worse off from changing their rankings. For the last two cases, the set of constraints should be reconsidered to take into account only changes that benefit (or not) to voters pushing up or down a candidate for MLP or LMP. Possible combinations are also considered. Since [CW] and [CYC] are mutually exclusive and [B] and [W] as well, we only consider [MLP+CW]; [LMP+CW]; [M+CW]; [MLP+B]; [LMP+B]; [M+CW]; [MLP+CW+B]; [LMP+CW+B] and [M+CW+B] where [M] stands for the monotonicity failure.

5.1. Monotonicity failure with [CW]

It is easy to conjecture that a cycle in the majority relation favors monotonicity failures and, indeed, most of the examples of such failures that we can find in the literature contain a majority cycle. *A contrario*, assuming the existence of a Condorcet winner should reduce the vulnerability of SER's to monotonicity paradoxes. We evaluate in this subsection the extent of this reduction.

1) More-is-Less Paradox with [CW]

[7] Felsenthal and Tideman (2013) call *dynamic voters* those voters who change their preference in the voting situation under consideration.

Only voting situations at which the positional elimination rule F_λ exhibits MLP in the presence of a Condorcet winner are considered. Such voting situations are identified by the MLP set of constraints presented in Proposition 3.1 conjointly with the constraints that a Condorcet winner exists. More precisely, when a is the winner against b in the runoff and loses against c after being moved up by some voters, only c can be the Condorcet winner. The probability of [MLP+CW] is given by:

Result 5.1 For $0 \leq \lambda \leq \frac{1}{2}, Pr(MLP + CW, F_\lambda) =$

$$\frac{(1-2\lambda)^3(1296\lambda^{12}+1404\lambda^{11}-10368\lambda^{10}-39093\lambda^9+92536\lambda^8+68065\lambda^7-244480\lambda^6+71028\lambda^5+169312\lambda^4-121504\lambda^3-13552\lambda^2+33216\lambda-7872)}{1728(1+\lambda)(2+\lambda)(1-\lambda)^3(2-\lambda)^2(2-3\lambda)^4(4\lambda^2-\lambda-2)}$$

For $\frac{1}{2} \leq \lambda \leq 1,\ Pr(MLP + CW, F_\lambda) = \frac{(1-2\lambda)^3(2\lambda^2-13\lambda-1)}{432\lambda^3}$

2) Less-is-More Paradox with [CW]

Only voting situations at which the positional elimination rule F_λ exhibits LMP in the presence of a Condorcet winner are considered. Such voting situations are identified by the LMP set of constraints presented in Proposition 3.3 conjointly with the constraints that a Condorcet winner exists. More precisely, when a is the winner against b in the runoff and b wins against c after some voters move down b in their rankings, only a can be the Condorcet winner. The probability of [LMP+CW] is as follows:

Result 5.2 For $0 \leq \lambda \leq \frac{1}{2}, Pr(LMP + CW, F_\lambda) = \frac{(1-2\lambda)^2(48\lambda^5+160\lambda^4+151\lambda^3-123\lambda^2-3216\lambda+204)}{5184(2-\lambda)(1-\lambda)^3(2-3\lambda)}$

For $\frac{1}{2} \leq \lambda \leq 1,\ Pr(MLP + CW, F_\lambda) = \frac{(1-2\lambda)^2(1+5\lambda+54\lambda^2-18\lambda^3+27\lambda^4-6\lambda^5)}{1296\lambda^4}$

3) Monotonicity failure with [CW]

For three-candidate elections, it turns out that no scoring elimination rule simultaneously exhibits MLP and LMP at the same voting situation when a Condorcet winner exists. Given $\lambda \in [0,1]$, assume that F_λ exhibits MLP at a voting situation that admits a Condorcet winner. Without loss of generality, suppose that a is wins against b in the runoff and that a loses against c after some voters move up a in their rankings. Then necessarily c beats a. Thus a is not the Condorcet winner; nor b. Hence c is the Condorcet winner and b is the Condorcet loser. To see that LMP cannot occur, first suppose that some voters move down c in their rankings. Since c initially has the lowest score, c is still ruled out in the first round and cannot win at the new voting situation. Now suppose that some voters move down b in their rankings. Since b is initially losing against a and c, b cannot win at the new voting situation even when b is qualified for the runoff. In both cases, LMP cannot occur[8].

We then deduce that the probability of monotonicity failure with [CW] is the sum $Pr(MLP, F_\lambda, CW) + Pr(LMP, F_\lambda, CW)$ given by:

Result 5.3 For $0 \le \lambda \le \frac{1}{2}$, $Pr(GMP + CW, F_\lambda) =$

$$\frac{(1-2\lambda)^2(2592\lambda^{13}-6264\lambda^{12}-35640\lambda^{11}-119577\lambda^{10}+637043\lambda^9-131110\lambda^8-1661042\lambda^7)}{5184(1+\lambda)(2+\lambda)(1-\lambda)^3(2-\lambda)^2(2-3\lambda)^4(2+\lambda-4\lambda^2)}$$
$$+\frac{(1-2\lambda)^2(1637340\lambda^6+633512\lambda^5-1569488\lambda^4+512368\lambda^3+270864\lambda^2-206592\lambda+36672)}{5184(1+\lambda)(2+\lambda)(1-\lambda)^3(2-\lambda)^2(2-3\lambda)^4(2+\lambda-4\lambda^2)}$$

For $\frac{1}{2} \le \lambda \le 1$, $Pr(GMP + CW, F_\lambda) = \frac{(1-2\lambda)^2(1+2\lambda+21\lambda^2+66\lambda^3+15\lambda^4-6\lambda^5)}{1296\lambda^4}$

Note that $Pr(GMP + CYC, F_\lambda) = Pr(GMP, F_\lambda) - Pr(GMP + CW, F_\lambda)$.

Computed values of $Pr(MLP + CW, F_\lambda)$, $Pr(LMP + CW, F_\lambda)$ and $Pr(GMP + CW, F_\lambda)$ in terms of percentage are shown in Table 6.

λ	$Pr(MLP + CW, F_\lambda)$	$Pr(LMP + CW, F_\lambda)$	$Pr(GMP + CW, F_\lambda)$	$\frac{Pr(GMP + CW, F_\lambda)}{Pr(GMP, F_\lambda)}$
0	1.78	0.98	2.76	48.11
0.1	1.36	0.90	2.26	42.18
0.2	0.90	0.73	1.63	33.39
0.3	0.45	0.49	0.94	21.15
0.4	0.11	0.20	0.31	7.24
0.5	0	0	0	0
0.6	0.07	0.54	0.61	11.17
0.7	0.39	1.56	1.95	28.44
0.8	0.99	2.67	3.66	43.32
0.9	1.80	3.79	5.59	55.35
1	2.78	4.86	7.64	65.56

Table 6. Vulnerability to monotonicity paradoxes when a Condorcet Winner exists

Once again, due to the presence of a Condorcet winner, BER is the only SER that does not give rise to monotonicity failure. Moreover, we observe that the existence of a Condorcet winner divide approximately by 2 the PER (global) vulnerability to monotonicity paradoxes whereas the NPER vulnerability is only divided by 1.5.

[8] This observation stands for every SER in three-candidate elections and generalizes a result given by Miller (2012) for PER and by Felsenthal and Tideman (2014) for NPER and BER.

5.2. Monotonicity with [B]

Suppose that the voters are perfectly informed and that their preferences can be considered as stable. In such a context, the only reasons susceptible to justify changes in individual rankings are of strategic nature. Consequently, when a voting situation can (potentially) give rise to a monotonicity paradox, it is of interest to know whether the dynamic voters are better off or worse off from changing their preference; in the first case (dynamic voters better off), the effective realization of the paradox is considerably more likely than in the second one.

