

A Dynamic AutoRegressive Expectile for Time-Invariant Portfolio Protection Strategies[☆]

Benjamin Hamidi^a, Bertrand Maillet^{b,*}, Jean-Luc Prigent^c

^a*Neuflize OBC Investissements (ABN AMRO).*

^b*A.A.Advisors-QCG (ABN AMRO), Variances, Univ. La Reunion and Orleans (CEMOI, LEO/CNRS and LBI).*

^c*Univ. Cergy (THEMA).*

Abstract

“Constant proportion portfolio insurance” (CPPI) is nowadays one of the most popular techniques for portfolio insurance strategies. It simply consists of reallocating the risky part of a portfolio with respect to market conditions, *via* a leverage parameter - called the multiple - guaranteeing a predetermined floor. We propose to introduce a conditional time-varying multiple as an alternative to the standard unconditional CPPI method, directly linked to actual risk management problematics. This *ex ante* approach for the conditional multiple aims to diversify the risk model associated, for example, with the expected shortfall (ES) or extreme risk measure estimations. First, we recall the portfolio insurance principles, and main properties of the CPPI strategy, including the time-invariant portfolio protection (TIPP) strategy, as introduced by Estep and Kritzman (1988). We emphasize the existence of an upper bound on the multiple, for example to hedge against sudden drops in the market. Then, we provide the main properties of the conditional multiples for well-known financial models including the discrete-time portfolio rebalancing case and Lévy processes to describe the risky asset dynamics. For this purpose, we precisely define and evaluate different gap risks, in both conditional and unconditional frameworks. As a by-product, the introduction of discrete or random time portfolio rebalancing allows us to determine and/or estimate the density of durations between rebalancements. Finally, from a more practical and statistical point of view due to trading restrictions, we present the class of Dynamic AutoRegressive Expectile (DARE) models for estimating the conditional multiple. This latter approach provides useful complementary information about the risk and performance associated with probabilistic approaches to the conditional multiple.

Keywords: CPPI, VaR, Expected Shortfall, Expectile, Quantile Regression, Dynamic Quantile Model, Extreme Value.

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*Corresponding author: Pr. Bertrand B. Maillet, University of La Réunion, CEMOI, 15 avenue Rene Cassin - CS 92003 - 97744 Saint-Denis Cedex 9, France. Email: bertrand.maillet@univ-reunion.fr.

Email addresses: benjamin.hamidi@fr.abnamro.com (Benjamin Hamidi), bertrand.maillet@univ-reunion.fr (Bertrand Maillet), jean-luc.prigent@u-cergy.fr (Jean-Luc Prigent)

1. Introduction

A portfolio insurance trading strategy is designed to guarantee a *minimum* level of wealth at a pre-specified time horizon, and to participate in the potential gains of a reference portfolio (see Perold, 1986; Grossman and Villa, 1989; Black and Perold, 1992; Basak, 2002). Using this type of strategy, the investor can reduce her downside risk and may partially participate in market rallies. Amongst the various existing variants of such a general strategy, the main portfolio insurance methods are: 1) the option-based portfolio insurance (OBPI) strategies based on synthetic¹ puts, which have been introduced by Leland and Rubinstein (1976); 2) the constant proportion portfolio insurance (CPPI), introduced by Perold (1986) and Black and Jones (1987).

Such strategies, using insurance properties, are thus rationally preferred by individuals that are especially concerned by extreme losses and completely risk averse for values below the guarantee (defined by the so-called floor). Applications of portfolio insurances are very common in markets and well-known in the literature (see Leland and Rubinstein, 1976; Basak, 2002; Cesari and Cremonini, 2003). The CPPI was introduced by Perold (1986) on fixed income assets (see also Perold and Sharpe, 1988). Black and Jones (1987, 1988) extended this method by using equity-based underlying assets. In this case, the CPPI is invested in various proportions in a risky asset and in a riskless one, in order to keep the unconditional risk exposure constant. CPPI strategies are very popular nowadays: they are commonly used in hedge funds, retail products and life-insurance products. A variation of the CPPI, called time-invariant portfolio protection (TIPP), is also very popular (Estep and Kritzman, 1988) and it can be seen as a simple modification of CPPI: the portfolio (theoretically) cannot decline below a pre-set floor (corresponding to a given percentage of the initial capital which is guaranteed over the investment horizon). The floor is then continuously adjusted to a specified percentage of the highest portfolio value (classically 90% of the *maximum* portfolio value). Both CPPI and TIPP strategies can be seen as trend followers' strategies invested in a risky asset and in cash to respect a certain level of *maximum* acceptable loss or *minimum* acceptable return (MAR), according to a parameter that defines the leverage that can be used: the so-called multiple.

However, the optimality of an investment strategy always depends on the risk profile of the investor. In order to determine the optimal rule, one has to decide which strategy to adopt according, for example, to an expected utility *criterion*. Thus, portfolio insurers can be modelled by utility maximizers where the optimization problem is given under the additional constraint that

¹The term "synthetic" is understood here as a strategy in traded assets that duplicates a put. In a complete financial market model, a perfect hedge exists (*i.e.* a self-financing and duplicating strategy). In incomplete markets, other concepts of hedging must be considered (see *e.g.* Schweizer, 2001, for a review about quadratic hedging and Föllmer and Leukert, 1999, for the quantile hedging approach).

the value of the strategy is above a specified wealth level. In a complete market, proportion portfolio insurance (PPI) can be characterized as expected utility maximizing, when the utility function is a piecewise hyperbolic absolute risk aversion (HARA) type and the guaranteed level increases with the risk-free interest rate (Kingston, 1989). If, furthermore, the risky asset follows a geometric Brownian motion, then the HARA optimal portfolio corresponds to a CPPI portfolio as shown by Black and Perold (1992). In this case, the multiple is equal to the product of an instantaneous Sharpe-type ratio, multiplied by the inverse of the relative risk aversion (see Prigent, 2001). This argument is no longer valid if additional frictions are introduced, such as, for example, trading restrictions².

When the guarantee constraint is (exogenously) introduced, the solution of the expected utility maximization problem is given by the combination of the solution of the unconstrained problem and of a put option written on it (see Bertrand *et al.*, 2001; El Karoui *et al.*, 2005). Obviously, this is in the spirit of the OBPI method. However, the introduction of various sources of market incompleteness, in terms of stochastic volatility and trading restrictions, makes the determination of an optimal investment rule under *minimum* wealth constraints rather difficult (if not impossible). For example, if the payoff of a put (or a call) option is not attainable, the standard OBPI approach is no more viable. A dynamic (not perfect) hedging must be introduced. It explains why the dynamic PPI method has become so popular among practitioners. Additionally, in incomplete markets, hedging strategies depend upon the choice of a decision *criterion*. In this framework, we need precise definitions of such a *criterion*, adapted to control gap risks and thus related to value-at-risk or expected shortfalls (see Föllmer and Leukert, 1999).

The main difficulty of the CPPI strategy is, therefore, to determine the crucial parameter defining the portfolio risk exposure, known as the multiple. As mentioned, for instance, by Balder *et al.* (2009), the introduction of market incompleteness and model risk may just “impede the concept of dynamic portfolio insurance”. Measuring the risk that the value of a CPPI portfolio is below the floor (or the guaranteed amount) is then of practical importance. For example, the introduction of unexpected jumps, trading restrictions or liquidity problems are justifications to take account of the gap risk in the sense that a CPPI strategy may not be adequately adjusted. In other words, a residual risk is hidden in the strategy - which is called a “gap risk”, that results when the value of the CPPI strategy falls below the floor. Banks directly bear the risk of the guaranteed portfolios they sell: at maturity, if the guaranteed floor is not reached, they have to compensate the loss with their own capital. Therefore, the sharpest determination of the multiple

²For works with the completeness assumption, we refer to Leland (1980), Cox and Huang (1989), Brennan and Schwartz (1989), Grossman and Villa (1989), Grossman and Zhou (1993), Basak (1995), Browne (1999), Tepla (2000 and 2001), and Bertrand and Prigent (2005).

is the main actual challenge of the CPPI strategy.

Unconditional multiple determination methods have been developed in the literature: for example, in a discrete-time setting, Balder *et al.* (2009) analyze CPPI effectiveness using quantile conditions when the risky asset follows a geometric Brownian motion (GBM) discretized at deterministic times; the extreme value approach is applied to the CPPI method by Prigent (2001) and Bertrand and Prigent (2002). Cont and Tankov (2009) examine CPPI strategies for exponential Lévy processes. However, these traditional methods do not sufficiently take account of the underlying asset risk changes, according for instance to market conditions exposure. Furthermore, the final result of the strategy, related to the choice of the implicit exposure, is highly dependent of the market performance over the period, as illustrated in figures 1 and 2. In these latter figures, several unconditional TIPP strategies introduced by Estep and Kritzman (1988) (see also Grossman and Zhou, 1993) are applied on the Dow Jones Index and their performances are presented (with constant multiples ranging for $m=3$ to $m=13$) in two distinguished market situations (one bullish market – from March 2009 to May 2010, and one bearish market – from May 2008 to March 2009).

- Please insert Figures 1 and 2 somewhere here -

But let us first grasp our intuition here. If the portfolio manager anticipates that the market will be upwards, he will decide to go for a high multiple to outperform, thus exploiting the up-trend of the bullish market (see Figure 1). On the contrary, if he anticipates a bearish market, he will go for a low multiple to limit the losses linked to a bear market (see Figure 2). Obviously, *ex post*, the chosen high level will be the best one if up-trends in the markets occur during the backtest period (case 1) and, conversely, the winner is the low profile when turbulences and crises happen in the period of the study (case 2).

We propose an alternative to these standard unconditional CPPI (or TIPP) methods, directly linked to a risk management approach. For this purpose, following Ben Ameur and Prigent (2007), Lee *et al.* (2008), Jiang *et al.* (2009) and Hamidi *et al.* (2009a and 2009b), we introduce in the traditional picture the notion of a conditional multiple in a general framework of Lévy processes with patterns in risk dynamics. One of the main reasons to consider such an extension is to allow the investor to better adapt his portfolio strategy to market fluctuations. Suppose, for example, that for the first year of a global management period of five years, the anticipation on the stock index (the reference risky asset in this example) is that there may be sudden significant drops of value. As illustrated in what follows, the multiple must be relatively low (say equal to 4). Even if significant rises occur in the future, the exposition corresponding to a small value of the chosen multiple initially will not provide the opportunity to benefit better from a bullish market. On the contrary, if the initial value of the multiple is relatively high (say equal to 7), sudden significant

drops will imply that the portfolio may become fully monetarized from a given date, meaning usually that at maturity the investor might only recover his initial capital. The possibility of reducing the multiple can potentially prevent such an unfavourable event.

Furthermore, even in an unconditional multiple framework, the guarantee depends on the estimation of the *maximum* potential loss that the risky asset can reach before the portfolio manager is able to rebalance his position. Actually, in continuous time, the PPI strategies provide a value above a floor level unless the price dynamic of the risky asset has too big jumps. In practice, it is mainly caused by liquidity constraints and large price movements. If the potential loss is underestimated, the predefined guarantee of the portfolio is no longer insured.

Both unconditional and conditional PPI methods can be modelled in a setup where the price of the risky asset is described by a continuous-time stochastic process, but, for instance, in which trading is restricted to discrete-time or randomly triggered according to a tolerance to some given market fluctuations. In this setting where the dynamics of the risky reference asset is fully parametrized, the theoretical upper bounds on a conditional time-varying multiple can be determined in various situations, using quantiles or expected shortfalls, to measure gap risks. Using such approaches, we can evaluate the gap risks and compare conditional and unconditional strategies in a PPI framework. We also examine the theoretical gap risk duration distribution, which can be considered as a useful complementary information about the risk associated to probabilistic approaches of the conditional multiple.

Secondly, from a more practical and statistical point of view due to trading restrictions, this paper also provides a general methodology to deal with conditional multiple CPPI (TIPP) strategies. For this purpose, we present and compare different ways for estimating the multiple, conditionally to market evolutions, defined for instance by the expected shortfall. We use in particular the notion of a τ -expectile (following Taylor 2008a and 2008b, who makes explicit the link between an expectile and the expected shortfall). Indeed, taking account of the magnitude of potential losses, the expectile may be a better measure for tail risk than value-at-risk (VaR).³

Additionally, for each τ -expectile, there is a corresponding α -quantile (see Efron, 1991; Jones, 1994; Abdous and Rémillard, 1995; Yao and Tong, 1996). We can then use conditional autoregressive expectile (CARE) models based on quantile regression estimations (see Koenker and Basset, 1978; Mukherjee, 1999; Koenker, 2005) for estimating dynamic conditional quantile models as extensions of CAViaR models (see Engle and Manganelli, 2004). Moreover, since quantile estimates can be linearly combined through a dynamic additive quantile (DAQ) model (see Gouriéroux and Jasiak, 2008), the ES and then the conditional multiple can therefore be expressed as a function of a quantile combination whose associated probabilities can be precisely estimated using a dynamic

³Recall that the τ -expectile is the solution to the minimization of asymmetrically weighted mean squared errors.

autoregressive expectile (DARE) approach. In this article, we thus introduce, develop and apply this original DARE approach for the ES, to propose a new way to model the conditional multiple diversifying the risk model of extreme risk measures (see Hamidi *et al.*, 2014). We illustrate it for both the CPPI strategy and one of its extensions, namely the time-invariant portfolio protection (TIPP) strategy.

The main advantage of the conditional strategy we propose in this article is to adapt itself according to the risk evaluation, and a unique and fixed unconditional multiple does not have to be chosen anymore. Since we control and manage potential losses in a systematic way, this conditional approach is thus coherent with an actual risk management problematic. With the proposed approach, you do not need to choose anymore, once for all, your multiple (and risk exposure) for the whole life of your product, whatever the market conditions you are exposed to. The relevant problematic of our work is indeed to determine how the product designer can fix the multiple on an *ex ante* basis.

The paper is organized as follows. First, we recall portfolio insurance principles, the CPPI method developed in Perold (1986) and Black and Perold (1992) and its extension, the TIPP strategy (section 2). We also recall the existence of upper bounds on the unconditional or conditional multiple. Secondly, for main parametrized financial models, we evaluate different gap risks according to given risk measures. We then determine the gap risk duration distribution for a conditional multiple (section 3), focusing on when the multiple is time-varying (section 4). Thirdly, from a more practical point of view, we propose a new time-varying PPI strategy based on actual risk management evolutions, in particular on dynamic ES estimations, improving the existing unconditional approaches. In this framework, we justify, describe and explain how to estimate the conditional multiple DARE model (section 5). We finally illustrate the DARE time-varying proportion portfolio insurance approach in a TIPP framework, and evaluate its properties and performances on a long stock database (section 6). We compare its performances to those of unconditional CPPI strategies, using various performance measures. The last section of the paper concludes, whilst Appendices are devoted to figures, tables, definitions and *addenda* of complementary results.