1) More-is-Less Paradox with [B]

Only voting situations at which the positional elimination rule F_λ exhibits MLP while dynamic voters are better off from changing their rankings are considered. Given a voting situation x, suppose without loss of generality that a wins against b in the runoff and loses against c after being moved up by some voters in their rankings. Since these changes should be profitable for their instigators, only voters of type bca, cab or cba are concerned.

Proposition 5.4 [MLP+B-a] *Let x be a voting situation at which a wins against b in the runoff and loses against c after being moved up by some voters in their rankings. Then MLP occurs under F_λ in favor of voters changing their preferences if and only if c wins when all voters of type cba submit cab while an appropriate proportion t of bca type voters submit abc.*

Proof. *Sufficiency.* Suppose c wins when all voters of type cba submit cab while an appropriate proportion t of bca type voters submit abc. Since both types of changes move a up in individual preferences, MLP occurs.

Necessity. Suppose that MLP occurs under F_λ in favor of voters changing their preferences. Then there exists another voting situation y at which c wins after some $R_4 = bca$, $R_5 = cab$ or $R_6 = cba$ type voters move a up. Let $y_{j,k}$ stand for the proportion of R_j voters who move up a over k candidates with $j = 4, 5$ *or* 6 and $k = 1$ *or* 2. Then $y = \left(x_1 + y_{4,2}, x_2 + y_{5,1} + y_{6,2}, x_3 + y_{4,1}, x_4 - y_{4,1} - y_{4,2}, x_5 - y_{5,1} + y_{6,1}, x_6 - y_{6,1} - y_{6,2}\right)$.

We prove that if c wins at y, then c also wins at

$$z = (x_1 + y_{4,2}, x_2, x_3, x_4 - y_{4,2}, x_5 + x_6, 0).$$

For this purpose let

$$\Delta = S_\lambda(y,c) - S_\lambda(y,b) - \left(S_\lambda(z,c) - S_\lambda(z,b)\right) \text{ and } \Delta' = [cMAJ(y)a] - [cMAJ(z)a]$$

where $[cMAJ(y)a]$ is the difference between the proportion of voters who prefer c to a and the proportion of voters who prefer a to c with respect to y; $[cMAJ(z)a]$ is defined in the same way. After basic algebraic simplifications,

$$\begin{aligned}\Delta &= -\mathrm{t}y_{4,1} - ty_{4,2} - (1-t)y_{5,1} + ty_{6,1} - (1-2t)y_{6,2} - tx_6 \\ &\leq -\mathrm{t}(y_{4,1} + y_{4,2} - y_{6,1}) - (1-t)y_{5,1} - (1-2t)y_{6,2} - t(y_{6,1} + y_{6,2}) \text{ since } x_6 \\ &= -\mathrm{t}y_{4,1} - ty_{4,2} - (1-t)y_{5,1} + ty_{6,1} - (1-t)y_{6,2} \leq 0. \\ &\qquad\quad \geq y_{6,1} + y_{6,2}\end{aligned}$$

Since $\Delta \leq 0$ and c wins at y, we deduce that

$$S_\lambda(z,c) - S_\lambda(z,b) \geq S_\lambda(y,c) - S_\lambda(y,b) > 0. \qquad (1)$$

Thus $S_\lambda(z,c) > S_\lambda(z,b)$. Moreover, a wins at x and gains more points from x to z while the score of c does not increase. Thus $S_\lambda(z,a) \geq S_\lambda(x,\mathrm{a}) > S_\lambda(x,\mathrm{c}) \geq S_\lambda(z,\mathrm{c}) > S_\lambda(z,b)$. Therefore b is ruled out at the first stage given z. In the runoff at z, note that

$$\Delta' = -2y_{4,1} - y_{4,2} - 2y_{5,1} - 2y_{6,2} \leq 0.$$

Since $\Delta' \leq 0$ and c beats a at y, we deduce that

$$[cMAJ(z)a] \geq [cMAJ(y)a] > 0.$$

Therefore c also wins at z. Finally from x to z, all voters of type cba submit cab while a proportion $t = y_{4,2}$ of bca type voters submit abc.■

The following result is derived from Proposition 5.4 [MLP+B-a].

Proposition 5.5 [MLP+B] *Let x be a voting situation at which a wins against b in the runoff and loses against c after being moved up by some voters in their rankings. Then MLP occurs under F_λ in favor of voters changing their preferences if and only if*

$$-x_1 + (\lambda-1)x_2 - \lambda x_3 + \lambda x_4 + (1-\lambda)x_5 + x_6 = \mathrm{q}_1 < 0 \qquad \text{(MLP1)}$$
$$-\lambda x_1 + \lambda x_2 - x_3 + (\lambda-1)x_4 + x_5 + (1-\lambda)x_6 = \mathrm{q}_2 < 0 \qquad \text{(MLP2)}$$
$$-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 = \mathrm{q}_3 < 0 \qquad \text{(MLP3)}$$
$$x_1 + x_2 + x_3 - x_4 - x_5 - x_6 = \mathrm{q}_4 < 0 \qquad \text{(MLP4)}$$

and

1. *For* $\lambda \in \left[0,\frac{1}{2}\right]$,
$$x_1 + (1-4\lambda)x_2 + (3-2\lambda)x_3 + x_4 + (2\lambda-3)x_5 + (2\lambda-3)x_6 = \mathrm{q}_5 < 0 \qquad \text{(MLP5)}$$
2. *For* $\lambda \in \left[\frac{1}{2},1\right]$,
$$\lambda x_1 - \lambda x_2 + x_3 + (1-\lambda)x_4 - x_5 - x_6 = q_6 < 0 \qquad \text{(MLP6)}$$

Proof. Let x be a voting situation at which a wins against b in the runoff and loses against c after being moved up by some voters in their rankings. Then by Proposition [MLP+B-a], MLP occurs under F_λ in favor of voters changing their preferences if and only if there exists $\mathrm{t} \in [0, x_4]$ such that

$$S_\lambda(x,a) > S_\lambda(x,\mathrm{c}), S_\lambda(x,a) > S_\lambda(x,c) \text{ and } aMAJ(x)b^9 \qquad \text{(MLP7)}$$

$$S_\lambda(z,a) > S_\lambda(z,c), S_\lambda(z,a) > S_\lambda(z,c) \text{ and } cMAJ(z)a \qquad \text{(MLP8)}$$

with $z = (x_1 + t, x_2, x_3, x_4 - t, x_5 + x_6, 0)$. Note that (MLP7) is equivalent to (MLP1), (MLP2) and (MLP3). Then we only have to prove that given (MLP1), (MLP2) and (MLP3), (MLP8) holds if and only if (MLP4) and (MLP5) hold for $\lambda \in \left[0, \frac{1}{2}\right]$; and that (MLP4) and (MLP6) hold for $\lambda \in \left[\frac{1}{2}, 1\right]$. Clearly, $S_\lambda(z,a) > S_\lambda(z,c)$ is a consequence of $S_\lambda(x,a) > S_\lambda(x,c)$ since some voters move a up from x to z. Therefore given (MLP1), (MLP2) and (MLP3), (MLP8) is now equivalent to $S_\lambda(z,a) > S_\lambda(z,c)$ and $cM(z)a$. That is

$$(2\lambda - 1)\mathrm{t} < -\lambda x_1 + \lambda x_2 - x_3 - (1-\lambda)x_4 + x_5 + x_6 = \mathrm{T}_1$$

$$2\mathrm{t} < -x_1 - x_2 - x_3 + x_4 + x_5 + x_6 = \mathrm{T}_2$$

Taking into consideration the sign of the coefficient $2\lambda - 1$ and the fact that $\mathrm{t} \in]0, x_4]$, it appears that t exists if and only if for $\lambda \in \left[0, \frac{1}{2}\right[$, $\max(\frac{\mathrm{T}_1}{2\lambda-1}, 0) < t < min\,(\frac{\mathrm{T}_2}{2}, x_4)$ and for $\lambda \in \left]\frac{1}{2}, 1\right]$, $0 < t < min\,(\frac{\mathrm{T}_1}{2\lambda-1}, \frac{\mathrm{T}_3}{2}, x_4)$.

Now, for $\lambda \in \left[0, \frac{1}{2}\right[$, there exists t such that $\max(\frac{\mathrm{T}_1}{2\lambda-1}, 0) < t < min\,(\frac{\mathrm{T}_2}{2}, x_4)$ if and only if

$$0 - \frac{\mathrm{T}_2}{2} = \frac{\mathrm{q}_4}{2} < 0 \qquad \text{(MLP9)}$$

$$\frac{\mathrm{T}_1}{2\lambda-1} - \frac{\mathrm{T}_2}{2} = \frac{\mathrm{q}_5}{2(1-2\lambda)} < 0 \qquad \text{(MLP10)}$$

$$\frac{\mathrm{T}_1}{2\lambda-1} - x_4 = \frac{\mathrm{q}_7}{1-2\lambda} < 0 \qquad \text{(MLP11)}$$

$$0 - x_4 < 0 \qquad \text{(MLP12)}$$

Since (MLP9) and (MLP10) are respectively equivalent to (MLP4) and (MLP5), to complete the proof for $\lambda \in \left[0, \frac{1}{2}\right[$, we have to prove that (MLP11) and (MLP12) can be discarded. We omit (MLP12) as it has no influence on the 5-dimensional volume computed with $0 \le x_4$. To see that (MLP12) is redundant, we simply rewrite q_7 as a sum of non positive terms:

$$\mathrm{q}_7 = \frac{1-\lambda}{\lambda^2-\lambda+1}(\mathrm{q}_1 + \mathrm{q}_2) - \frac{\lambda^2+\lambda}{\lambda^2-\lambda+1}x_2 - \frac{\lambda^2+\lambda}{\lambda^2-\lambda+1}x_3 + \frac{5\lambda^2-5\lambda+2}{2(\lambda^2-\lambda+1)}\frac{\mathrm{q}_5}{1-2\lambda} + \frac{3\lambda(1-\lambda)}{\lambda^2-\lambda+1}\frac{\mathrm{q}_6}{2} < 0.$$

Similarly, for $\lambda \in \left]\frac{1}{2}, 1\right]$, there exists t such that $0 < t < min\,(\frac{\mathrm{T}_1}{2\lambda-1}, \frac{\mathrm{T}_2}{2}, x_4)$ if and only if

$$0 - \frac{\mathrm{T}_2}{2} = \frac{\mathrm{q}_4}{2} < 0 \qquad \text{(MLP13)}$$

$$0 - \frac{\mathrm{T}_1}{2\lambda-1} = \frac{\mathrm{q}_6}{2\lambda-1} < 0 \qquad \text{(MLP14)}$$

[9] aMAJ(x)b means that A beats B at x.

$$0 - x_4 < 0 \qquad \text{(MLP15)}$$

As mentioned above, (MLP15) has no influence on the evaluation of the 5-dimensional volume computed for $x_j \geq 0$, *j=1,2,…,6*. Finally (ML13) and (MLP14) are respectively equivalent to (MLP4) and (MLP6).

For $= \frac{1}{2}$, (MLP1), (MLP2), (MLP3) and (MLP4) hold from (MLP7) and (MLP8). Moreover, (MLP5) and (MLP6) are now equivalent.■

From Proposition 5.5, the probability of [MLP+B] is:

Result 5.6 For $0 \leq \lambda \leq \frac{1}{2}, Pr(MLP + B, F_\lambda) =$

$$\frac{192\lambda^{15}-1864\lambda^{14}+6970\lambda^{13}-16218\lambda^{12}+36786\lambda^{11}-46938\lambda^{10}-94200\lambda^{9}+422061\lambda^{8}-516717\lambda^{7}+96282\lambda^{6}+331613\lambda^{5}-291197\lambda^{4}+41232\lambda^{3}+59820\lambda^{2}-33318\lambda+55}{864(1+\lambda)(2+\lambda)(3-\lambda)(1-\lambda)^3(2-\lambda)^3(4\lambda^2-\lambda-2)(3\lambda^2-7\lambda+3)}$$

For
$\frac{1}{2} \leq \lambda \leq 1$,
$$Pr(MLP + B, F_\lambda) = \frac{32\lambda^9-448\lambda^8+2183\lambda^7-5282\lambda^6+7180\lambda^5-5588\lambda^4+2261\lambda^3-301\lambda^2-85\lambda+24}{432\lambda^3(2-\lambda)^2(4\lambda-2\lambda^2-3)}$$

2) Less-is-More Paradox with [B]

Voting situations in consideration here are only those at which the SER F_λ exhibits LMP while dynamic voters are better off from changing their rankings. We proceed as in the case of MLP. Given a voting situation x, suppose without loss of generality that a wins against b in the runoff and that b wins after being moved down by some voters in their rankings. Since these changes should be profitable for their instigators, only voters of type bac, bca or cba are concerned. As with MLP, we identify changes that are necessary and sufficient for this specific type of LMP, that is [LMP+B].

Proposition 5.7 [LMP+B] *Let x be a voting situation such that a wins against b in the runoff and b wins after being moved down by some voters in their rankings. Then MLP occurs under* F_λ *in favor of voters changing their preferences if and only if b wins when some proportion t of bac voters submit acb while some proportion s of bca voters report cba.*

Proof. Very similar to the proof of Proposition 5.2 [MLP+B].

Consequently LMP occurs under the assumption [B] if and only if there exists a proportion $y_{3,2}$ of bac voters submitting acb and a proportion $y_{4,1}$ of bca voters submitting cba in such a way that b now wins at the new voting situation which is then giving by

$$y = \left(x_1, x_2 + y_{3,2}, x_3 - y_{3,2}, x_4 - y_{4,1}, x_5, x_6 + y_{4,1}\right).$$

Necessary and sufficient conditions for [LMP+B] are then derived by looking at the set of constraints on $y_{4,1}$ and $y_{3,2}$ under which b wins at y. Collecting all constraints on $y_{4,1}$, one obtains [LMP+B] existence conditions depending only on the parameter $y_{3,2}$ and variables x_j. The same operation is applied on $y_{3,2}$to obtain the following characterization:

Proposition 5.8 [LMP+B] *Let x be a voting situation at which a wins against b in the runoff and b wins after being moved down by some voters in their rankings. Then LMP occurs under F_λ in favor of voters changing their preferences if and only if*

$$-x_1 + (\lambda - 1)x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda)x_5 + x_6 < 0$$
$$-x_1 - x_2 + x_3 + x_4 - x_5 + x_6 < 0$$

$$(2 - \lambda)(x_1 + x_2) - (1 - 2\lambda)(x_3 + x_4) - (1 + \lambda)(x_4 + x_6) < 0$$

and

1. *For* $\lambda \in \left[0, \frac{1}{2}\right]$,
$$x_1 + (1 - \lambda)x_2 + \lambda x_3 + x_4 + (\lambda - 1)x_5 - (1 + \lambda)x_6 < 0$$
$$(1 + \lambda)(x_1 - x_4 - x_6) + (1 - \lambda)(3x_3 - x_5) - (1 - 3\lambda)x_3 < 0$$

2. *For* $\lambda \in \left[\frac{1}{2}, \frac{2}{3}\right]$,
$$(1 + \lambda)(x_1 - x_4 - x_6) + (1 - \lambda)(3x_2 - x_5) - (1 - 3\lambda)x_3 < 0$$