2. On Proportion Portfolio Insurance Principles

Portfolio insurance is designed to allow investors to recover, at maturity, a given proportion of their initial capital. One of the standard portfolio insurance methods is the CPPI. This strategy is based on a specific dynamic allocation on a risky asset and a riskless one to guarantee a predetermined floor. The properties of CPPI strategies have been extensively studied in the literature (see

Black and Perold, 1992)⁴. The literature also deals with the effects of jump processes, stochastic volatility models and extreme value approaches on the CPPI method with an unconditional multiple (Prigent, 2001; Bertrand and Prigent, 2002 and 2003; Cont and Tankov, 2009). The management of a cushioned portfolio follows a dynamic strategic portfolio allocation. The floor, denoted by \tilde{F}_t , is the *minimum* value of the portfolio that is acceptable for an investor at any time t during the management period $[0, T]$. Its initial value \tilde{F}_0 , capitalized at the non-risky rate r , is usually equal to a predetermined percentage of the capital deposit (after applying management costs) at the beginning of the management period (*i.e.* $\tilde{F}_0 = pV_0$) where, most of the time, the percentage p is equal to e^{-rT} in order to be sure to recover at least V_0 at maturity T . The value of the covered portfolio, denoted V_t , is invested in risky and non-risky assets, whose prices are respectively denoted by S_t and B_t . The portfolio strategy is determined from the proportion invested in the risky asset. The latter one must be chosen so that the floor is guaranteed at any time, even if the market is downward sloping. For this purpose, consider the cushion, denoted by C_t , and defined as the spread (which can vary across time) between the portfolio value and the value of the guaranteed floor. Therefore, it satisfies, $\forall t \in [0, \dots, T]$:

$$C_t = V_t - \tilde{F}_t.$$

Keeping the portfolio value higher than its floor (the dynamic guarantee) is equivalent to ensuring that the cushion is always positive. For the standard CPPI method, the key assumption is that the position in the risky asset, denoted by e_t , is proportional to the cushion. Thus, we have, $\forall t \in [0, \dots, T]$:

$$e_t = m \times C_t,$$

for a given non-negative parameter m , called the multiple.

Assume for example a discrete-time trading between two rebalancing times t_l and t_{l+1} . Then, according to previous strategy, the portfolio value evolves as follows:

$$\Delta V_{t_{l+1}} = (V_{t_l} - mC_{t_l}) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + mC_{t_l} \frac{\Delta S_{t_{l+1}}}{S_{t_l}},$$

with the notation: $\Delta X_{t_{l+1}} = X_{t_{l+1}} - X_{t_l}$.

Thus, the cushion satisfies:

$$\begin{aligned} \Delta C_{t_{l+1}} &= \Delta V_{t_{l+1}} - \Delta \tilde{F}_{t_{l+1}} = (V_{t_l} - mC_{t_l}) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + mC_{t_l} \frac{\Delta S_{t_{l+1}}}{S_{t_l}} - \Delta \tilde{F}_{t_{l+1}} \\ &= (C_{t_l} + \tilde{F}_{t_l} - mC_{t_l}) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + mC_{t_l} \frac{\Delta S_{t_{l+1}}}{S_{t_l}} - \Delta \tilde{F}_{t_{l+1}}. \end{aligned}$$

⁴Comparisons of OBPI and CPPI are provided in Bookstaber and Langsam (2000), Bertrand and Prigent (2005, 2011), Annaert *et al.* (2009), and Zagst and Kraus (2011).

But, since the floor evolves as the riskless asset, we have: $\Delta F_{t_{l+1}} = F_{t_l} \Delta B_{t_{l+1}}/B_{t_l}$. Therefore, we get:

$$\Delta C_{t_{l+1}} = C_{t_l} \times \left[(1 - m) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + m \frac{\Delta S_{t_{l+1}}}{S_{t_l}} \right],$$

and thus:

$$C_{t_{l+1}} = C_{t_l} \times \left[1 + (1 - m) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + m \frac{\Delta S_{t_{l+1}}}{S_{t_l}} \right]. \quad (1)$$

Equation (1) proves that, if the guarantee holds at time t_l , it will also hold at time t_{l+1} provided that the term $\left[1 + (1 - m) \frac{\Delta B_{t_{l+1}}}{B_{t_l}} + m \frac{\Delta S_{t_{l+1}}}{S_{t_l}} \right]$ is positive. Assuming that $(\Delta S_{t_{l+1}}/S_{t_l} - \Delta B_{t_{l+1}}/B_{t_l})$ may be non positive, we get the following equivalent condition:

$$m \leq \left[\underset{t_{l+1} \in \mathbb{R}^+ \text{ with } \Delta S_{t_{l+1}}/S_{t_l} < \Delta B_{t_{l+1}}/B_{t_l}}{\text{Inf}} \frac{(1 + \Delta B_{t_{l+1}}/B_{t_l})}{(\Delta B_{t_{l+1}}/B_{t_l} - \Delta S_{t_{l+1}}/S_{t_l})} \right].$$

For a very short time period (one or two days), $\Delta B_{t_{l+1}}/B_{t_l}$ is small and negligible. Then, the previous condition essentially corresponds to the positivity of $\left[1 + m \frac{\Delta S_{t_{l+1}}}{S_{t_l}} \right]$ and leads to:

$$m \leq \left[\underset{t_{l+1} \in \mathbb{R}^+ \text{ with } \Delta S_{t_{l+1}} < 0}{\text{Sup}} \left(-\Delta S_{t_{l+1}}/S_{t_l} \right) \right]^{-1}. \quad (2)$$

For example, if the *maximum* drop in the risky asset prices, $\Delta S_{t_{l+1}}/S_{t_l}$, is equal to -10% , m must be smaller than 10; if it is equal to -20% , m should be lower than 5. This indicates the set of standard values for the multiple.

The multiple therefore reflects the maximal exposure of the portfolio to the risky asset. The standard cushioned management strategy aims at keeping the proportion m of risk exposure constant. This means that the amount invested in the risky asset is determined by multiplying the cushion by the multiple. Nevertheless, the problem of cushion management is the determination of the target multiple. If, for instance, the risky asset price drops, the value of the cushion has, by definition, to remain higher or equal to zero. Therefore, a portfolio based on the cushion method will (theoretically) always have a value higher or equal to the floor. In the case of a drop of the risky underlying asset price, exposure rapidly tends to be zero. Thus, before the manager can rebalance his portfolio, the cushion should allow the portfolio to absorb main shocks. As illustrated by equation (1), the multiple of the insured portfolio has to remain smaller than the inverse of the underlying asset *maximum* drawdown, until the portfolio manager can rebalance his position (since, for short time periods, $\Delta B_{t_{l+1}}/B_{t_l} \simeq 0$). Thus, the multiple m must be upper bounded. Conversely, since the investor seeks to benefit from market rises, she claims a convex cash flow with respect to the risky asset return. Therefore, the investor has to require a multiple higher than

one⁵.

Standard results about CPPI strategies are based on the assumption that the floor \tilde{F}_t evolves just like the riskless asset. But this assumption is quite stringent. For example, it does not allow the investor to keep past benefits. The time-invariant protected portfolio (TIPP, see Estep and Kritzman, 1988) strategy is a modified version of CPPI which has this property. The main difference between CPPI and TIPP is in the stochastic time-varying definition of the floor. Actually, the TIPP floor is defined as the *maximum* between the usual CPPI floor and a percentage of the *maximum* past portfolio value. The new floor \tilde{F}_t^* in this approach satisfies, $\forall t \in [0, \dots, T]$ (with usual notations):

$$\tilde{F}_t^* = \text{Max} \left[\tilde{F}_t, \eta \text{Sup}_{s \leq t} (V_s) \right], \quad (3)$$

where η is a given non-negative parameter, usually defined as equal to .9. Thus, the modified TIPP floor includes some of the past gains throughout the life of the product and tends to protect the original CPPI floor completely.

In what follows, we now allow the multiple to vary at any time according to market fluctuations. In this framework, we provide computations of the CPPI portfolio value with a conditional multiple for various parametric financial models including those of Balder *et al.* (2009) for the discrete-time rebalancing, and of Cont and Tankov (2009) for continuous-time rebalancing when the risky asset follows an exponential Lévy process.

Let us suppose that the market evolves in a continuous-time, and consider a filtered probability space $[\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P}]$. We assume that the risky asset prices process S_t is a diffusion with jumps given by:

$$dS_t = S_t dZ_t^S, \quad (4)$$

with $dZ_t^S = \mu(t, S_t)dt + \sigma(t, S_t)dW_t + \delta(t, S_t)dN_t$ and where $(W_t)_{t \in \mathbb{R}^+}$ is a standard Brownian motion, independent from the Poisson process $(N_t)_{t \in \mathbb{R}^+}$. The functions $\mu(\cdot)$, $\sigma(\cdot)$ and $\delta(\cdot)$ satisfy usual conditions to ensure positivity, existence and uniqueness of the solution of the previous stochastic differential equation (see Jacod and Shiryaev, 2003). This means that jumps occur at random times $(T_n)_{n \in \mathbb{R}^+}$, so that interarrival times $(T_{n+1} - T_n)$ are identically and independently distributed, following an exponential distribution associated to parameter λ . The left-hand limit of process X at time T_n will be hereafter denoted by X_{T_n-} . The relative jumps $(\Delta S_{T_n}/S_{T_n-})$ of the risky asset returns are equal to $\delta(T_n, S_{T_n})$, and are all assumed to be higher than -1 , to

⁵By definition, the multiple is strictly positive. With a multiple equal to 1, the protection is absolute: the risky asset exposition is then equal to the value of the cushion. Note that $m = 1$ implies a static portfolio insurance. A multiple inferior to one is therefore irrational (see Perold and Sharpe, 1988).

imply positivity of S_t . In this framework, the value of the portfolio that follows a CPPI strategy is presented in the following proposition.

Proposition 1. *When the multiple is supposed to follow a stochastic process, such that $m_t: \Omega \times [0, T] \mapsto \mathbb{R}^+$, is adapted and locally bounded ⁶, the CPPI portfolio value is given by:*

$$V_t^{CPPI} = C_t + \tilde{F}_0 \exp(rt), \quad (5)$$

with:

$$C_t = C_0 \exp \left\{ \int_0^t \left\{ r + m_s \left[\mu - r - \frac{1}{2} m_s \sigma^2(s, S_s) \right] \right\} ds + \int_0^t m_s \sigma(s, S_s) dW_s \right\} \\ \times \prod_{0 \leq T_n \leq t} \left[1 + m_{T_n-} \delta(T_n, S_{T_n}) \right].$$

Proof. The CPPI portfolio value is indeed a solution of the following stochastic differential equation such that:

$$dV_t^{CPPI} = (V_t^{CPPI} - e_t) \frac{dB_t}{B_t} + e_t \frac{dS_t}{S_t}.$$

At any time, the exposure e_t is equal to $m_t C_t$. Thus, since the cushion value C_t satisfies:

$$dC_t = d(V_t^{CPPI} - \tilde{F}_t),$$

we get:

$$dC_t = C_t dY_t,$$

where the process $(Y_t)_{t \in \mathbb{R}^+}$ satisfies:

$$dY_t = \{r + m_t [\mu(t, S_t) - r]\} dt + m_t \sigma(t, S_t) dW_t + m_t \delta(t, S_t) dN_t.$$

Therefore, the process $(C_t/C_0)_{t \in \mathbb{R}^+}$ is the Doléans-Dade exponential of the process $(Y_t)_{t \in \mathbb{R}^+}$. Then application of the theorem (4.61) of Jacod and Shiryaev (2003) leads to the result. \square

Let us turn now to the conditions of guarantee according to specific hypotheses both on the risky asset process and the portfolio rebalancing timing. It leads us to distinguish the three following cases: a diffusion process without jumps and continuous-time rebalancing (case 1), with jumps and continuous-time rebalancing (case 2), and, with or without jumps with constrained discrete-time rebalancing (case 3).

Case 1. This is the standard case considered in Perold (1986) and in Black and Perold (1992). If the risky asset price follows a diffusion process without jumps (*i.e.* $\delta(t, S_t) = 0$), the cushion is

⁶This case encompasses the case where m is constant and bounded. This is also in relation to the empirical results of this paper about the multiple, *i.e.* it belongs in our case to values in the range [1, 13].

always positive whatever the choice of the multiple (if, of course, the initial cushion C_0 is positive). In this simplest (but unrealistic) case, the guarantee is perfect.

Case 2. If the risky asset price occasionally experiences jumps, according to relation (5), the guarantee condition in such a case is equivalent to the following condition: at any jump time T_n ,

$$\delta(T_n, S_{T_n}) \geq -m_{T_n}^{-1}. \quad (6)$$

Indeed, if all jumps are higher than a non-positive constant u , then the condition $0 \leq m_{T_n} \leq -1/u$ for all T_n allows us to guarantee the positivity of the cushion⁷. This condition does not depend either on jump times T_n , nor on specific assumptions on the probability distribution of ΔS_{T_n} (except the minimal bound u .)

Case 3. Suppose now that the portfolio is rebalanced in discrete-time (the more realistic case). In this case, the strategy is of a buy-and-hold type. We consider the same market evolution as in previous cases (with or without jumps), but now the portfolio strategy can only be modified at dates $t_0 < t_1 < \dots < T$. Thus, we have to modify the computation of the portfolio value, since the multiple can only be changed on a discrete-time basis. The multiple m_{t_l} is defined at time t_l and equal to the ratio e_{t_l}/C_{t_l} .

According to the framework defined in Case 3, we get a value of the CPPI portfolio as summarized in the following proposition.

Proposition 2. *When the portfolio is rebalanced in a discrete-time basis $(t_l)_{l \in \mathbb{N}^*}$, the CPPI portfolio value is given by, for any time $t \in [t_l, t_{l+1}[$:*

$$\begin{aligned} V_t^{CPPI} &= (V_{t_l}^{CPPI} - e_{t_l}) \exp[r(t - t_l)] \\ &+ e_{t_l} \exp \left\{ \int_{t_l}^t \left[\mu(s, S_s) - \frac{1}{2} \sigma^2(s, S_s) \right] ds + \int_{t_l}^t \sigma(s, S_s) dW_s \right\} \times \prod_{t_l < T_n \leq t} [1 + \delta(T_n, S_{T_n})], \end{aligned} \quad (7)$$

with:

$$e_{t_l} = m_{t_l} \left\{ V_{t_l}^{CPPI} - \tilde{F}_0 \exp[r(t_l - t_0)] \right\}.$$

Proof. In period $[t_l, t_{l+1}[$, the strategy is of a buy-and-hold type. Thus, the portfolio value at any time $t \in [t_l, t_{l+1}[$ is equal to the shares invested in the two basic assets multiplied by their respective returns. \square

⁷If, for instance, $u = -20\%$, then the sufficient condition to get the guarantee is $m \leq 5$.