3. *For* $\lambda \in \left[\frac{2}{3}, 1\right]$,
$$(1 + \lambda)(x_1 - x_4 - x_6) + (1 - \lambda)(3x_2 + 3x_3 - x_5) - (1 - \lambda)x_5 < 0$$
$$(3 - 2\lambda)(x_1 - x_6) + x_2 + x_3 + (1 - 4\lambda)x_4 - (3 - 4\lambda)x_5 < 0$$
$$(16\lambda - 9\lambda^2 - 5)x_1 + (1 - 2\lambda + 3\lambda^2)(x_2 + x_3)$$
$$+ (1 - 2\lambda - 3\lambda^2)(x_4 - x_6) + (1 - 8\lambda + 9\lambda^2)x_5 < 0$$

We derive from Proposition 5.8 that the probability of [LMP+B] is:

Result 5.9 For
$0 \le \lambda \le \frac{1}{2}$, $Pr([LMP + B], F_\lambda) = \frac{16\lambda^8 + 8\lambda^7 - 68\lambda^6 + 16\lambda^5 - 144\lambda^4 + 655\lambda^3 - 864\lambda^2 + 481\lambda - 102}{2592(1-\lambda)^4(\lambda-2)}$

For $\frac{1}{2} \le \lambda \le \frac{2}{3}$, $Pr([LMP + B], F_\lambda) = \frac{(1-\lambda)(8\lambda^8 - 92\lambda^7 + 446\lambda^6 - 732\lambda^5 + 347\lambda^4 - 83\lambda^3 + 13\lambda^2 - 5\lambda + 2)}{1296\lambda^5(2-\lambda)}$

For $\frac{2}{3} \le \lambda \le 1$, $Pr([LMP + B], F_\lambda) =$

$$\frac{56\lambda^{13}-804\lambda^{12}+4786\lambda^{11}-15522\lambda^{10}+30388\lambda^{9}-37701\lambda^{8}+30137\lambda^{7}-15144\lambda^{6}+4388\lambda^{5}-475\lambda^{4}-128\lambda^{3}+69\lambda^{2}-17\lambda+2}{1296\lambda^{5}(2-\lambda)(1-3\lambda+\lambda^{2})}$$

3) Monotonicity failure with [B]

We now focus our attention on voting situation at which the positional elimination rule F_λ^* simultaneously exhibits LMP and MLP – or double monotonicity failure - while dynamic voters are better off from changing their rankings.

Given a voting situation x, suppose without loss of generality that a wins against b in the runoff. Then MLP and LMP jointly occur under assumption [B] if and only if (i) a loses against c after being moved up by some voters who are better off from changing their rankings; and (ii) b wins after being moved down by some voters in their rankings. Such voting situations are identified by the two sets of constraints presented in Proposition 5.2 and Proposition 5.4. The evaluation of the corresponding volume yields the following probability that F_λ^* both LMP and MLP at the same voting situation while all dynamic voters are better off from changing theirs rankings:

Result 5.10 For $0 \leq \lambda \leq \frac{1}{2}, Pr\left(MPL + LMP + B,\ F_\lambda\ \right) =$

$$\frac{13392\lambda^{17}-151218\lambda^{16}+710178\lambda^{15}-1635907\lambda^{14}+981368\lambda^{13}+5246294\lambda^{12}-18571308\lambda^{11}+32814574\lambda^{10}-37581933\lambda^{9}+29118925\lambda^{8}}{1296(2\lambda^{3}+2\lambda^{2}-5\lambda+2)(1+\lambda)(1-\lambda)^{4}(2-\lambda)^{3}(3\lambda^{2}-7\lambda+3)(2-3\lambda)^{2}(4\lambda-3)}$$

$$+\frac{-14401003\lambda^{7}+3048779\lambda^{6}+1455765\lambda^{5}-1604349\lambda^{4}+717238\lambda^{3}-186681\lambda^{2}+27720\lambda-11836}{1296(2\lambda^{3}+2\lambda^{2}-5\lambda+2)(1+\lambda)(1-\lambda)^{4}(2-\lambda)^{3}(3\lambda^{2}-7\lambda+3)(2-3\lambda)^{2}(4\lambda-3)}$$

For $\frac{1}{2} \leq \lambda \leq \frac{2}{3},\ Pr\left(MLP + LMP + B,\ F_\lambda\ \right) = \frac{(1-\lambda)\left(3+38\lambda-219\lambda^{2}+403\lambda^{3}-391\lambda^{4}+201\lambda^{5}-47\lambda^{6}\right)}{1296\lambda^{4}(\lambda-2)(2\lambda^{2}-4\lambda+3)}$

For $\frac{2}{3} \leq \lambda \leq 1,\ Pr\left(MLP + LMP + B,\ F_\lambda\ \right) =$

$$\frac{262145\lambda^{15}-2806405\lambda^{14}+14018622\lambda^{13}-43323481\lambda^{12}+92617090\lambda^{11}-145109259\lambda^{10}+172250955\lambda^{9}-157940181\lambda^{8}+112979163\lambda^{7}}{1296\lambda^{3}(2-\lambda)^{2}(7\lambda^{2}-7\lambda+2)^{2}(5\lambda^{2}-5\lambda+2)(2\lambda^{2}-4\lambda+3)(5\lambda^{2}-10\lambda+4)}$$

$$+\frac{-63187783\lambda^{6}+27477878\lambda^{5}-9151584\lambda^{4}+2267216\lambda^{3}-395792\lambda^{2}+43680\lambda-2304}{1296\lambda^{3}(2-\lambda)^{2}(7\lambda^{2}-7\lambda+2)^{2}(5\lambda^{2}-5\lambda+2)(2\lambda^{2}-4\lambda+3)(5\lambda^{2}-10\lambda+4)}$$

To obtain the probability $Pr\left(GMP + B,\ F_\lambda\ \right)$ that F_λ exhibits a monotonicity paradox while dynamic voters are better off from changing their rankings, note that

$$Pr\left(GMP + B,\ F_\lambda\ \right) = Pr\left(MLP + B,\ F_\lambda\ \right) + Pr\left(LMP + B,\ F_\lambda\ \right) - Pr\left(MLP + MLP + B,\ F_\lambda\ \right).$$

After algebraic simplifications, the result is the following:

Result 5.11 For $0 \le \lambda \le \frac{1}{2}, Pr\left(GMP + B,\ F_\lambda\ \right) =$

$$\frac{13824\lambda^{22}-43776\lambda^{21}-365856\lambda^{20}+1724416\lambda^{19}-229432\lambda^{18}-6007556\lambda^{17}-6224444\lambda^{16}+30150396\lambda^{15}+80042172\lambda^{14}}{2592(3-4\lambda)(2-3\lambda)^2(2\lambda^3+2\lambda^2-5\lambda+2)(2+\lambda)(1-\lambda)^3(3-\lambda)(1+\lambda)(3\lambda^2-7\lambda+3)(2-\lambda)^3(4\lambda^2-\lambda-2)}$$

$$+\frac{-342217609\lambda^{13}+252096276\lambda^{12}+573002396\lambda^{11}-1338479620\lambda^{10}+908077340\lambda^{9}+375040562\lambda^{8}-1105263361\lambda^{7}}{2592(3-4\lambda)(2-3\lambda)^2(2\lambda^3+2\lambda^2-5\lambda+2)(2+\lambda)(1-\lambda)^3(3-\lambda)(1+\lambda)(3\lambda^2-7\lambda+3)(2-\lambda)^3(4\lambda^2-\lambda-2)}$$

$$+\frac{+791048112\lambda^{6}-158072500\lambda^{5}-141402888\lambda^{4}+125915520\lambda^{3}-46522080\lambda^{2}+8823168\lambda-705024}{2592(3-4\lambda)(2-3\lambda)^2(2\lambda^3+2\lambda^2-5\lambda+2)(2+\lambda)(1-\lambda)^3(3-\lambda)(1+\lambda)(3\lambda^2-7\lambda+3)(2-\lambda)^3(4\lambda^2-\lambda-2)}$$

For $\frac{1}{2} \le \lambda \le \frac{2}{3},\ Pr\left(GMP + B,\ F_\lambda\ \right) =$

$$\frac{-8\lambda^{10} + 68\lambda^9 - 162\lambda^8 + 180\lambda^7 - 353\lambda^6 + 549\lambda^5 - 286\lambda^4 + 112\lambda^3 + 42\lambda^2 + 18\lambda - 4}{1296\lambda^5(\lambda-2)^2}$$

For $\frac{2}{3} \le \lambda \le 1,\ Pr\left(GMP + B,\ F_\lambda\ \right) =$

$$\frac{68600\lambda^{22}-1406300\lambda^{21}+12829670\lambda^{20}-69452725\lambda^{19}+251061702\lambda^{18}-647047199\lambda^{17}+1240417731\lambda^{16}-1822285490\lambda^{15}}{1296\lambda^5(2-\lambda)^2(5\lambda^2-10\lambda+4)(5\lambda^2-5\lambda+2)(7\lambda^2-7\lambda+2)^2(\lambda^2-3\lambda+1)^2}$$

$$+\frac{2095718215\lambda^{14}-1913651176\lambda^{13}+1396641104 1\lambda^{12}-811844407\lambda^{11}+367749071\lambda^{10}-121125045\lambda^{9}+21583778\lambda^{8}+4200519\lambda^{7}}{1296\lambda^5(2-\lambda)^2(5\lambda^2-10\lambda+4)(5\lambda^2-5\lambda+2)(7\lambda^2-7\lambda+2)^2(\lambda^2-3\lambda+1)^2}$$

$$+\frac{-5309200\lambda^{6}+2474472\lambda^{5}-770232\lambda^{4}+172176\lambda^{3}-27008\lambda^{2}+2688\lambda-128}{1296\lambda^5(2-\lambda)^2(5\lambda^2-10\lambda+4)(5\lambda^2-5\lambda+2)(7\lambda^2-7\lambda+2)^2(\lambda^2-3\lambda+1)^2}.$$

Some computed values of $Pr([MLP + B], F_\lambda)$, $Pr([LMP + B], F_\lambda, CW)$ and $Pr([GMP + B], F_\lambda)$ are shown in Table 7.

λ	$Pr([MLP + B], F_\lambda)$ %	$Pr([LMP + B], F_\lambda)$ %	$Pr\left(\left[\begin{matrix} MLP + LMP \\ +B \end{matrix}\right], F_\lambda\right)$	$Pr([GMP + B], F_\lambda)$ %	$\frac{Pr([GMP+B],F_\lambda)}{Pr(GMP,F_\lambda)}$ (%)
0	2.21	1.97	0.24	3.94	68.5
0.1	2.01	1.92	0.20	3.72	69.6
0.2	1.80	1.85	0.17	3.48	71.2
0.3	1.65	1.79	0.16	3.28	73.7
0.4	1.60	1.80	0.19	3.22	76.9
0.5	1.74	2.08	0.34	3.47	79.0
0.6	2.24	2.22	0.41	4.04	74.3
0.7	2.90	2.04	0.38	4.56	66.6
0.8	3.72	2.14	0.41	5.46	64.5
0.9	4.63	2.36	0.40	6.59	65.3
1	5.56	2.70	0.38	7.87	67.6

Table 7. Vulnerability to monotonicity paradoxes when dynamic voters are better off

It turns out from Table 7 that dynamic voters are better off (and hence incited to effectively change their preferences) in a large proportion of those situations than can give rise to monotonocity failures. This proportion is maximal at $\lambda = 0.51423028$ with a proportion of 79.19% which is almost the performance of BER.

5.3. Monotonicity failure with [CW+B]

1) More-is-Less Paradox with [CW+B]

Let us recall that under the assumption [MLP+CW+B], only voting situations that simultaneously meet the following requirements are in consideration: (i) the positional elimination rule F_λ exhibits MLP; (ii) there exists a Condorcet winner; and (iii) dynamic voters are better off from changing their rankings. When a is wins against b in the runoff and loses against c after being moved up by some voters, voting situations that satisfy [CW+B] conditions are those that simultaneously satisfy constraints provided in Proposition [MLP] and Proposition [MLP+B] conjointly with the fact that c is the Condorcet winner. From the corresponding volume, the probability of [MLP+CW+B] is given by:

Result 5.12 For $0 \leq \lambda \leq \frac{1}{2}, Pr(MLP + CW + B, F_\lambda) =$

$$\frac{(1-2\lambda)^3(1728\lambda^{13}-6912\lambda^{12}-10980\lambda^{11}+36352\lambda^{10}+123535\lambda^9-270907\lambda^8-170673\lambda^7+636223\lambda^6-203782\lambda^5-398176\lambda^4+296832\lambda^3+25248\lambda^2-77472\lambda+19008)}{1728(3-4\lambda)(1+\lambda)(2+\lambda)(1-\lambda)^3(2-3\lambda)^3(2-\lambda)^2(3-\lambda)(2+\lambda-4\lambda^2)}$$

For $\frac{1}{2} \leq \lambda \leq 1,\ Pr(MLP + CW + B, F_\lambda, CW) = \frac{(1-2\lambda)^3(2\lambda^2-13\lambda-1)}{432\lambda^3}$

2) Less-is-More Paradox with [CW+B]

Under the assumption [LMP+CW+B], only voting situations that simultaneously meet the following requirements are in consideration: (i) the positional elimination rule F_λ exhibits LMP; (ii) there exists a Condorcet winner; and (iii) dynamic voters are better off from changing their rankings. When a wins against b in the runoff and b wins against c in the runoff after being moved down by some voters, voting situations that satisfy [CW+B] conditions are those that simultaneously satisfy constraints provided in Proposition [LMP] and Proposition [LMP+B] conjointly with the fact that a is the Condorcet winner. We compute the corresponding volume to obtain the probability of [LMP+CW+B] given by:

Result 5.13

For $0 \leq \lambda \leq \frac{1}{2}, Pr(LMP + CW + B, F_\lambda) = \frac{(1-2\lambda)^2(24\lambda^6+44\lambda^5-52\lambda^4-85\lambda^3-154\lambda^2+423\lambda-204)}{5184(2-\lambda)(\lambda-1)^3(2-3\lambda)}$

For $\frac{1}{2} \leq \lambda \leq \frac{2}{3}, Pr(LMP + CW + B, F_\lambda) = \frac{(1-\lambda)(1-2\lambda)^2(2\lambda^4-17\lambda^3+56\lambda^2+6\lambda+1)}{1296\lambda^4}$

For $\frac{2}{3} \leq \lambda \leq 1, Pr(LMP + CW + B, F_\lambda) = \frac{(1-2\lambda)^2(1+\lambda+42\lambda^2-84\lambda^3+75\lambda^4-14\lambda^5)}{1296\lambda^4}$

3) Monotonicity failure with [CW+B]

As shown above, for three-candidate elections, no SER simultaneously exhibits MLP and LMP at the same voting situation that admits a Condorcet winner. Thus the probability of monotonicity failure under the assumption [CW+B] is simply the sum

$$Pr(GMP + CW + B, F_\lambda) = Pr(MLP + CW + B, F_\lambda) + Pr(LMP + CW + B, F_\lambda)$$

given by:

Result 5.14 For $0 \leq \lambda \leq \frac{1}{2}$, $Pr(GMP + CW + B, F_\lambda) =$

$$\frac{(1-2\lambda)^2(3456\lambda^{15}+1728\lambda^{14}-84600\lambda^{13}+44732\lambda^{12}+36128\lambda^{11}+461583\lambda^{10}-2424349\lambda^9-118551\lambda^8+6077149\lambda^7-4816800\lambda^6-2750348\lambda^5)}{5184(3-4\lambda)(1-\lambda)^3(2+\lambda)(1+\lambda)(2-3\lambda)^3(3-\lambda)(2-\lambda)^2(4\lambda^2-\lambda-2)}$$

$$+\frac{(1-2\lambda)^2(5084368\lambda^4-1471344\lambda^3-908688\lambda^2+654336\lambda-115776)}{5184(3-4\lambda)(1-\lambda)^3(2+\lambda)(1+\lambda)(2-3\lambda)^3(3-\lambda)(2-\lambda)^2(4\lambda^2-\lambda-2)}$$

For $\frac{1}{2} \leq \lambda \leq \frac{2}{3}$, $Pr(GMP + CW + B, F_\lambda) = \frac{(1-2\lambda)^2(1+2\lambda+17\lambda^2+11\lambda^3+7\lambda^4-2\lambda^5)}{1296\lambda^4}$

For $\frac{2}{3} \leq \lambda \leq 1$, $Pr(GMP + CW + B, F_\lambda) = \frac{(1-2\lambda)^2(1-2\lambda+9\lambda^2+63\lambda^4-14\lambda^5)}{1296\lambda^4}$

Table 8 displays some computed values of $Pr(MLP + CW + B, F_\lambda)$, $Pr(LMP + CW + B, F_\lambda)$ and $Pr(GMP + CW + B, F_\lambda)$ in terms of percentage.

λ	$Pr(MLP + CW + B, F_\lambda)$	$Pr(LMP + CW + B, F_\lambda)$	$Pr(GMP + CW + B, F_\lambda)$	$\frac{Pr(GMP + CW + B, F_\lambda)}{Pr(GMP + CW, F_\lambda)}$	$\frac{Pr(GMP + CW + B, F_\lambda)}{Pr(GMP + B, F_\lambda)}$
0	0.95	0.98	1.93	70.16	49.26
0.1	0.75	0.86	1.61	71.38	43.28
0.2	0.52	0.68	1.20	73.31	34.38
0.3	0.27	0.45	0.72	76.89	22.08
0.4	0.07	0.18	0.25	84.67	7.96
0.5	0	0	0	---	0
0.6	0.07	0.20	0.27	44.90	6.75
0.7	0.39	0.47	0.86	44.26	32.79
0.8	0.99	0.80	1.79	48.84	32.79
0.9	1.80	1.78	2.98	53.30	45.19
1	2.78	1.62	4.40	57.58	55.88

Table 8. Vulnerability to monotonicity paradoxes with a Condorcet Winner and dynamic voters better off

6. Monotonicity Paradox in Three-Alternative Close Elections

Miller (2012) have shown, with the help of simulations, that the frequency of MLP paradox can be very high (up to 50%) under PER when elections are close. This issue is investigated in the present section.

Election closeness is measured by the average number of points (denoted by) obtained by the last ranked candidate, i.e. by his (her) score divided by the number *n* of voters. We suppose that every voting situation with a specified value of α is equally likely to occur (IAC type assumption). We only focus here on PER, BER and NPER and we assume large electorates.

6.1. Plurality Elimination Rule

Under PER, α can be interpreted as the percentage of votes obtained by the last ranked candidate (plurality loser)[10] and elections become closer and closer when α increases and tends to its maximal value 1/3. We aim to compute the probability of monotonicity failures as a function of this parameter α.

a) More-is-Less Paradox

We use here the possibility offered by Barvinok's algorithm of obtaining quasi-polynomials as functions of more than one parameter. Let k be the number of votes obtained by the plurality loser in the first round when PER is implemented (note that $\alpha = \frac{k}{n}$). We are able to obtain representations for $|D(MLP, F_0, n, k)|$ and for $|D(n, k)|$, that give for each value of n and k the number of voting situations giving rise to MLP under PER and the total number of possible voting situations (respectively). We obtain the probability $Pr(MLP, F_0, n, k)$ by dividing $|D(MLP, F_0, n, k)|$ by $|D(n, k)|$. The resulting representation is very complex but we can easily obtain close form relations by considering the limiting case in n: for that purpose, we replace k with αn in $Pr(MLP, F_0, n, k)$, and making n tend to infinity, we have just to consider the coefficient of the leading term in n to obtain the limiting representation $Pr(MLP, F_0, \infty, \alpha)$ that gives the desired probability as a function of α.

The probability of MLP under PER is thus given as:

Result 6.1 $(MLP, F_0, \infty, \alpha) = \frac{\frac{(1-\alpha)(3\alpha-1)(6\alpha^2-6\alpha+1)}{24}}{\frac{\alpha(3\alpha-1)(3\alpha^2-1)}{12}} = \frac{(1-\alpha)(6\alpha^2-6\alpha+1)}{2\alpha(3\alpha^2-1)}$, for $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$,

$$Pr(MLP, F_0, \infty, \alpha) = 0 \text{ , for } 0 \leq \alpha < \frac{1}{4}.$$

b) Less-is-More Paradox

Proceeding as for MLP, we obtain :

Result 6.2 $(LMP, F_0, \infty, \alpha) = \frac{126\alpha^3-174\alpha^2+77\alpha-10}{27\alpha(1-3\alpha^2)}$, for $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$,

$$Pr(LMP, F_0, \infty, \alpha) = \frac{(6\alpha-1)^2(12\alpha^2+4\alpha+1)}{216\alpha(3\alpha-1)(3\alpha^2-1)} \text{ , for } \frac{1}{6} \leq \alpha \leq \frac{1}{4},$$

$$Pr(LMP, F_0, \infty, \alpha) = 0 \text{ , for } 0 \leq \alpha < \frac{1}{6}.$$

c) Monotonicity Paradox (GMP)

The probability of having both MLP and LMP is :

[10] This is one of the closeness measures used by Miller (2012).

Result 6.3 $Pr(MLP + LMP, F_0, \infty, \alpha) = \frac{126\alpha^3 - 174\alpha^2 + 77\alpha - 10}{54\alpha(1-3\alpha^2)} = \frac{1}{2} Pr(LMP, F_0, \alpha)$, for $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$, and $Pr(MLP + LMP, F_0, \infty, \alpha) = 0$, for $0 \leq \alpha < \frac{1}{6}$.

We deduce from the above representations that the probability of Global Monotonicity Paradox is given as:

Result 6.4 $(GMP, F_0, \infty, \alpha) = \frac{288\alpha^3 - 498\alpha^2 + 266\alpha - 37}{54\alpha(1-3\alpha^2)}$, for $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$,

$Pr(GMP, F_0, \infty, \alpha) = \frac{(6\alpha-1)^2(12\alpha^2+4\alpha+1)}{216\alpha(3\alpha-1)(3\alpha^2-1)}$, for $\frac{1}{6} \leq \alpha \leq \frac{1}{4}$, and $Pr(GMP, F_0, \alpha) = 0$, for $0 \leq \alpha < \frac{1}{6}$.

Table 9 displays some computed values of $Pr(MLP, F_0, \infty, \alpha)$, $Pr(LMP, F_0, \infty, \alpha)$ and $Pr(GMP, F_0, \infty, \alpha)$ in percentages.