3. On the Evaluation of Gap Risks for Discrete-time Rebalancing

Since the guarantee cannot be perfect due to discrete-time rebalancing, we adopt hereafter a probabilistic approach to provide an upper bound on the multiple. Indeed, by construction, if a large drop in the market (or a jump) happens, the guarantee is not perfect since the rebalancing is not continuous and is thus associated to a tiny level of probability of not respecting the absolute guarantee. The risk of violating the floor protection is called a “gap risk”. We propose, in the following, several definitions of the so-called gap risks, depending on both the characteristics of a potential loss in terms of probability of occurrence, and the size of a potential first loss. Whatever the effective risk measure, these gap risks can be both explained by the presence of jumps in stock prices (Prigent, 2001; Cont and Tankov, 2009) and/or merely by relaxing the continuous-time trading assumption (Black and Perold, 1992; Bertrand and Prigent, 2002). To illustrate these gap risk definitions, we will focus hereafter on the discrete-time case (Case 3 - see above) without jumps (which corresponds to the case investigated by Balder *et al.*, 2009). In this case, the underlying process can evolve in continuous-time (with or without jumps), but portfolio rebalancing is in discrete-time and happens at dates $t_0 < t_1 < \dots < T$; it is therefore only possible to “locally” manage the gap risk (*i.e.* knowing that the cushion was positive at times t_l). This means that, conditional on the information available at time t_l and on the fact that the cushion is positive at the time, we can only manage the probability of going below the floor, and/or, as a complementary measure, the expectation of the cushion given the loss, for the time-period $[t_l, t_{l+1}[$. These two types of gap risks are herein respectively denoted by $GR_{loc,1,t_l}$ and $GR_{loc,2,t_l}$ in the context of Case 3. In what follows, we first indicate the results for the geometric Brownian case⁸, and, secondly, provide similar results in a more general case with jumps. We denote also by $(\bar{m}_l)_{l \in \mathbb{N}_*}$ a sequence of adapted target multiples that can be chosen with respect to different *criteria* (for the standard CPPI method, we always have $\bar{m}_l = m$ which is a constant). This corresponds to successive multiples applied on a sequence of time subperiods.

3.1. Control of the probability of a potential loss: a first gap risk definition

The first gap risk, denoted $GR_{loc,1,t_l}$ (discrete-time case), is defined as the probability of going below the floor. It corresponds to the following inequality:

$$GR_{loc,1,t_l} = \mathbb{P}_{t_l} [C_{t_{l+1}} > 0 \mid C_{t_l} > 0] \geq (1 - \alpha), \quad (8)$$

where $\mathbb{P}_{t_l}(\cdot)$ denotes the conditional probability with respect to \mathcal{F}_{t_l} , depending on information available at time t_l , and the parameter α corresponds to a given (low) probability level (for example, $\alpha = 1\%$ or 5% usually).

⁸Computations of global gap risks, $GR_{1,t}$ and $GR_{2,t}$, are provided in Balder *et al.* (2009).

Remark 1. If inequality (8) is satisfied, then we get a global quantile gap risk evaluation, since we have:

$$\mathbb{P}[\forall l, C_{t_l} > 0] \geq (1 - \alpha)^T.$$

As an illustration, consider the case where the risky asset returns follow a geometric Brownian motion and, for any time $t \in [t_l, t_{l+1}[$, $m_t = m_{t_l} = \bar{m}_l$. Then, we have the following result for the upper bound on the conditional multiple \bar{m}_l .

Proposition 3. *The global quantile guarantee condition (defining the first gap risk) when the risky asset price follows a geometric Brownian motion (i.e. $\mu(\cdot, \cdot)$, $\sigma(\cdot, \cdot)$ are constant and $\delta = 0$) and the rebalancing is only discrete (discrete-time case without jumps), such as:*

$$\mathbb{P}_{t_l} [C_{t_{l+1}} > 0 \mid C_{t_l} > 0] \geq (1 - \alpha), \quad (9)$$

is equivalent to the following condition on the conditional multiple \bar{m}_l :

$$\bar{m}_l \leq \left\{ 1 - \exp \left[\Phi^{-1}(\alpha) \sigma (t_{l+1} - t_l)^{1/2} + \left(\mu - r - \frac{1}{2} \sigma^2 \right) (t_{l+1} - t_l) \right] \right\}^{-1}, \quad (10)$$

where $\Phi^{-1}(\cdot)$ denotes the Gaussian quantile function and r is the constant riskless rate.

Proof. See hereafter, and Balder *et al.* (2009). Using relation (7), valid in the geometric Brownian case, the cushion value at time t_{l+1} is then given by:

$$\begin{aligned} C_{t_{l+1}} &= C_{t_l} \{ 1 + (1 - \bar{m}_l) \{ \exp [r(t_{l+1} - t_l)] - 1 \} \\ &\quad + \bar{m}_l \left\{ \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) (t_{l+1} - t_l) + \sigma (W_{t_{l+1}} - W_{t_l}) \right] - 1 \right\} \}. \end{aligned}$$

Denoting the rescaled variation of the Brownian motion as follows:

$$X_{t_{l+1}} = \frac{W_{t_{l+1}} - W_{t_l}}{\sqrt{(t_{l+1} - t_l)}},$$

the previous $GR_{loc,1,t_l}$ inequality (9) is equivalent to:

$$\mathbb{P}_{t_l} \left\{ X_{t_{l+1}} \leq \frac{\ln \{ \bar{m}_l^{-1} (\bar{m}_l - 1) \exp [r(t_{l+1} - t_l)] \} - (\mu - \frac{1}{2} \sigma^2) (t_{l+1} - t_l)}{\sigma \sqrt{(t_{l+1} - t_l)}} \right\} \leq \alpha,$$

from which we deduce the result. \square

3.2. Control of the potential loss size: a second gap risk definition

At this stage, we can now examine the expectation of a loss defining the second gap risk, denoted $GR_{loc,2,t_l}$, approached using two complementary quantities: the local conditional and the unconditional expectations of a loss (denoted, respectively, CEL_{t_l} and UEL_{t_l}).

Definition 1. *The conditional and unconditional expectations of a loss (defining the second gap risk), when the rebalancing is only in discrete-time (discrete-time case without jumps), are defined by:*

$$\begin{cases} CEL_{t_l} = \mathbb{E}_{t_l} \left[C_{t_{l+1}} / \tilde{F}_{t_{l+1}} | (C_{t_{l+1}} < 0) \cap (C_{t_l} > 0) \right] \\ \text{and} \\ UEL_{t_l} = \mathbb{E}_{t_l} \left[\left(C_{t_{l+1}} / \tilde{F}_{t_{l+1}} \right) 1I_{\{C_{t_{l+1}} < 0\}} | (C_{t_l} > 0) \right], \end{cases} \quad (11)$$

where $1I_{\{\cdot\}}$ is the Heaviside function.

In what follows, we set a function $g(\cdot, \cdot, \cdot)$ such as:

$$g(\bar{m}_l, t_l, t_{l+1}) = \frac{\ln \left\{ \bar{m}_l^{-1} (\bar{m}_l - 1) \exp [r(t_{l+1} - t_l)] \right\} - (\mu - \frac{1}{2}\sigma^2)(t_{l+1} - t_l)}{\sigma \sqrt{(t_{l+1} - t_l)}}. \quad (12)$$

When the risky asset price follows a geometric Brownian motion, we get the following result for discrete-time portfolio rebalancing.

Proposition 4. *The conditional and unconditional expectations of a loss (discrete-time case without jumps) are given by:*

$$\begin{cases} CEL_{t_l} = \left(C_{t_l} / \tilde{F}_{t_{l+1}} \right) \left\{ (1 - \bar{m}_l) \exp [r(t_{l+1} - t_l)] \right. \\ \quad \left. + \bar{m}_l \exp [\mu(t_{l+1} - t_l)] \Phi \left[g(\bar{m}_l, t_l, t_{l+1}) - \sigma \sqrt{(t_{l+1} - t_l)} \right] \Phi \left[g(\bar{m}_l, t_l, t_{l+1}) \right]^{-1} \right\} \\ \text{and} \\ UEL_{t_l} = \left(C_{t_l} / \tilde{F}_{t_{l+1}} \right) \left\{ (1 - \bar{m}_l) \exp [r(t_{l+1} - t_l)] \right. \\ \quad \left. + \bar{m}_l \exp [\mu(t_{l+1} - t_l)] \Phi \left[g(\bar{m}_l, t_l, t_{l+1}) - \sigma \sqrt{(t_{l+1} - t_l)} \right] \right\}. \end{cases} \quad (13)$$

Proof.

We have:

$$\begin{aligned} & \mathbb{E}_{t_l} [C_{t_{l+1}} | (C_{t_{l+1}} < 0) \cap (C_{t_l} > 0)] \\ &= C_{t_l} \left\{ (1 - \bar{m}_l) \exp [r(t_{l+1} - t_l)] 1I_{\{X_{t_l} < g(\bar{m}_l, t_l, t_{l+1})\}} \Phi [g(\bar{m}_l, t_l, t_{l+1})]^{-1} \right. \\ & \quad \left. + \bar{m}_l \exp \left[\left(\mu - \frac{1}{2}\sigma^2 \right) (t_{l+1} - t_l) + \sigma \sqrt{(t_{l+1} - t_l)} X_{t_l} \right] 1I_{\{X_{t_l} < g(\bar{m}_l, t_l, t_{l+1})\}} \Phi [g(\bar{m}_l, t_l, t_{l+1})]^{-1} \right\} \\ &= C_{t_l} \left\{ (1 - \bar{m}_l) \exp [r(t_{l+1} - t_l)] \right. \\ & \quad \left. + \bar{m}_l \exp [\mu(t_{l+1} - t_l)] \Phi \left[g(\bar{m}_l, t_l, t_{l+1}) - \sigma \sqrt{(t_{l+1} - t_l)} \right] \Phi [g(\bar{m}_l, t_l, t_{l+1})]^{-1} \right\}. \end{aligned}$$

□

3.3. Gap risk duration

Conditions on the multiple depend upon the assumptions we make on the risky asset process. We introduce hereafter a sequence of times $(\Theta_l)_{l \in \mathbb{N}_*}$ at which the target multiple can be modified, used to define the so-called “gap risk duration”. It can be formally defined so that, for $t \in [\Theta_l, \Theta_{l+1}[$, the gap risk duration at time Θ_l , denoted d_l , satisfies:

$$GRD_{\Theta_l} = d_l = T_{NG} - \Theta_l, \quad (14)$$

where T_{NG} is the first time in the period $[\Theta_l, \Theta_{l+1}[$ at which the cushion becomes negative (no guarantee applies any longer) with a target multiple always equal to \bar{m}_l .

These definitions of gap risk characteristics are examined for different standard cases in the next section.

4. On Gap Risks for Triggered Trading

Assuming continuous-time rebalancing and that the multiple follows an adapted stochastic process, relation (5) provides the portfolio value in this case. Due to the simplicity associated with case 1 of the previous section (rebalancing can be done continuously and the risky asset price does not jump), the guarantee always holds whatever the choice of the multiple. For all other more realistic cases, the multiple needs to be upper bounded. We can develop several continuous-time or discrete-time versions of the conditional multiple⁹.

In what follows, since for a very short time period (one or two days), $\Delta B_{t_{l+1}}/B_{t_l}$ is small (close to zero), we neglect it when computing the upper bounds on the multiplier¹⁰.

4.1. First and second gap risk measures with a continuous time-varying multiple when underlying asset moves in continuous-time with jumps

We consider a sequence of stopping times $(\Theta_l)_{l \in \mathbb{N}_*}$ which corresponds to the times at which the target multiple is modified. They can be, for example, deterministic. In what follows, we assume that the target multiple is constant in intervals between two stopping times and we have on $[\Theta_l; \Theta_{l+1}[$:

$$m_t = m_{\Theta_l} = \bar{m}_l.$$

⁹In the paper, we only expose some cases (mainly the discrete-time case). Complementary results are available from the authors upon request.

¹⁰Formulae that do not assume the approximation $\Delta B_{t_{l+1}}/B_{t_l} \approx 0$ are available on request. However, from the numerical point of view and for practical applications, the approximation is shown to be very accurate.

Remark 2. We could also assume that there is no trading as the multiple (defined by the ratio of the exposure upon the cushion) remains between certain given bounds, corresponding for example to a tolerance ρ . In this case, we define the stopping times and the sequence of random variables of discrete multiples $(\bar{m}_l)_{l \in \mathbb{N}_*}$ by induction so that:

$$\Theta_0 = 0, \text{ and } m_0 \in [1, m_{\max}],$$

then:

$$\Theta_l = \underset{t \in \mathbb{R}_*}{\text{Min}}\{t \mid t > \Theta_{l-1}, m_t \geq \bar{m}_{l-1} (1 + \rho) \text{ or } m_t \leq \bar{m}_{l-1} (1 - \rho)\}, \quad (15)$$

and:

$$m_{\Theta_l} = \bar{m}_l.$$

As usual, if the set $\text{Min}\{t \mid t > \Theta_{l-1}, m_t \geq \bar{m}_{l-1}(1 + \rho) \text{ or } m_t \leq \bar{m}_{l-1}(1 - \rho)\}$ is empty, we set $\Theta_l = T$. Such a case, which is more convenient in practice (“buy-and-hold” strategy for each time period $[\Theta_l, \Theta_{l+1}[$), is empirically investigated in Section 6, together with the choice of \bar{m}_l .¹¹

The risky asset price S_t is assumed to be an exponential Lévy process given by:

$$dS_t = S_t dZ_t^S,$$

where Z_t^S is a Lévy process (defined in equation 4 where μ and σ are constant and the relative jumps, $\Delta S_{T_n}/S_{T_n}$, are i.i.d. random variables).

The target multiple \bar{m}_l is defined in the time interval $[\Theta_l; \Theta_{l+1}[$. Recall that we assume here $m_t = \bar{m}_l$ for every $t \in [\Theta_l; \Theta_{l+1}[$. Then, the portfolio value satisfies (with the previous notations):

$$dV_t^{CPPI} = \left[V_{t^-}^{CPPI} - \bar{m}_l (V_{t^-}^{CPPI} - \tilde{F}_t) \right] r dt + \bar{m}_l (V_{t^-}^{CPPI} - \tilde{F}_t) \frac{dS_t}{S_{t^-}},$$

which implies:

$$dC_t = C_{t^-} \left[\bar{m}_l dZ_t^S + (1 - \bar{m}_l) r dt \right].$$

That leads to the following definitions and propositions about the first and second gap risks in such a framework.

Proposition 5. The global quantile guarantee condition (defining the first gap risk $GR_{loc,1,\Theta_l}$) is as such:

$$\mathbb{P}_{\Theta_l}[C_t \geq 0, \forall \Theta_l < t \leq \Theta_{l+1} | C_{\Theta_l} > 0] \geq 1 - \alpha, \quad (16)$$

which is equivalent to:

$$\mathbb{P}_{\Theta_l} \left\{ \forall k, \Theta_l < T_k \leq \Theta_{l+1}, \left[1 + \bar{m}_l \left(\Delta S_{T_k} / S_{T_k^-} \right) \right] \geq 0 | C_{\Theta_l} > 0 \right\} \geq 1 - \alpha. \quad (17)$$

¹¹For more details about this strategy, see also Mkaouar and Prigent (2010a).

Proof.

We just have to take into account all possible jumps at random times T_k along the time period $]\Theta_l, \Theta_{l+1}]$.

□

Recall here that we denote by T_{NG} the first time at which the guarantee is no longer satisfied in a given time period; we can compute both CEL_{Θ_l} and UEL_{Θ_l} in such a setting.