α	$Pr(MLP, F_0, \alpha)$	$Pr(LMP, F_0, \alpha)$	$Pr(MLP + LMP, F_0, \alpha)$	$Pr(GMP, F_0, \alpha)$
≤1/6	0	0	0	0
.17	0	.005	0	.005
.18	0	.084	0	.084
.19	0	.273	0	.273
.20	0	.600	0	.600
.21	0	1.100	0	1.100
.22	0	1.825	0	1.825
.23	0	2.847	0	2.847
.24	0	4.275	0	4.275
.25	23.077	6.268	3.134	26.211
.26	27.562	8.437	4.219	31.780
.27	31.595	10.279	5.140	36.734
.28	35.236	11.836	5.918	41.154
.29	38.540	13.146	6.573	45.113
.30	41.553	14.240	7.121	48.672
.31	44.316	15.147	7.574	51.889
.32	46.868	15.890	7.945	54.813
1/3	½	1/6=16.667	1/12=8.333	7/12=58.333

Table 9. Vulnerability to monotonicity paradoxes under PER as a function of the closeness parameter (large electorate)

6.2. Borda Elimination Rule

Parameter α is now the average number of points obtained by the last ranked candidate when the candidate score is computed with the Borda Rule. In order to compare the BER results to those obtained with PER, we consider here that each voter gives 2/3 point for a first position, 1/3 for a second position and 0 point for a third and last position. This rescaling allows Parameter α to range from 0 to 1/3.

Using the same approach as for PER, we obtain that the probability of MLP under BER is given as:

Result 6.5 $(MLP, F_{0.5}, \infty, \alpha) = \frac{2(4536\alpha^3 - 4680\alpha^2 + 1572\alpha - 173)}{621\alpha^3 - 459\alpha^2 + 99\alpha - 7}$, for $\frac{11}{36} \leq \alpha \leq \frac{1}{3}$,

$$Pr(MLP, F_{0.5}, \infty, \alpha) = \frac{(18\alpha - 5)^2(324\alpha^2 - 108\alpha - 1)}{27(1 - 3\alpha)(621\alpha^3 - 459\alpha^2 + 99\alpha - 7)}, \text{ for } \frac{5}{18} \leq \alpha \leq \frac{11}{36},$$

$$Pr(MLP, F_{0.5}, \infty, \alpha) = 0, \text{ for } 0 \leq \alpha < \frac{5}{18}.$$

We obtain for LMP:

Result 6.6 $(LMP, F_{0.5}, \infty, \alpha) = \frac{2(1 - 4\alpha)(180\alpha^2 - 96\alpha + 13)}{621\alpha^3 - 459\alpha^2 + 99\alpha - 7}$, for $\frac{5}{18} \leq \alpha \leq \frac{1}{3}$,

$$Pr(LMP, F_{0.5}, \infty, \alpha) = \frac{(6\alpha - 1)^4}{24(3\alpha - 1)(621\alpha^3 - 459\alpha^2 + 99\alpha - 7)}, \text{ for } \frac{2}{9} \leq \alpha \leq \frac{5}{18},$$

$$Pr(LMP, F_{0.5}, \infty, \alpha) = -\frac{(6\alpha - 1)^4}{12(2835\alpha^4 - 1836\alpha^3 + 432\alpha^2 - 48\alpha + 2)}, \text{ for } \frac{1}{6} \leq \alpha \leq \frac{2}{9},$$

$$Pr(LMP, F_{0.5}, \infty, \alpha) = 0, \text{ for } 0 \leq \alpha < \frac{1}{6}.$$

The probability of having both MLP and LMP is :

Result 6.7 $Pr(MLP + LMP, F_{0.5}, \infty, \alpha) = \frac{2(18\alpha - 7)(756\alpha^2 - 450\alpha + 67)}{3(621\alpha^3 - 459\alpha^2 + 99\alpha - 7)} =$ for $\frac{11}{36} \leq \alpha \leq \frac{1}{3}$,

$$Pr(MLP + LMP, F_{0.5}, \infty, \alpha) = \frac{(18\alpha - 5)^3(18\alpha - 7)}{27(1 - 3\alpha)(621\alpha^3 - 459\alpha^2 + 99\alpha - 7)}, \text{ for } \frac{5}{18} \leq \alpha \leq \frac{11}{36},$$

$$Pr(MLP + LMP, F_{0.5}, \infty, \alpha) = 0, \text{ for } 0 \leq \alpha < \frac{5}{18}.$$

Hence:

Result 6.8 $(GMP, F_{0.5}, \infty, \alpha) = -\frac{2(2160\alpha^3 - 1044\alpha^2 + 84\alpha + 11)}{3(621\alpha^3 - 459\alpha^2 + 99\alpha - 7)}$, for $\frac{5}{18} \leq \alpha \leq 1$,

$$Pr(GMP, F_{0.5}, \infty, \alpha) = \Pr(LMP, F_{0.5}, \infty, \alpha), \text{ for } \frac{1}{6} \leq \alpha \leq \frac{5}{18},$$

$$Pr(GMP, F_{0.5}, \infty, \alpha) = 0, \text{ for } 0 \leq \alpha < \frac{1}{2}.$$

Computed values are given in Table 10.

α	$Pr(MLP, F_{0.5}, \alpha)$	$Pr(LMP, F_{0.5}, \alpha)$	$Pr(MLP + LMP, F_{0.5}, \alpha)$	$Pr(GMP, F_{0.5}, \alpha)$
≤1/6	0	0	0	0
.17	0	.000+	0	.000+
.18	0	.001	0	.001
.19	0	.008	0	.008
.20	0	.028	0	.028
.21	0	.073	0	.073
.22	0	.156	0	.156
.23	0	.293	0	.293
.24	0	.510	0	.510
.25	0	.848	0	.848
.26	0	1.356	0	1.356
.27	0	2.149	0	2.149
.28	.132	3.408	.002	3.539
.29	4.005	5.460	.309	9.156
.30	13.633	8.681	2.058	20.257
.31	30.830	13.452	07.475	36.818
.32	56.406	20.212	17.366	59.252
1/3	1	1/3	1/3	1

Table 10. Vulnerability to monotonicity paradoxes under BER as a function of the closeness parameter (large electorate)

6.3. Negative Plurality Elimination

Assuming that each voter gives ½ point for a first position, ½ point for a second position and 0 point for a last position, Parameter α ranges from 0 to 1/3.

The results are the following:

Result 6.9 $(MLP, F_1, \alpha) = -\frac{3(4\alpha-1)^2}{2(6\alpha^2-6\alpha+1)}$, for $\frac{1}{4} \le \alpha \le \frac{1}{3}$,

$$Pr(MLP, F_1, \alpha) = 0 \text{ , for } 0 \le \alpha < \frac{1}{4}.$$

Result 6.10 $(LMP, F_1, \alpha) = \frac{(4\alpha-1)(1008\alpha^3-852\alpha^2+229\alpha-20)}{54(1-2\alpha)(6\alpha^2-6\alpha+1)}$, for $\frac{1}{4} \le \alpha \le \frac{1}{3}$,

$$Pr(LMP, F_1, \alpha) = \frac{(6\alpha-1)^2(48\alpha^2-20\alpha+1)}{432\alpha^3(2\alpha-1)} \text{ , for } \frac{1}{6} \le \alpha \le \frac{1}{4},$$

$$Pr(LMP, F_1, \alpha) = 0 \text{ , for } 0 \le \alpha < \frac{1}{6}.$$

Result 6.11 $Pr(MLP + LMP, F_1, \alpha) = \frac{(4\alpha-1)(288\alpha^2-120\alpha+11)}{108(2\alpha-1)(6\alpha^2-6\alpha+1)}$ for $\frac{7}{24} \leq \alpha \leq \frac{1}{3}$,

$$Pr(MLP + LMP, F_1, \alpha) = \frac{(4\alpha-1)^4}{8(1-2\alpha)(3\alpha-1)(6\alpha^2-6\alpha+1)} \text{ for } \frac{1}{4} \leq \alpha \leq \frac{7}{24},$$

$$Pr(MLP + LMP, F_1, \alpha) = 0 \text{ for } 0 \leq \alpha < \frac{1}{4}.$$

Hence:

Result 6.12 $Pr(GMP, F_1, \alpha) = \frac{8352\alpha^3-7656\alpha^2+2242\alpha-213}{108(1-2\alpha)(6\alpha^2-6\alpha+1)}$ for $\frac{7}{24} \leq \alpha \leq \frac{1}{3}$,

$$Pr(GMP, NPER, \alpha) = \frac{36288\alpha^4-48816\alpha^3+23652\alpha^2-4936\alpha+377}{216(1-2\alpha)(3\alpha-1)(6\alpha^2-6\alpha+1)} \text{ for } \frac{1}{4} \leq \alpha \leq \frac{7}{24},$$

$$Pr(GMP, F_1, \alpha) = Pr(LMP, NPER, \alpha) \text{ for } \frac{1}{6} \leq \alpha \leq \frac{1}{4},$$

$$Pr(GMP, F_1, \alpha) = 0 \text{ for } 0 \leq \alpha < \frac{1}{6}.$$

Computed values are displayed in Table 11.

α	$Pr(MLP, F_1, \alpha)$	$Pr(LMP, F_1, \alpha)$	$Pr(MLP + LMP, F_1, \alpha)$	$Pr(GMP, F_1, \alpha)$
≤1/6	0	0	0	0
.17	0	.029	0	.029
.18	0	.415	0	.415
.19	0	1.139	0	1.139
.20	0	2.083	0	2.083
.21	0	3.156	0	3.156
.22	0	4.281	0	4.281
.23	0	5.397	0	5.397
.24	0	6.454	0	6.454
.25	0	7.407	0	7.407
.26	1.554	8.460	.002	10.013
.27	5.257	9.708	.032	14.936
.28	10.305	11.028	.176	21.167
.29	16.313	12.345	.637	28.021
.30	23.077	13.604	1.638	35.043
.31	30.487	14.752	3.047	42.190
.32	38.482	15.733	4.928	49.286
1/3	½	1/6	1/12	7/12

Table 11. Vulnerability to monotonicity paradoxes under NPER as a function of the closeness parameter (large electorate)

6.4. Comments

As expected, the probability of Monotonicity paradox(es) increases when elections become closer and closer[11] and may reach very high values when the three candidates obtain approximately the same score (7/12 for PER and NPER and 1 for BER!). Of course, such (quasi) tied elections are very rare. However, elections where the last ranked candidate obtains 25% of the total score appears to be more plausible and it is rather worrying to observe that, in such situations, PER exhibits a significant vulnerability to monotonicity failures (more than .26). This observation corroborates Miller's findings about PER (Miller, 2012).

In order to compare PER, BER and NPER, it is important to point out that the frequency distributions of the various values of parameter α are not similar for these three voting rules. For each of them, we have computed the proportion of voting situations having an election closeness measure lower than α, for α in [0,1/3] (to save space, the associated representations are omitted). Thanks to these results, we are able to propose the following Table, that allows a fair comparison between PER, BER and NPER.

Monotonicity Paradox (GMP) Vulnerability	**Percentage of voting situations**		
	PER	BER	NPER
>50%	1,71%	2,00%	1,21%
>20%	13,28%	7,28%	22,29%
>10%	13,28%	11,33%	36,04%
>1%	28,42%	34,00%	78,64%
>0%	53,47%	84,37%	86,42%
Average value[12]	5,74%	4,40%	11,65%

Table 12. Proportion of situations associated to various levels of monotonicity failure

To illustrate, Table 12 indicates that 13,28% of the voting situations exhibits a risk of monotonicity failure higher that 20% when PER is used. On the other hand, PER is, among the three rules under consideration, the one for which the proportion of voting situations with a zero risk is the highest (46,53% for PER, 15,63% for BER and 13,58% for NPER).

7. Conclusion

The most salient conclusions that emerge from our calculations can be summarized as follows:

[11] We note that LMP occurrence implies α > 1/6 (for the three rules under consideration) whereas MLP occurrence implies closer elections: α ≥ 1/4 for PER (in accordance with Miller, 2012), α ≥ 5/18 for BER and α > 1/4 for NPER.

[12] These values come from Section 2.

- Contrary to what intuition could have suggested, BER (λ = 1/2) is not the SER that minimizes the vulnerability to monotonicity paradoxes: minimizing the probability that MLP or LMP occur implies $\lambda \approx .42$. However, BER is optimal when a Condorcet Winner exists (i.e. in 93.75% of the voting situations under the IAC assumption): in these situations, BER is immune to monotonicity failure in three-candidate elections and it is the only SER for which this is true.

- The very poor performance of NPER (or Coombs rule *i.e.* λ = 1) is to be pointed out: in almost each of the "scenarii" we have been considering, NPER is the SER that *maximizes* the probability of monotonicity paradoxes. Notice however that NPER minimizes the likelihood of double monotonicity failure.

- The picture is partly modified when strategic aspects are taken in consideration: in 79% of the voting situations that are susceptible to give rise to monotonicity paradoxes under BER, moving up the winner (or moving down a loser) is beneficial for some voters; for PER and NPER, these percentages are only 68.5% and 67.6%. However, BER remains less vulnerable than PER and NPER even when attention is restricted to situations where some voters are strategically incited to move up the winner or move down a loser in their preferences.

- When three-alternative elections are close, the risk on monotonicity failure is surprisingly high for PER, BER and NPER. This is particularly true for PER: when the score in the first round of the last ranked candidate exceeds 25% (such elections are not so infrequent), the probability of GMP is higher than 32% under PER! This result is in accordance with Miller (2012) and supports his conclusion that "monotonicity failure should not be dismissed as a rare phenomenon".

Finally, it is important to emphasize that all our probability results are dependent on the IAC assumption, that has been considered throughout this paper. We do not claim here that these results give realistic estimates of monotonicity failures in real-world elections. We conjecture however that the hierarchy of the voting rules we obtain is robust. This conjecture is supported by a recent paper by Gehrlein and Plassmann (2014), who compare theoretical probabilities based on the IAC assumption and empirical probabilities obtained from observed and simulated data, regarding the Condorcet Efficiencies[13] of five voting rules; they find that, although theoretical and empirical probabilities are fairly different, the two sets of probabilities lead to the same qualitative conclusions.

[13] A voting rule's Condorcet Efficiency is defined as the conditional probability that the rule will elect the Condorcet winner in an election, given that a Condorcet winner exists.

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Appendix

Proof of Proposition 2.1

1) By definition of $D(MLP, F_{0.5}, n)_{\nearrow(a,c)}$, situation x is characterized by:

$S_{0.5}(a,x) > S_{0.5}(c,x)$, $S_{0.5}(b,x) > S_{0.5}(c,x)$ and $aMAJb$ in x (1)

and $S_{0.5}(a,y) > S_{0.5}(b,y)$, $S_{0.5}(c,y) > S_{0.5}(b,y)$ and $cMAJa$ in y (2)

where $aMAJb$ means that a majority of voters prefer a to b.

In the extreme case, (1) must be transformed into (2) by all the following improvements (for a) :

- All voters of type $R_3 = bac$ change their preferences to $R_1 = abc$,
- All voters of type $R_6 = cba$ change their preferences to $R_5 = cab$.

Note that, with BER, the passage from R_4 to R_1 has no effect on the difference between the scores of b and c.

Hence, in the extreme case, situation y is given by: $y = (n_1 + n_3 +, n_2, 0, n_4, n_5 + n_6, 0)$ and we have: $S_{0.5}(a,y) = S_{0.5}(a,x) + (n_3 + n_6)$, $S_{0.5}(b,y) = S_{0.5}(b,x) - (n_3 + n_6)$ and $S_{0.5}(c,y) = S_{0.5}(c,x)$.

It follows that the first inequality in (1) is redundant since: $S_{0.5}(a,y) > S_{0.5}(a,x) > S_{0.5}(c,x) = S_{0.5}(c,y) > S_{0.5}(b,y)$.

We get the first characterization system by writing the five remaining inequalities.

2) The proof can be obtained as a particular case of the proof of Proposition 3.1 by replacing the x_i's by n_i and by taking $\lambda = 0.5$.