Proposition 6. *The conditional and unconditional expectations of a loss (defining the second gap risk denoted $GR_{loc,2,\Theta_l}$), when stock prices experience jumps and the rebalancing is in continuous-time, are given by (with the previous notations):*

$$\left\{ \begin{array}{l} CEL_{\Theta_l} = \mathbb{E}_{\Theta_l} \left[C_{\Theta_{l+1}} / \tilde{F}_{\Theta_{l+1}} | (\Theta_l < T_{NG} \leq \Theta_{l+1}) \cap C_{\Theta_l} > 0 \right] \\ \quad = \mathbb{E}_{\Theta_l} \left[C_{\Theta_{l+1}} / \tilde{F}_{\Theta_{l+1}} | \{ \exists k, \Theta_l < T_k \leq \Theta_{l+1}, [1 + \bar{m}_l (\Delta S_{T_k} / S_{T_{k-}})] < 0 \} \cap C_{\Theta_l} > 0 \right] \\ \text{and} \\ UEL_{\Theta_l} = \mathbb{E}_{\Theta_l} \left[\left(C_{\Theta_{l+1}} / \tilde{F}_{\Theta_{l+1}} \right) 1I_{\{\Theta_l < T_{NG} \leq \Theta_{l+1}\}} | C_{\Theta_l} > 0 \right] \\ \quad = \mathbb{E}_{\Theta_l} \left[\left(C_{\Theta_{l+1}} / \tilde{F}_{\Theta_{l+1}} \right) 1I_{\{\exists k, \Theta_l < T_k \leq \Theta_{l+1}, [1 + \bar{m}_l (\Delta S_{T_k} / S_{T_{k-}})] < 0\}} | C_{\Theta_l} > 0 \right]. \end{array} \right. \quad (18)$$

Proof. The event $\Theta_l < T_{NG} \leq \Theta_{l+1}$ corresponds exactly to condition:

$$\exists k, \Theta_l < T_k \leq \Theta_{l+1}, [1 + \bar{m}_l (\Delta S_{T_k} / S_{T_{k-}})] < 0.$$

□

These conditions can now be examined more precisely for different (realistic) standard cases. In what follows, Case (i) is related to a deterministic sequence $(\Theta_l)_{l \in \mathbb{N}_*}$ (*i.e.* $\Theta_l = t_l$); in case (ii), the sequence $(\Theta_l)_{l \in \mathbb{N}_*}$ is linked to the sequence of jump occurrence times of S (*i.e.* $\Theta_l = T_l$); finally, case (iii) corresponds to both discrete-time varying multiple and portfolio strategies. We present hereafter calculations for case (i), whilst those of cases (ii) and (iii) are reported in the Appendices.

4.2. First and second gap risks when the multiple can be only modified on a deterministic time basis (Case i)

In this setting, the portfolio manager can only modify his target multiple at fixed cut-off times (daily closes for instance). Here, the sequence $(\Theta_l)_{l \in \mathbb{N}_*}$ is deterministic ($\Theta_l = t_l$). However, portfolio rebalancing is in continuous-time. In this case, condition (17) can be detailed as follows, when risky asset price follows a geometric Lévy process.

Proposition 7. *The global quantile guarantee condition (defining the first gap risk) when the risky asset price follows a geometric Lévy process and the rebalancing time is deterministic, written such as:*

$$\mathbb{P}[C_t \geq 0, \forall t \leq T] \geq 1 - \alpha, \quad (19)$$

is equivalent to the following condition on the conditional multiple \bar{m}_l :

$$\bar{m}_l \leq - \left\{ F^{-1} \left\{ -[\lambda(t_{l+1} - t_l)]^{-1} \ln(1 - \alpha) \right\} \right\}^{-1}, \quad (20)$$

where $F^{-1}(\cdot)$ denotes the quantile function related to the density of the risky asset relative jumps $\Delta S_{T_n}/S_{T_{n-}}$.

Proof. Condition (19) is shown to be equivalent to:

$$\mathbb{P}[\forall t_l < t \leq t_{l+1}, \Delta S_t/S_{t-} \geq -1/\bar{m}_l] \geq 1 - \alpha.$$

Recall that S_t is an exponential Lévy process (see description of dynamics of S_t in equation 4). Denote $F(\cdot)$ the cumulative density function of relative jumps $\Delta S_{T_n}/S_{T_{n-}}$ and consider its inverse $F^{-1}(\cdot)$. The number of jumps of the Lévy process Z^S in intervals $]t_l; t]$, which are in $] -\infty; -1/\bar{m}_l]$, is a Poisson variable with parameter $(t - t_l)\lambda F(-1/\bar{m}_l)$. Then, using the same arguments as in Prigent (2001) applied to the time-period $]t_l; t_{l+1}]$, the quantile guarantee condition (19) is equivalent to the following condition on the multiple \bar{m}_l :

$$\bar{m}_l \leq - \left\{ F^{-1} \left\{ -[\lambda(t_{l+1} - t_l)]^{-1} \ln(1 - \alpha) \right\} \right\}^{-1}. \quad (21)$$

□

This last condition determines a less stringent condition on the upper bound of the multiple, which decreases with respect to intensity λ . However, we can also search for better control of the expectation of losses. This can be done by introducing a measure of the second gap risk. To examine the expectation of loss, conditional on the fact that the loss occurs, we introduce the following function $\psi(\cdot)$, as in Cont and Tankov (2009), so that:

$$\psi(\bar{m}_l) = \bar{m}_l(\mu - r) + m\lambda \int_{x > -1/\bar{m}_l}^{+\infty} x dF(x).$$

We also denote function $\chi(\cdot, \cdot)$ as such:

$$\begin{aligned} \chi(\bar{m}_l, u) &= [\lambda F(-1/\bar{m}_l) + \bar{m}_l \lambda \int_{-1}^{-1/\bar{m}_l} x dF(x)] \\ &\times \{ \exp[-\lambda F(-1/\bar{m}_l)u + u\psi(\bar{m}_l)] - 1 \} [\psi(\bar{m}_l) - \lambda F(-1/\bar{m}_l)]^{-1}. \end{aligned}$$

Lemma 1. For a fixed multiple \bar{m}_l , the global expectation of loss conditional on a loss occurrence in the time-period $[0, T]$, denoted $CEL_{gl,T}$, is given by (with previous notations):

$$CEL_{gl,T} = \mathbb{E} \left[C_T / \tilde{F}_T \mid T_{NG} \leq T \right] = \left(C_0 / \tilde{F}_0 \right) \chi(\bar{m}_l, T) \{1 - \exp[-\lambda F(-1/\bar{m}_l)T]\}^{-1}, \quad (22)$$

whilst the global unconditional expected loss, denoted $UEL_{gl,T}$, is given by:

$$UEL_{gl,T} = \mathbb{E} \left[\left(C_T / \tilde{F}_T \right) 1_{T_{NG} \leq T} \right] = \left(C_0 / \tilde{F}_0 \right) \chi(\bar{m}_l, T). \quad (23)$$

Proof. See Cont and Tankov (2009). \square

Then, applying previous results locally, the local CEL_{t_l} and UEL_{t_l} can also be determined as follows.

Proposition 8. The local conditional and unconditional expectations of a loss, when stock prices follow a Lévy process and the rebalancing is deterministic, are given by :

$$\begin{cases} CEL_{t_l} = \left(C_{t_l} / \tilde{F}_{t_l} \right) \bar{m}_l \chi(\bar{m}_l, t_{l+1} - t_l) \{1 - \exp[-\lambda F(-1/\bar{m}_l)(t_{l+1} - t_l)]\}^{-1} \\ \text{and} \\ UEL_{t_l} = \left(C_{t_l} / \tilde{F}_{t_l} \right) \chi(\bar{m}_l, t_{l+1} - t_l). \end{cases} \quad (24)$$

Proof. Due to the Lévy property, we can apply results of Lemma (1) to the interval $[t_l, t_{l+1}[$, from which we deduce the result. \square

Then, if we set $(t_{l+1} - t_l)$ as the time-window in which the risk is re-evaluated, we obtain the required multiple \bar{m}_l by setting a given level of risk.

The gap duration can also be computed in this framework as in the following *lemma*.

Lemma 2. For the durations d_l smaller than $(t_{l+1} - t_l)$, the gap risk duration is given by:

$$\mathbb{P}_{t_l}[T_{NG} - t_l \leq d_l] = 1 - \exp[-d_l \lambda F(-1/\bar{m}_l)]. \quad (25)$$

Proof. As recalled in proof of Proposition (7), the number of jumps of the Lévy process Z^S in intervals $]t_l; t]$, which are in $] -\infty; -1/\bar{m}_l]$, is a Poisson variable with parameter $(t - t_l)\lambda F(-1/\bar{m}_l)$. Note also that $\mathbb{P}_{t_l}[T_{NG} - t_l \leq d_l] = 1 - \mathbb{P}[\forall t_l < t \leq t_{l+1}, \Delta S_t / S_{t-} \geq -1/\bar{m}_l]$. Then, using the same arguments as in Prigent (2001), we deduce the result. \square

Consider for instance the special case in which $J_n = \ln(1 + \Delta S_{T_n} / S_{T_n-})$ has an asymmetric Double Exponential Distribution (see Kou, 2002). The density $g(\cdot)$ of realizations of J_n , denoted j , is then given by:

$$g(j) = (1 - p)\kappa \exp(-\kappa j) 1_{\{j > 0\}} + p\gamma \exp(\gamma j) 1_{\{j < 0\}}, \quad (26)$$

with $\kappa > 1$, $\gamma > 0$ and $p \in]0, 1[$, and where parameters γ and κ are respectively the parameters determining the intensity of negative jumps and the number of jumps (positive or negative), and the probability p corresponds to the one that a given jump is negative.

We recall that: $E[J_n] = (1 - p)\kappa^{-1} - p\gamma^{-1}$. The assumption about the distribution of j (in fact, the existence of J_n as a random variable) implies the basic assumption that $\Delta S_{T_n}/S_{T_n-} > -1$ is almost everywhere satisfied. It ensures that the price of the risky asset will always be positive.

The Lévy measure ν is given by:

$$\nu(dj) = \lambda(1 - p)\kappa \exp(-\kappa j) \mathbb{1}_{\{j \geq 0\}} + \lambda p \gamma \exp(\gamma j) \mathbb{1}_{\{j < 0\}}.$$

Thus, we can deduce at any time t_l , the probability distribution function for the duration corresponding to the target multiple¹² \bar{m}_l , denoted by $f_{t_l}(\cdot)$:

$$f_{t_l}(d_l) = p\lambda(1 - 1/\bar{m}_l)^\gamma \exp\{-[p\lambda(1 - 1/\bar{m}_l)^\gamma] d_l\}. \quad (27)$$

In this framework, the sequence of durations of the gap risk, which has a shape of a weighted negative exponential, can be characterized by its first four moments. The expected value, variance, relative skewness and relative *kurtosis* of the duration are respectively denoted $M_{d_l,1}$, $M_{d_l,2}$, $M_{d_l,3}$ and $M_{d_l,4}$, and defined by¹³:

$$\begin{cases} M_{d_l,1} = [(p\lambda(1 - 1/\bar{m}_l)^\gamma)^{-1}] \\ M_{d_l,2} = [p\lambda(1 - 1/\bar{m}_l)^\gamma]^{-2} \\ M_{d_l,3} = 2 \\ M_{d_l,4} = 9. \end{cases} \quad (28)$$

In this setting, a gap risk-embedded option with a varying multiple can also be introduced so that the investor is protected against the gap risk. This implies that the loss is compensated by the bank that has issued the CPPI fund. In this case, the difference between insured and uninsured CPPI portfolios is an embedded option with payoff equal to $-C_T \mathbb{1}_{\{\tau \leq T\}}$. Using results in Cont and Tankov (2009), we can show that using put options with strikes $K_l = (m_{t_l} - 1) \bar{m}_l^{-1} \exp[r(t_{l+1} - t_l)] S_{t_l}$ in each subperiod $[t_l, t_{l+1}]$ allows us to get a CPPI without gap risk when $t_{l+1} - t_l$ converges asymptotically to 0. To estimate the gap risk in a PPI framework, we could thus use the multiple at any time to get the assumed estimation of the *maximum* potential

¹²If the target multiple \bar{m}_l is random, the unconditional density could be deduced by integrating function $f_{t_l}(\cdot)$ with respect to the distribution of \bar{m}_l .

¹³These computations are made without taking account of time t_{l+1} . Indeed, the idea is to use the $M_{d_l,i}$ to calibrate $t_{l+1} - t_l$, thus determining t_{l+1} from t_l and the first four moments of the duration distribution.

loss. To reach a perfect guarantee assuming no *maximum* potential loss *scenario*, a hedge of the gap risk can be obtained when buying a put whose maturity will be the rebalancing frequency. The strike will be defined each rebalancing time thanks to the assumed (or modeled) maximal potential loss at T . Thus, a hedging strategy can be proposed using short maturity put options in such an approach. An alternative method, introduced by Prigent and Tahar (2005), is to buy an option which is exercised as soon as the cushion becomes negative.

5. A DARE Model for a Conditional Multiple

In this empirical section, we now develop a general quantile hedging approach for the determination of a time-varying conditional multiple in a discrete-time setting encompassing the case developed in equation (20)¹⁴ in a traditional Lévy setting (i.i.d. increments). The purpose of this section is to determine a time-varying conditional multiple in a discrete-time setting, and to provide an efficient and accurate way to model it using real market data, which have a more complex and rich structure with characteristics such as, amongst others, non-stationarity, conditional autoregressive heteroskedasticity, volatility and extreme event clustering, regime switching, break and mean reversion *phenomena*. We first describe and justify conditional multiple models in such an empirical framework, adopting here a risk management philosophy. We then propose an original DARE approach for estimating the conditional multiple model, diversifying the risk model associated to usual extreme market risk measures. Finally, we briefly recall traditional benchmarks of extreme market risk measures mobilized in the paper.

Thus, we use a time-varying conditional multiple, denoted m_t hereafter, to adapt the portfolio exposition to the risky asset underlying risk using a quantitative systematic probabilistic approach. As shown in Section 2 for the single period case, in a time-varying framework, to be perfectly protected, a PPI-based portfolio has to fulfill, at time t , the following condition:

$$m_t \leq [-r_{t+1}^S]^{-1}, \quad (29)$$

where we denote by r_{t+1}^S the periodic return of the risky asset at time $t + 1$ (*i.e.* $r_{t+1}^S = (S_{t+1} - S_t)/S_t$). However, at time t , r_{t+1}^S is unknown and we have to adopt a probabilistic approach to determine the *maximum* potential loss of the risky asset. We propose in this section to estimate the potential loss by applying probabilistic approaches used in risk management for extreme risk measures based on quantiles estimated on samples of market data.

¹⁴See also (46) and (48) in Appendix for other cases.

At any rate, even in an unconditional multiple PPI framework, the guarantee depends on the estimation of the *maximum* potential loss that the risky asset can reach before the portfolio manager is able to rebalance his position. If the potential loss is underestimated, the predefined guarantee of the portfolio is not ensured anymore. The only way to be sure of reaching a perfect guarantee is to choose an unconditional multiple equal to 1 (*i.e.* the potential loss is assumed to be as large as 100%). For all other cases, there is a residual gap risk between a perfect insurance and the true one proposed when estimating potential losses.

At any time, the fluctuating conditional multiple may move away from the target value given by the systematic quantitative model. This is the reason why, in practical applications, a third parameter - called the tolerance and denoted by ρ , is introduced to determine whether the portfolio has to be rebalanced. If after the fluctuation of the risky asset, the remaining multiple, denoted by $m_{t,\rho}$, moves away from its target value m_t by a higher percentage than the tolerance, adjustments are needed so that the multiple tends to the value given by the model and associated with transaction fees. Therefore, the portfolio is rebalanced as soon as we have:

$$m_{t,\rho} \notin [m_t \times (1 - \rho), m_t \times (1 + \rho)]. \quad (30)$$

5.1. Conditional multiple models

For a capital guarantee constraint at a significance level α , the multiple must be lower than the inverse of the α conditional-quantile of the asset return distribution, that is, $\forall t \in [0, \dots, T]$:

$$\mathbb{P}_t(C_{t+1} \geq 0 | C_t > 0) \geq 1 - \alpha \Leftrightarrow m_t(1 + \rho) \leq -\{VaR_{1-\alpha,t+1}\}^{-1}, \quad (31)$$

with:

$$\begin{cases} VaR_{1-\alpha,t+1} = \underset{r_{t+1}^s \in]-1, +\infty[}{Inf} \{ r_{t+1}^s \mid G_{t+1}(r_{t+1}^S) > \alpha \} \\ m_t(1 - \rho) \geq 1, \end{cases}$$

where T is the terminal date, $VaR_{1-\alpha,t+1}$ is the α -quantile related to the conditional cumulative distribution function of the risky asset returns¹⁵ denoted $G_{t+1}(\cdot)$. Note that condition $m_t(1 - \rho) \geq 1$ is introduced to avoid considering multiples lower than 1.

The target multiple can here be interpreted as the inverse of the *maximum* loss that the cushioned portfolio can bear before the rebalancing of its risky component occurs, at a given confidence level. For a given parametric model, explicit solutions for the upper bound on the

¹⁵Note that we here expressed the VaR in general terms (returns or quantiles) and not in terms of losses (*i.e.* $L_t = r_t \times V_{t-1}$ where L_t is the actual loss corresponding to the VaR and V_{t-1} the previous portfolio value).

multiple can be provided from the relation (31). However, without specific assumptions on market dynamics, upper bounds on the multiple can also be determined from market conditions, as related below.

On the one hand, Ben Ameur and Prigent (2007) first express the conditional multiple in the special case of an ARCH-type model. Lee *et al.* (2008) propose an Exponential PPI based on a dynamic multiple, whilst Jiang *et al.* (2009) use a first VaR-based definition of it, and Hamidi *et al.* (2009a and 2009b) propose an experimental aggregated quantile modelling for computing the conditional multiple model. In this general setting, the hedged portfolio, for keeping a constant exposure to extreme risk measured by the VaR, should respect the following condition:

$$m_t = |VaR_{1-\alpha,t}(r_t^S; \beta_t)|^{-1}, \quad (32)$$

where $VaR_{1-\alpha,t}(r_t^S; \beta_t)$ is the VaR of the conditional distribution of returns of the underlying asset r_t^S , corresponding to the periodic return of the risky part of the portfolio covered, and β_t is the (time-varying) vector of unknown parameters of the conditional VaR model.

On the other hand, various approaches have been proposed for estimating conditional tail quantiles of financial returns. The common approach for modelling dynamic quantiles is to specify a conditional quantile at a risk level α as a function of conditioning variables, so that the quantile function can be written as the following:

$$q_{\alpha,t}(\cdot) = G_{\alpha,t}^{-1}(\cdot) := VaR_{1-\alpha,t}(\cdot), \quad (33)$$

where $q_{\alpha,t}(\cdot)$ is the quantile function and $G_{\alpha,t}^{-1}(\cdot)$ is the inverse of the cumulative distribution function of the risky asset returns associated to probability α .

This quantile function has to be an increasing function with respect to the risk level defined by α . Quantile estimators also have to fulfill the monotonicity property with α . Thus, a well specified quantile model is expected to provide estimators that behave like true quantiles and to increase for any value of parameters and conditioning variables according to the risk level α . Quantile functions can be positively combined and used to derive new quantile functions from other ones (Gouriéroux and Jasiak, 2008). But the VaR has recently been largely criticized for only reporting a quantile and thus disregarding outcomes beyond the quantile. Other risk measures, such as expected shortfall (denoted ES hereafter), overcome these weaknesses and are becoming widely used. The ES (also called conditional VaR or expected tail loss) is the average loss that a portfolio can suffer at a predefined horizon for a specified level of probability, such as (with the previous notations):

$$ES_{1-\alpha,t} = -\mathbb{E}[r_t^S | r_t^S \leq VaR_{1-\alpha,t}]. \quad (34)$$

If we consider that the distribution of r_t^S is known and continuous (see Acerbi and Tasche, 2002), we have:

$$ES_{1-\alpha,t} = -(1/\alpha) \int_{-\infty}^{VaR_{1-\alpha,t}} r_t^S f_t(r_t^S) dr_t^S, \quad (35)$$

where $f_t(\cdot)$ is the probability density function of r_t^S at time t .

ES was introduced for its ability to probe how serious the losses of a specific sample of worst cases are. The superiority of ES lies in revealing extreme losses and the severity of the tail losses more precisely. As shown by Artzner *et al.* (1999) and Acerbi and Tasche (2002), the $(1 - \alpha)$ -Expected Shortfall, defined as the average of the $\alpha\%$ worst losses of a portfolio, is a coherent risk measure. ES can be used as a basic object for obtaining new risk measures (see Acerbi, 2002; Acerbi and Tasche, 2002). It is in fact natural to think of the one-parameter family as a basis for expansions which define a larger class of risk measures. Following these works, the risky asset exposition in our setting is driven by a conditional multiple determined by the inverse of a shortfall constraint, precisely quantified by the ES. The hedging depends here, in fact, upon this risk measure. The conditional multiple thus allows the hedged portfolio to keep a constant exposure to the risk defined by the ES.

The inverse of the ES is a better way to model the multiple. Actually, to be protected in a PPI setting, the multiple of the insured portfolio has to stay inferior to the inverse of the underlying asset *maximum* period drawdown, at least until the portfolio manager can rebalance his position, as previously explained above. Thus, the multiple can here be modelled as such:

$$m_t = |ES_{1-\alpha,t}(r_t^S; \beta_t)|^{-1}, \quad (36)$$

where $ES_{1-\alpha,t}(r_t^S; \beta_t)$ is the ES of the conditional distribution of returns of the underlying asset r_t^S , corresponding to the periodic return of the risky part of the covered portfolio, and β_t is the time-varying vector of unknown parameters of the conditional ES model.

We have shown that the multiple can be modeled to manage a pre-determined level of risk measure. In the following section, we introduce a specific way to model this conditional multiple.

5.2. The DARE approach for the conditional multiple

We propose, hereafter, a way to aggregate well specified expectile and quantile models for obtaining a good estimation of the conditional multiple models based on risk measures such as VaR or ES.

We also have to provide a unified framework which enables us to aggregate quantile estimation

methods, involving the introduction hereafter of the dynamic autoregressive expectile (DARE) approach. After having introduced the DARE approach for the ES, we briefly review the main methods for estimating the ES.

We first recall the definition of expectiles, and their uses to estimate VaR and ES with conditional AutoRegressive expectile (CARE) class of models. Then, we introduce our DARE methodology to aggregate CARE models with a unified quantile estimation method.

As shown in Taylor (2008a, 2008b), both VaR and ES can be estimated with expectiles calculated by the minimization¹⁶ of:

$$\hat{\mu}_{\tau,t}^* = \underset{\hat{\mu}_{\tau,t} \in \mathbb{R}}{\text{ArgMin}} \left\{ \mathbb{E} \left[\left| \tau_t - \mathbb{1}_{\{r_t^S < \hat{\mu}_{\tau,t}\}} \right| (r_t^S - \hat{\mu}_{\tau,t})^2 \right] \right\}, \quad (37)$$

where the τ -expectile of r_t^S is estimated by the parameter $\hat{\mu}_{\tau,t}$ at time t .

Parameters of a conditional model for expectiles can be estimated by using the asymmetric least square (ALS) regression, which is the least square analogue for quantile regression:

$$\hat{\beta}_t^* = \underset{\hat{\beta}_t \in \mathbb{R}^n}{\text{ArgMin}} \left\{ \sum_{j=1}^t \left\{ \left| \tau_t - \mathbb{1}_{\{r_j < \hat{\mu}_{\tau,t}(r_j^S; \hat{\beta}_t)\}} \right| \times \left[r_j^S - \hat{\mu}_{\tau,t}(r_j^S; \hat{\beta}_t) \right]^2 \right\} \right\}, \quad (38)$$

where $\hat{\mu}_{\tau,t}$ is the estimation of $\mu_{\tau,t}$ and $\hat{\beta}_t$ is the estimated parameter vector of a specific expectile conditional model at time t .

We can use expectiles as quantile estimators, given that there is a corresponding α -quantile for each τ -expectile (see Efron, 1991; Jones, 1994; Abdous and Rémillard, 1995; Yao and Tong, 1996), such as (with the previous notations):

$$\tau_t = \frac{\alpha \times VaR_{1-\alpha,t} - \int_{-\infty}^{VaR_{1-\alpha,t}} r_t^S df_t(r_t^S)}{\mu_t - 2 \int_{-\infty}^{VaR_{1-\alpha,t}} r_t^S df_t(r_t^S) - (1 - 2\alpha) \times VaR_{1-\alpha,t}}, \quad (39)$$

where $f_t(\cdot)$ is the probability density function of returns at time t and μ_t is the expected return at time t .

Using an expectile as the relevant risk measure, the confidence level α is not *a priori* determined, but the parameter τ can be estimated so that the α -quantile and the τ -expectile match perfectly.

¹⁶For the explanation of such a result, see also the definition and general properties of the expectiles in Appendix C, in particular Equation (54).

Taylor (2008a and 2008b) and Kuan *et al.* (2009) extensively study the link between expectiles and ES, that leads to (with the previous notations):

$$ES_{1-\alpha,t} = \left[1 + \tau_t (1 - 2\tau_t)^{-1} \alpha^{-1}\right] \mu_{\tau,t} (r_t^S; \beta_t). \quad (40)$$

Thus, the conditional ES for a given value of α is proportional to the conditional γ -quantile model, which is estimated by the τ -expectile. Additionally, the quantile estimations can be linearly combined through the dynamic additive quantile (DAQ) model proposed by Gouriéroux and Jasiak (2008). This class of dynamic quantile models is defined by:

$$VaR_{1-\alpha,t} = \sum_{k=1}^K a_k (r_{t-1}^S, y_{t-1}, \beta_{k,t-1}) \times VaR_{1-\alpha,t}^{(k)} (\beta_{k,t-1}), \quad (41)$$

where the $VaR_{1-\alpha,t}^{(k)}(\cdot)$ are some path-independent quantile functions, with $k = [1, 2, \dots, K]$ a finite number, and the $a_k(\cdot)$ are non-negative functions of past returns and of past exogenous variables, denoted respectively r_{t-1}^S and y_{t-1} .

Thus, the DAQ model can use different quantile functions to model a given one. We can also combine these functions into a multi-quantile method (see Kim *et al.*, 2008) to increase the accuracy of the conditional model. Actually, every quantile function can be extended to define a simple class of parametric dynamic quantile models.

The ES can therefore be expressed as a combination of quantiles whose associated probabilities are defined through the estimation of equation (38), such as:

$$ES_{1-\alpha,t} = \underset{[1 \times 1]}{[\mathbf{W}]'_{\alpha,t}} \underset{[K \times 1]}{[\mathbf{VaR}]_{1-\gamma,t}}, \quad (42)$$

with:

$$\begin{cases} \mathbf{W}_{\alpha,t} = \left[w_{1,t} \times b_{1,t} & w_{2,t} \times b_{2,t} & \dots & w_{K,t} \times b_{K,t} \right]' \\ \mathbf{VaR}_{1-\gamma,t} = \left[VaR_{1-\gamma_1,t}^{(1)} & VaR_{1-\gamma_2,t}^{(2)} & \dots & VaR_{1-\gamma_K,t}^{(K)} \right]' \end{cases},$$

and:

$$\begin{cases} ES_{1-\alpha,t}^{(k)} = b_{k,t} VaR_{1-\gamma_k,t}^{(k)} (\mu_{\tau_k}, \beta_{k,t}) \\ b_{k,t} = \left[1 + \tau_{k,t} (1 - 2\tau_{k,t})^{-1} \alpha^{-1}\right] \\ \sum_{k=1}^K w_{k,t} = 1, \end{cases}$$

where the $VaR_{1-\gamma_k,t}^{(k)}(\cdot)$ are a finite number of quantile functions associated with probability γ_k at time t and related to the model k , with $k = [1, 2, \dots, K]$.

The estimation of the $(1-\alpha)$ -ES can be found by linearly aggregating K quantile functions and estimating the right correspondence between the probability α associated with the ES of each

model (denoted $ES_{1-\alpha}^{(k)}$) and the probability γ_k associated with each quantile function (thanks to the τ -expectile defined by the equation 42 for each model k involved).

Keeping in mind that using this DARE model, the ES can be expressed as a combination of quantile functions, the quantile hedging approach used in the previous section also justifies the use of the ES to model the conditional multiple (only the probability level is changed but that can be precisely estimated). We review, hereafter, the main quantile models which can be used within the DARE approach for the conditional multiple.

5.3. Estimation methods for the conditional multiple

One goal of this section is to aggregate several conditional estimation methods to get the target multiple within the cushioned portfolio framework. Within our framework using a DARE ES model, the multiple can be modeled by the inverse of the expected shortfall conditional on the distribution of the asset return estimated by a combination of several quantiles. We briefly present below the main VaR models and estimation methods used in this article for estimating the model of a multiple depending on the expected shortfall expressed as a combination of quantile functions.

The main approaches for dynamic quantile estimations can be classified into three main categories: non-parametric, parametric and semi-parametric methods (Engle and Manganelli, 1999; Jorion, 2006; Nieto and Ruiz, 2008).

We first recall the most popular non-parametric method, which is based on the so-called historical approach. The latter only requires mild distributional assumptions and it implies the estimation of the VaR as the quantile of the empirical distribution of historical returns from a moving window of the most recent period (the “naive” approach). The main problem of this non-parametric approach is the way of defining the width of this window: few observations will lead to an important sampling error, whereas too many observations will slow down estimates when reacting to changes in the true distribution of financial returns. Moreover, it only takes realized past returns on a predefined data sample into consideration. This approach cannot model tail returns that have never been seen.

The second class of estimation methods is the parametric approach. Assuming that returns follow a specific probability distribution (as for example a normal or a Student t -distribution), parametric approaches have the advantage of modelling tail returns that have never been realized according to the return distribution that has been chosen. The parameters of the distribution are

specified and the risk measure is deduced thanks to the quantile of the estimated distribution. Consequently, the risk measure mainly depends on estimation of the parameters and on the shape of the chosen distribution. At any rate, parametric approaches are affected by the assumed return probability distribution specification, which does not provide a perfect representation of the main stylized facts of financial series all the time.

The third class of estimation approach groups semi-parametric estimation techniques, which combines the two previous approaches. Cornish-Fisher, *RiskMetrics* and quantile regression belong to this family. First, the Cornish-Fisher approach (1937) is based on statistical expansions of the return probability density function around a reference density, which integrate the higher-order moments of the distribution for computing corrected quantiles (Favre and Galeano, 2002). Secondly, the *RiskMetrics* model can be considered as another semi-parametric approach. It assumes that asset returns follow a normal distribution with a volatility estimated by the exponential weighted moving average method, which corresponds to an integrated GARCH model, obtained by using historical data. Many other volatility estimation methods, belonging for example to the GARCH family, have been already tested in the literature (Engle and Rangel, 2008). These approaches are also called conditional normal methods, as we use a Gaussian distribution to describe the returns associated to a variance estimated by conditional methods. This approach can also be adapted to other distribution assumptions.

Lastly, the main semi-parametric approach presented in this article is based on quantile regression methods. These estimation methods only require mild distributional assumptions. They provide a convenient approach for estimating conditional quantiles. They also have the great virtue of robustness to distributional assumptions and make no prior assumption about the symmetry of the innovation process. Within this family, Engle and Manganelli (2004) directly define the dynamics of risk by the mean of an autoregressive model involving the lagged-VaR and the lagged values of endogenous variables, in a model named CAViaR. They present four CAViaR specifications: a model with a symmetric absolute value, an asymmetric slope, an indirect GARCH(1,1) and an adaptive form, denoted respectively: $VaR_{1-\alpha,t}^{SAV}(r_{t-1}^S, \beta)$, $VaR_{1-\alpha,t}^{AS}(r_{t-1}^S, \beta)$, $VaR_{1-\alpha,t}^{IG}(r_{t-1}^S, \beta)$, $VaR_{1-\alpha,t}^A(r_{t-1}^S, \beta)$ where:

$$\left\{ \begin{array}{l} VaR_{1-\alpha,t}^{SAV}(r_{t-1}^S, \beta) = \beta_1 + \beta_2 \times VaR_{1-\alpha,t-1}^{SAV}(r_{t-1}^S, \beta) + \beta_3 \times |r_{t-1}^S| \\ VaR_{1-\alpha,t}^{AS}(r_{t-1}^S, \beta) = \beta_1 + \beta_2 \times VaR_{1-\alpha,t-1}^{AS}(r_{t-1}^S, \beta) \\ \quad + \beta_3 \times \max(0, r_{t-1}^S) + \beta_4 \times [-\min(0, r_{t-1}^S)] \\ VaR_{1-\alpha,t}^{IG}(r_{t-1}^S, \beta) = \beta_1 + \beta_2 \times [VaR_{1-\alpha,t-1}^{IG}(r_{t-1}^S, \beta)]^2 + \beta_3 \times r_{t-1}^S{}^2 \\ VaR_{1-\alpha,t}^A(r_{t-1}^S, \beta) = VaR_{1-\alpha,t-1}^A(r_{t-1}^S, \beta) - \alpha + \beta_1 [1 + \\ \quad \exp\{.5 \times [r_{t-1}^S - VaR_{1-\alpha,t-1}^A(r_{t-1}^S, \beta)]\}]^{-1}, \end{array} \right. \quad (43)$$

with β_i the specific CAViaR model parameters.

In the symmetric absolute value CAViaR specification, the VaR reacts symmetrically to positive or negative returns. The asymmetric slope CAViaR varies differently according to positive or negative return realizations. We notice also that the indirect GARCH(1,1) CAViaR model is correctly specified if the underlying data are generated by a GARCH(1,1) model with an independently and identically distributed residual. The intuition associated to the adaptive specification is the following: as long as the daily return is not inferior to the VaR estimation, the VaR can increase by a small amount. On the contrary, when VaR is exceeded, the portfolio has to be more prudent at the following date. Thus, this model adapts itself to its past errors and it reduces the probability for the VaR to be consecutively under-estimated.

A conditional quantile function is well defined if its parameters can be considered as quantile functions too. In fact, CAViaR models weight different baseline quantile functions at each date and they can therefore be considered as quantile functions. Adding a non-negative autoregressive component of VaR, the CAViaR conditional quantile function becomes a linear combination of quantile functions weighted by non-negative coefficients. Thus, the CAViaR model satisfies the properties of a quantile function, even if the indirect GARCH(1,1) or adaptive specifications do not satisfy the monotonicity property. CAViaR model parameters are estimated by using the quantile regression minimization (denoted QR Sum) presented by Koenker and Bassett (1978):

$$\hat{\beta}^* = \underset{\hat{\beta} \in \mathbb{R}^n}{ArgMin} \left\{ \sum_{t=1}^T \left\{ \left[r_t^S + \widehat{VaR}_{1-\alpha,t}(r_{t-1}^S, \hat{\beta}) \right] \times \left[\alpha - \mathbb{1}_{\{r_t^S < -\widehat{VaR}_{1-\alpha,t}(r_{t-1}^S, \hat{\beta})\}} \right] \right\} \right\}, \quad (44)$$

with $\widehat{VaR}_{1-\alpha,t}(r_{t-1}^S, \hat{\beta})$ is the VaR estimation according to CAViaR models (see equation 43) and $\hat{\beta}$ is a vector of estimated parameters.

Where the quantile model is linear, this minimization can be formulated as a linear program for which the dual problem is conveniently solved. Koenker and Bassett (1978) show that the resulting quantile estimator, $\widehat{VaR}_{1-\alpha,t}$, essentially partitions the r_t^S observations so that the proportion lower than the corresponding quantile estimate is α .

The optimization algorithm that can be used to estimate the parameters of the quantile models according to quantile regression objective function (*i.e.* the QR Sum) is the procedure proposed by Engle and Manganelli (2004). This procedure used to estimate CAViaR or CARE models (see also Taylor 2008a and 2008b) is to generate vectors of parameters from a uniform random number generator between 0 and 1, or between -1 and 0 (depending on the appropriate sign of the parameters). For each of these vectors, the quantile regression sum of squared errors is then evaluated. The ten vectors that produced the lowest values for the function are used as initial values in a quasi-Newton algorithm. The QR Sum is calculated for each of the ten resulting vectors, and the one which produce the lowest value of the QR Sum is chosen as the final starting parameter vector.

CAViaR models only provide a model for the quantile, but computing by the corresponding ES is not straightforward. The ES can be conveniently computed using conditional expectile models. Following Taylor (2008a), we use CAViaR models to build CARE models, replacing the conditional VaR by the conditional expectile in equation (43). We estimate CARE model parameters using the same procedure as for CAViaR models, except that the quantile regression sum is replaced by the asymmetric least square sum (see equation 38). The τ -expectile, for which the proportion α of in-sample observations are below the expectile, is the estimator of the α -quantile (see Efron, 1991; Granger and Sin, 2000). According to equation (40), the ES is then directly obtained.

In the following empirical illustration, we aggregate CARE-based models with respect to equation (36) in order to estimate conditional time-varying multiples for a TIPP-based PPI Strategy. This conditional multiple estimation method is performed and the performance of its PPI is then compared to classical unconditional strategies with various levels of fixed multiples.

6. The DARE Approach for PPI: Backtest Results

We first describe in the following the database and the main properties of the conditional multiple model. We then compare the performances of a cushioned portfolio using traditional unconditional multiples, and the DARE conditional multiple¹⁷.

¹⁷The comparison of TIPP strategies could be more complex than the comparison of CPPI strategies, since the final guarantee depends on the whole path of the portfolio value dynamics. However, note that all TIPP strategies not only guarantee 90% of the past *maximum* value of the portfolio but also at least the same floor that usually corresponds to 100% of the initial portfolio value V_0 (see formula 3). Therefore, they are still comparable through this same *minimum* capital guarantee.

6.1. A DARE risk measure and the related conditional multiples

The sample period used in our study consists of 61 years of daily quotes of the Dow Jones Index, from the 2nd of January 1950 to the 1st of March 2011. This period contains 15,057 daily returns. We use a rolling window of 1,000 returns to dynamically estimate method parameters.

The multiple depends at each date on the inverse of the 99% expected shortfall (denoted $ES_{99\%}$). Using this model, the portfolio $ES_{99\%}$ is controlled and extreme returns are taken into consideration¹⁸. As mentioned in the previous section, the DARE approach can be used to estimate the ES and then the conditional multiple. Although there are many VaR estimation methods, we restrict our empirical study to commonly used main benchmark methods and aggregate them into a DARE approach for reducing the potential model error. Eight methods of ES calculation are hereafter combined: one non-parametric method using the “naive” historical simulation approach, three methods based on distributional assumptions, and four are ES CARE based on conditional autoregressive specifications (CAViAR). More precisely, the normal ES is built on an assumption of normality of returns. It is computed with the empirical mean and standard deviation of the returns of the in-sample period. The *RiskMetrics* ES is a standard for practitioners: an exponential moving-average is used to forecast the volatility and to compute the ES assuming a Gaussian distribution. Another very popular method based on volatility forecasts is presented as the GARCH ES. To compute it, we implement a GARCH(1,1) model¹⁹. We estimate the model parameters using the *maximum* likelihood based on a Student *t*-distribution of standardized residuals. Using the procedure proposed by Engle and Manganelli (2004) with the ALS function (see sub-section 5.3 for a brief presentation of the implementation method), we estimate the four CARE models (the symmetric absolute value, the asymmetric slope, the IGARCH(1,1) and the adaptive). Figure 3 presents the data used for the empirical analysis (daily return on the DJI), as well as the time-variation of the resulting DARE ES.

The $ES_{99\%}$ models are now aggregated and used for estimating the DARE ES-related conditional multiple²⁰ represented in Figure 4. As also shown in Figure 5 and Table 1 where the density of the multiple is provided, with several over-weights on specific multiples over the sample, the estimations of the conditional multiple spread between 1 and 13: almost 80% of the multiples estimates are between 3.03 and 8.37, 90% are inferior to 8.37 and 90% are superior to 3.03, whilst the mean

¹⁸The probability of 1% associated to the ES is chosen in order to focus on extremes and having at the same time enough points for recovering good estimations. It is also a standard in Risk Management for defining extreme loss. We test several higher probability levels in the following.

¹⁹Our choice of the AGARCH(1,1) specification is based on the analysis of the initial in-sample period of 1,000 returns and on the popularity of this order for GARCH models.

²⁰

Multiples have been limited to an extreme value of 13 for the sake of realism.

conditional multiple estimations is 5.84.

- Please insert Figure 3 somewhere here -
- Please insert Figure 4 somewhere here -
- Please insert Figure 5 somewhere here -
- Please insert Table 1 somewhere here -

This is compatible with multiple values used in the market by practitioners (generally between 3 and 8) and by the literature (under 13, see for example Annaert *et al.*, 2009).²¹ We can also notice some extreme values of multiples, either very protective (around a value of 1) when markets are highly turbulent and other very aggressive ones (9 or so) when all market conditions are fine, which illustrates the main advantage of the method: flexibility.

An illustration of the “multiple-change durations” is also provided within Figure 6 and Table 2, corresponding to times (in numbers of days) separating two dates where the multiple had to be changed and portfolio rebalanced. We have estimated both the empirical probability distribution function and a theoretical one corresponding to the gap risk durations²² defined by equation (25), also considering several ranges of risk levels (defined by given tolerance rates denoted ρ) of conditional multiples determined within the DARE approach.²³

- Please insert Figure 6 -
- Please insert Table 2 -

We can here note that, even if this framework is still simple, the empirical and approximate theoretical distributions have somehow comparable characteristics in terms of their shapes as perceived using the first moments of durations (see Figure 6 and Table 2). In other words, despite the simple assumptions of the theoretical framework (for instance, absence of autoregressivity in the shocks; Engle and Russel, 1998; Bauwens and Giot, 2003), this approach seems to be realistic enough for evaluating the gap risk of proportion portfolio insurance strategies. But let us turn now to evaluation of the performance of the time-varying insurance strategy.

²¹It happens that none of the various conditional or unconditional strategies presented here experienced a floor violation in the sample.

²²For a null tolerance, multiple-change durations should be very close to gap risk ones, since the multiple must change as soon as the probability of not respecting the guarantee is higher than the confidence threshold. When there is a tolerance, durations may differ (with a longer multiple-change duration), since in this case there is no immediate answer to a potential gap risk occurrence.

²³When the sequence $(\Theta_t)_{t \in \mathbb{R}^+}$ corresponds to times at which the current multiple is no longer in the tolerance corridor, explicit solutions can be provided for the Laplace transform of the distribution of time interarrivals $(\Theta_{t+1} - \Theta_t)$; see Mkaouar and Prigent (2010b). However, relation (17) is more demanding and simulations are necessary to get numerical values of gap risks.

6.2. Conditional versus unconditional portfolio insurance performances

Based on previous estimates of the conditional multiple, we can now evaluate the interest of our approach, comparing performances of the strategies using traditional unconditional and conditional DARE multiples of the DJIA. TIPP performances, based on traditional unconditional multiples and the DARE approach, are analyzed on 1 to 15 year horizons from 1950²⁴, and results are summarized in Figure 7.

- Please insert Figure 7 -

If we first compare, according to the Sharpe ratios, the DARE-conditional approach to the unconditional approaches for various levels of fixed multiples, it happens that the performance of the DARE-TIPP is quite stable with respect to horizons. With no *ex ante* definition of the level of risk, the DARE-TIPP strategy exhibits comparable performance to a multiple of 5 or 6 (compared with the best returns for a multiple equal to 10 to 12) and allows the investor to reach the best Sharpe ratios. When we more precisely look at conditional differences in performance (on TIPP strategies on five year overlapping horizons, in the period 03/01/1956 to 03/01/2011), we observe that the DARE-TIPP beats a multiple 7 (respectively 8) constant strategy in most cases, and, more interestingly, in 75% of the cases, when the market is down or up significantly (either with a return on the DJI inferior to 2.75% *per year* or superior to 8.05% *per year*), then the mean out-performance compared to a multiple 7 (respectively 8) constant strategy is equal to .49% *per year* (.58% *per year*). In other words, and as we could have expected before doing the test, when the market is “normal”, the DARE-TIPP normally gives an output very similar to a *medium* leverage constant strategy (as shown by the distribution of the time-varying multiple, which is most of the time at a reasonable value around 7 – see the previous Figure 5), but it reveals its advantage when markets are extreme. Consequently, we did not find here a cost to the feature of flexibility of the strategy.

Our strand of reasoning regarding the comparison of previous results is until now based on Sharpe ratios. However, since it is a mean-variance-based performance measure and that the underlying returns on the various strategies are non-Gaussian²⁵, we need to interpret these first results with caution and try to complement this approach using more general *criteria*. First, if the densities of returns are not normal, the volatility might not be the relevant risk metric; secondly, if some investors value skewness positively, a mean-variance based measure over-rates strategies that

²⁴To take the path dependency of the results of PPI strategies into account, we have here computed results of every possible strategie (taking every start and end dates between 1950 and 2011).

²⁵The assumption of normality of returns is rejected, for every strategy tested, at a 99.99% confidence level using usual normality tests (Jarque-Bera, Lilliefors and Anderson-Darling) not reported here for reasons of space.

reduce skewness (“value strategies”) and under-rates strategies that buy skewness (*momentum* or portfolio insurance strategies). We thus complement our first results and generalize the comparison using other *criteria* as in Table 3.

- Please insert Table 3 -

As a complement, we also provide, in Table 3, Sortino ratios (using as the reference threshold the mean of the returns), *Omega* measures (using an arbitrary reference threshold of zero) and the third-order *Kappa* measures (to take account of the skewness of the studied strategies). These measures generalize the Sharpe ratio in the case of non-normal distributions. Indeed, the Sortino ratio is a modified Sharpe ratio, the volatility being replaced by the semi-volatility in the denominator (Sortino and van der Meer, 1991). In case of asymmetry in the return distribution, the Sortino ratio reduces the excess return by a risk measure more linked to the investor potential loss (return inferior to its mean value). The *Omega* performance measure, introduced by Keating and Shadwick (2002), is defined as the ratio between the surface under the return cumulative distribution function below and the one above a given return threshold. A higher *Omega* measure means that the return density is more pronounced above the threshold than below. In fact, this ratio takes the skewness of returns into consideration, but also the *kurtosis* of returns of the strategy under study, through a synthetic measure using main moment characteristics of the return distribution. Finally, the performance measure *Kappa*, introduced by Kaplan and Knowles (2004), uses a more general risk measure; the Sortino ratio is equivalent to the second-order *Kappa* measure, and the *Omega* measure is equal to unity *plus* the first-order *Kappa* measure. The *Kappa* measure associated to order n is the ratio between the excess return at a given threshold, and the lower partial moment (LPM) of order n with respect to the same threshold.²⁶

According to Table 3, the *ex ante* conditional DARE-based cushioned portfolio is indeed always in the first half of the ranking of portfolios (determined *ex post*) according to Sharpe as mentioned earlier, but also according to Sortino and *Kappa* performance *criteria* (and in the first part of the ranking according to *Omega*). These performance measures are in line²⁷ with unconditional strategies associated to a multiple between 5 and 8.

However, values of cushioned portfolio are also path-dependent. The performances of these guaranteed portfolios, determined by using different estimation methods of the conditional multiple, are unfortunately not so easy to compare. Actually, the performance of the insured portfolio

²⁶See Bertrand and Prigent (2011) and the Appendix for a general definition of these measures.

²⁷Except for the one-year TIPP where the ranking is inconclusive, due to a limited cushion (explained by the short investment horizon) and by a high leverage bias (since portfolios with high multiples are never so penalized by a long monetarization period).

depends more on its start date, and on the investment horizon, than on the estimation of the conditional multiple. Moreover, some estimation methods are more adapted to a certain period. To investigate whether these different estimation methods lead to significantly different performances (for a long-term analysis), we also perform the same comparisons using extensive stationary bootstrap simulations (Politis and Romano, 1994) on sample returns. A stationary bootstrap allows us, in fact, to partially keep the dependence structure between the return series under study, whilst testing the robustness of our approach.²⁸ We thus compare portfolios managed with the DARE conditional multiple and traditional unconditional PPI with a fixed multiple in Table 4 over a five-year horizon²⁹.

- Please insert Table 4 -

The portfolio managed with a conditional DARE multiple is here first, according to the Sharpe ratio, and it also belongs to the first distribution quartile for every other performance measure. In other words, mitigating the impact of the starting date actually reinforces our previous conclusions drawn from a general test starting in 1950 (see previous Table 4). The portfolio managed with a DARE conditional multiple is thus always amongst the best PPIs with unconditional multiples. Moreover the DARE conditional multiple PPI provides more stable results through long investment horizons.

7. Conclusion

In this paper, we extend the standard CPPI method by introducing a conditional multiple, for which we provide and analyze various upper bounds. Such bounds are mainly determined from several quantile and expected shortfall local conditions depending on the hypotheses on the data generating process. In a first step, using different parametric models, we indicate the corresponding limits on the multiple and give analytical results of gap risks for various underlying processes. In a

²⁸Other classic simulations methods such as simple bootstrap and surrogate data procedure (Schreiber and Schmitz, 2000) have also been employed in preliminary works and lead to similar results and analogous conclusions. All results converge in the same way: the conditional dynamic strategy with constant-risk exposure most of the time dominates the traditional constant-asset exposure unconditional strategies in terms of return *per* unit of risk, combining a return close to the one of the best unconditional strategy, with a volatility amongst the lowest. While the risk of the conditional strategy is *ex ante* defined (with an almost constant risk exposure), it, however, appears *ex post* among the best portfolio strategies.

²⁹The strategies characteristics are calculated using 1,500,000 simulations of daily returns based on a stationary bootstrap technique: artificial series are composed with Dow Jones and day-to-day interest rate (daily US Fed Funds rate) random blocks of daily returns determined using a geometric probability law defined by a parameter of .9.

second step, we adopt an empirical approach, based on VaR and ES estimations, using real market data with a set of complex characteristics. In this framework, the multiple is modeled as a function of the ES determined by a combination of quantile functions. This has the advantage of dealing with a more robust and flexible multiple quantile estimations at the same time, in a coherent risk measure framework. The aggregated model for the conditional multiple proposed in this article, allows us to guarantee the insured floor according to market evolutions, on a large post sample period. This method provides a rigorous framework to determine the conditional multiple, which is the main crucial parameter of a cushioned portfolio.

To compute the conditional multiple, according to this model, appropriate methods to estimate the quantile of the risky asset return are combined to determine the ES. A dynamic auto-regressive expectile approach for the TIPP conditional multiple is introduced. According to a large set of performance *criteria*, it appears to be high in the ranking compared to most traditional unconditional methods. This DARE approach for the conditional multiple diversifies the risk model associated with extreme risk measures. Moreover, it allows the investor not to fix *ex ante* the unconditional risk limit, whilst targeting to *ex post* deliver the highest return under a *minimum* guarantee. Finally, we also illustrate a similar theoretical distribution of the duration associated with the gap risk thanks to a time changes framework, providing useful characteristics about probabilistic approaches of the multiple.

A complementary extension would be to make the framework more complex, taking the auto-regressivity of shocks into consideration as in autoregressive conditional duration models (Engle and Russel, 1998; Bauwens and Giot, 2003). Another research development would be to propose a management strategy explicitly integrating the duration of the gap risk. But this is left for further research.

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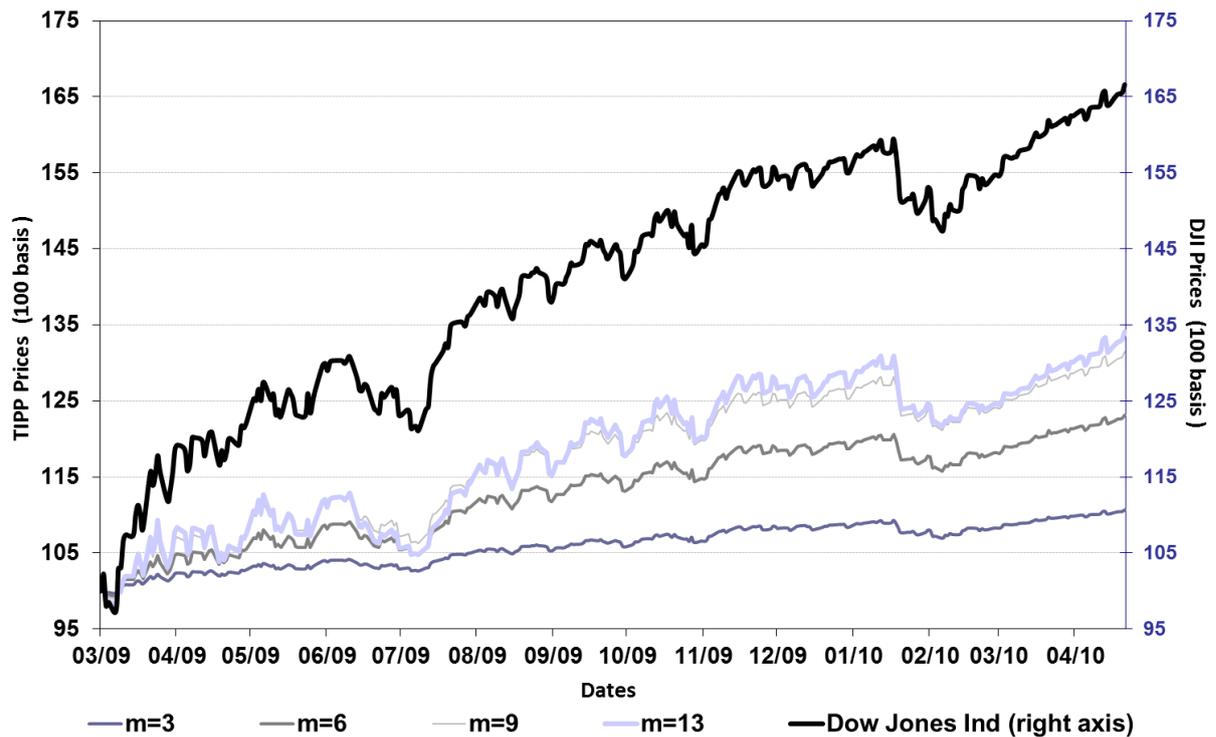
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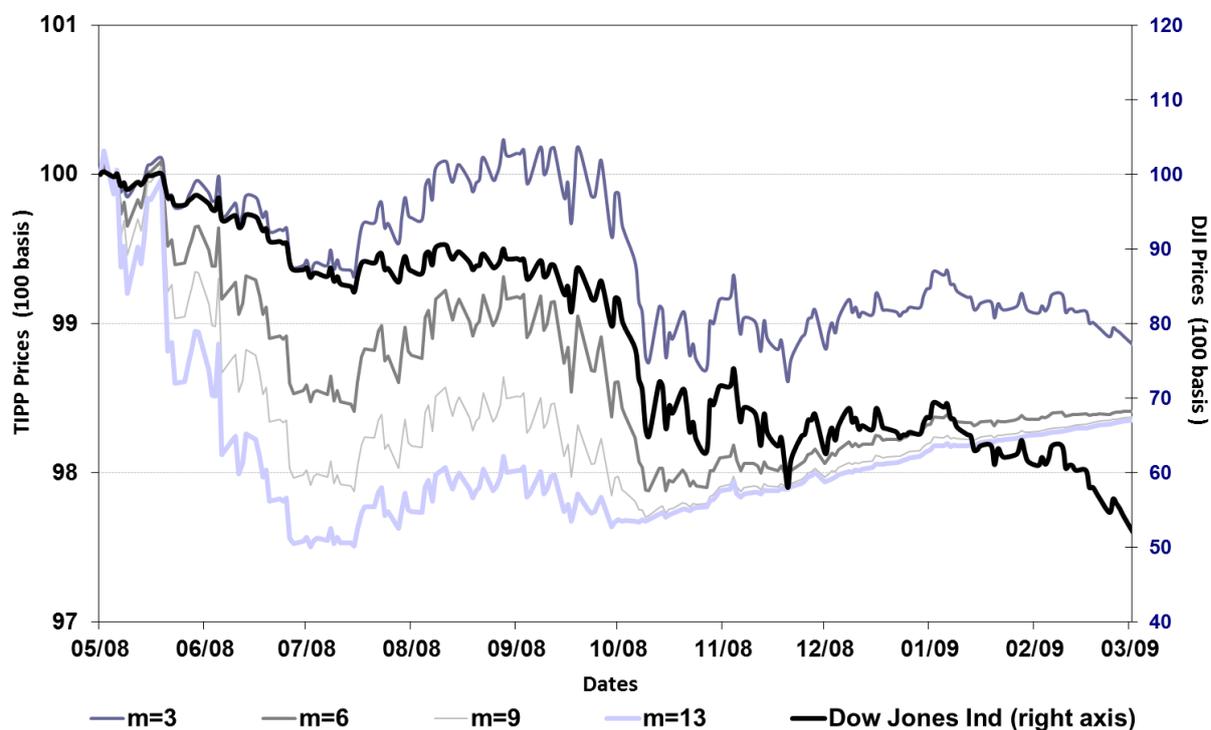
Appendix A: Figures

Figure 1: Illustration of TIPP Portfolios in a Bull Market



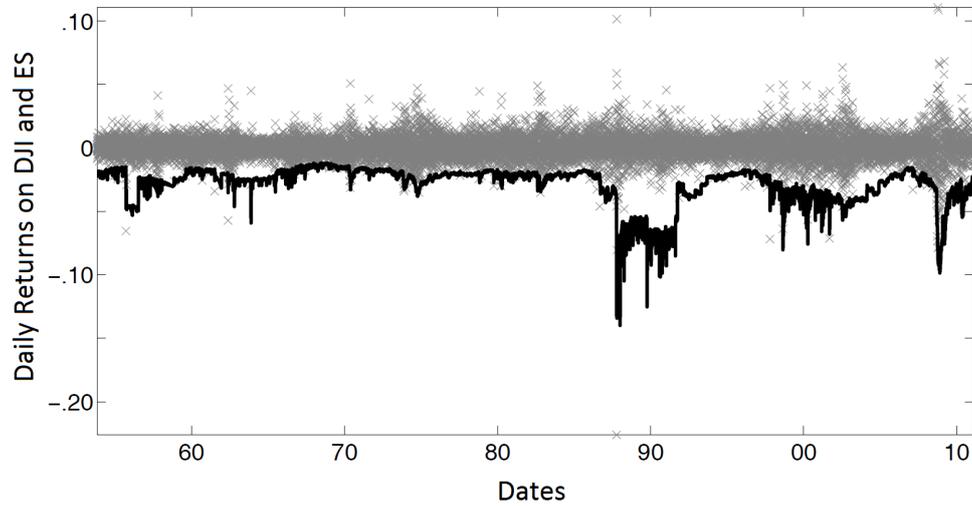
Source: *Bloomberg*, daily returns of the Dow Jones Index from 03/01/2009 to 05/01/2010; computations by the authors. The bold black line associated to the right axis represents the underlying risky asset: the Dow Jones Index. The four other lines represent, respectively, TIPP strategies managed with a multiple of 3, 6, 9 and 13. For all these TIPP strategies, the floor is continuously adjusted to preserve 90% of the highest portfolio value; the interest rate series used in this illustration is the daily US Fed Funds rate.

Figure 2: Illustration of TIPP Portfolios in a Bear Market



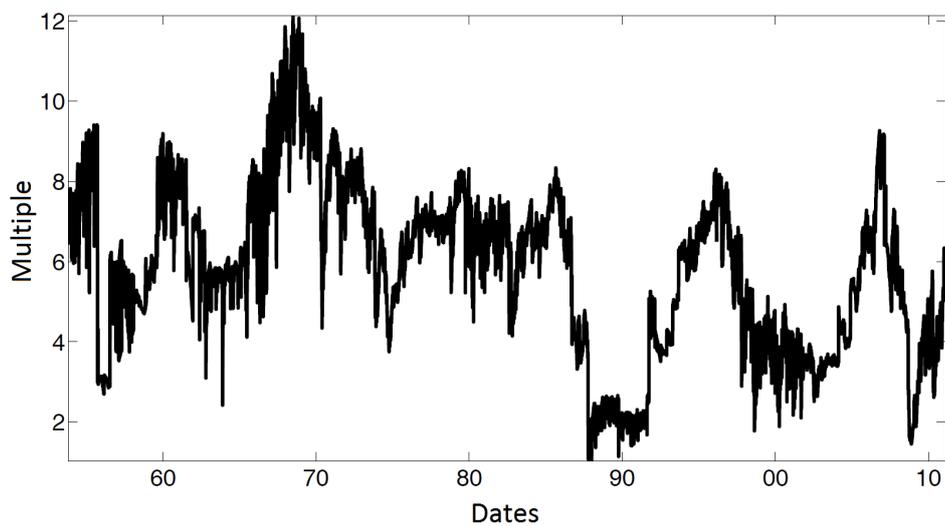
Source: *Bloomberg*, daily returns of the Dow Jones Index from 04/30/2008 to 05/01/2009; computations by the authors. The bold black line associated to the right axis represents the underlying risky asset: the Dow Jones Index. The four other lines represent, respectively, TIPP strategies managed with a multiple of 13, 9, 6 and 3. For all these TIPP strategies, the floor is continuously adjusted to preserve 90% of the highest portfolio value; the interest rate series used in this illustration is the daily US Fed Funds rate.

Figure 3: DARE ES_{99%} on the Dow Jones Index



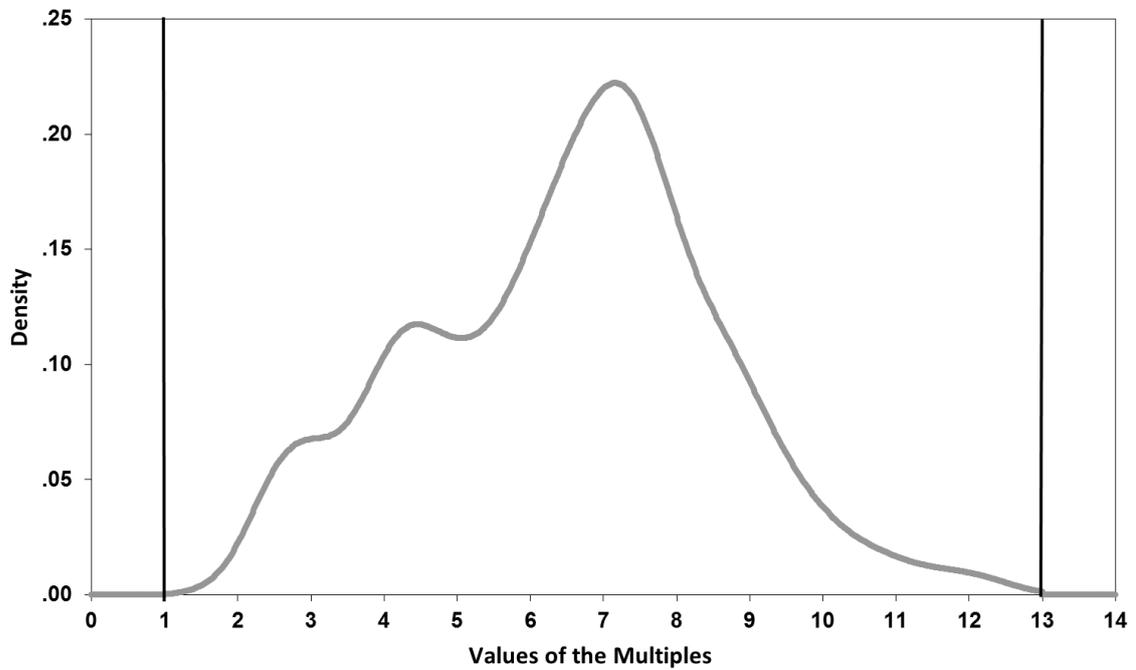
Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. The figure shows daily returns (grey stars) and the Dynamic Expected Shortfall computed at the 99%, with the Dynamic Autoregressive Expectile model (black bold line).

Figure 4: Multiple based on the DARE ES_{99%}



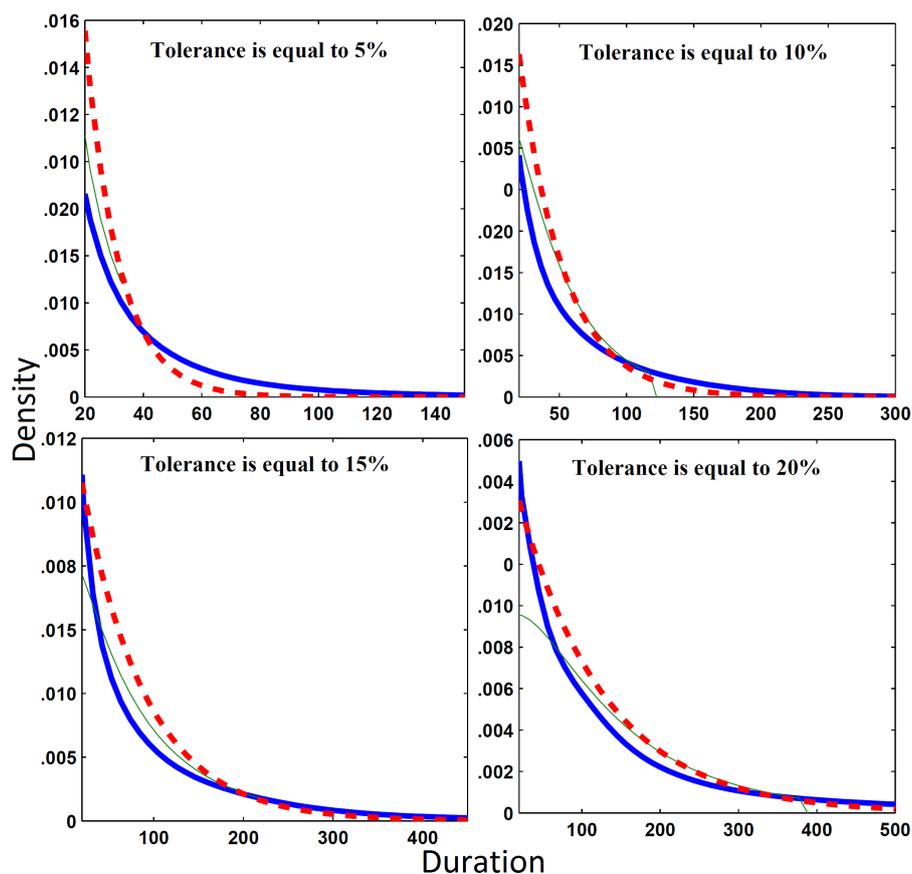
Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. Conditional multiple is dynamically estimated as the ES changes.

Figure 5: Conditional Multiples Empirical Frequencies based on DARE-ES_{99%} Estimates



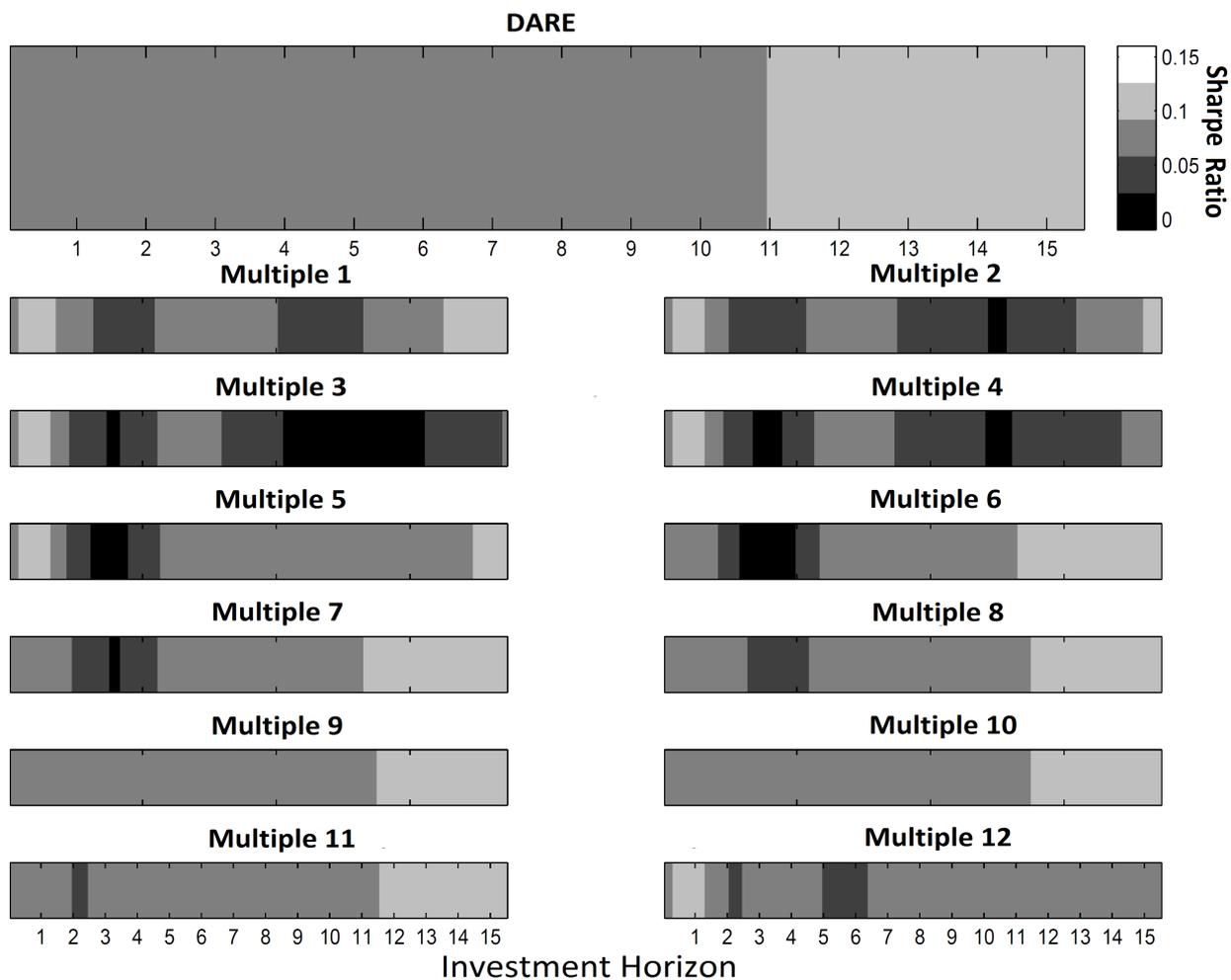
Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. Conditional multiples are dynamically estimated. The kernel of the conditional multiple density is estimated using the cross-validation *criterion* (Silverman, 1986). The figure plots the “DARE Multiple” density (*i.e.* for a multiple based on the DARE-modeled Expected Shortfall).

Figure 6: Estimated Densities of “Multiple-change Durations” of DARE TIPP Cushioned Portfolio Strategies determined by Four Tolerance Levels



Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. Durations are estimated from 01/02/1950 to 03/01/2011. The four figures plot densities of the “Multiple-change durations” of a DARE TIPP cushioned portfolio with several tolerance levels (denoted ρ) respectively equal to 5% (top left figure), 10% (top right figure), 15% (bottom left figure) and 20% (bottom right figure). For each plot, the bold continuous line represents the empirical density of durations, the dashed line a fitted exponential distribution (as an approximate of the “Gap-risk durations” density) and the thin continuous line a fitted normal inverse Gaussian. The empirical densities are estimated using the cross-validation *criterion* (Silverman, 1986). The probability distribution functions are estimated using respectively 3,330, 1,458, 850 and 496 durations associated to 5%, 10%, 15% and 20% tolerance levels.

Figure 7: Sharpe Ratios of DARE and Unconditional TIPP Cushioned Portfolio Strategy on a One to Fifteen-year Basis



Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computation by the authors. TIPP strategies are evaluated from 01/02/1950 to 03/01/2011. The figures plot the Sharpe ratios (variations of grey) of DARE and unconditional (with a fixed multiple from 1 to 12) TIPP cushioned portfolio strategies (y-axis) on a one to fifteen-year basis (x-axis).

Appendix B: Tables

Table 1: Moments and Quantiles for the Conditional Multiples based on DARE Estimates

	1%	5%	10%	25%	50%	75%	90%	95%	99%
Conditional Multiples Quantiles	1.76	2.20	3.03	4.23	6.04	7.16	8.37	9.11	10.92

	Mean	Variance	Skewness	Kurtosis
Conditional Multiples Moments	5.84	4.28	0.03	2.65

Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. Conditional multiples are dynamically estimated (*i.e.* for a multiple based on the DARE modeled Expected Shortfall).

Table 2: Empirical Moments of the “Multiple-change Durations” within the DARE Approach

	Mean	Variance	Skewness	<i>Kurtosis</i>
Tolerance of 5%				
Empirical duration	11.60	335.99	3.31	16.93
Exponential-shape estimation	11.60	134.33	2.00	9.00
Tolerance of 10%				
Empirical duration	33.14	1,762.32	1.91	6.50
Exponential-shape estimation	33.14	1,098.97	2.00	9.00
Tolerance of 15%				
Empirical duration	71.37	9,397.44	2.52	10.91
Exponential-shape estimation	71.37	5,092.23	2.00	9.00
Tolerance of 20%				
Empirical duration	111.11	18,303.92	2.00	7.08
Exponential-shape estimation	111.11	12,345.31	2.00	9.00

Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computations by the authors. Multiple-change durations are estimated from 01/02/1950 to 03/01/2011. Moments are estimated using durations associated to 5%, 10%, 15% and 20% tolerance levels ρ (see Figure 6 and equation 28).

Table 3: DARE and Unconditional TIPP Cushioned Portfolio Strategy Characteristics on the Dow Jones Index from 1950 to 2011 on a Five-year Basis

	Return	Volatility	VaR99%	ES99%	Skewness	Kurtosis	Sharpe	Sortino	Omega	Kappa
Risky Asset	6.60%	15.28%	-2.46%	-3.30%	-0.78	31.81	.08	.04	1.10	.02
DARE	5.59%	3.23%	-0.57%	-0.83%	-1.36	56.19	.07	.16	1.50	.07
Multiple 1	5.39%	.58%	-0.08%	-0.11%	1.32	19.57	.07	1.34	6.67	.68
Multiple 2	5.43%	1.10%	-0.18%	-0.24%	.92	24.89	.08	.55	2.83	.30
Multiple 3	5.48%	1.63%	-0.28%	-0.38%	.71	30.14	.08	.35	2.06	.18
Multiple 4	5.52%	2.17%	-0.38%	-0.54%	.54	35.73	.08	.25	1.74	.13
Multiple 5	5.57%	2.75%	-0.48%	-0.70%	.37	41.22	.08	.19	1.58	.10
Multiple 6	5.61%	3.35%	-0.60%	-0.88%	.18	46.29	.08	.16	1.47	.08
Multiple 7	5.65%	3.99%	-0.73%	-1.08%	-0.05	50.88	.08	.13	1.40	.07
Multiple 8	5.68%	4.66%	-0.85%	-1.29%	-0.31	55.22	.07	.11	1.35	.05
Multiple 10	5.72%	6.11%	-1.10%	-1.72%	-1.00	65.13	.06	.08	1.28	.04
Multiple 11	5.73%	6.88%	-1.23%	-1.97%	-1.44	72.84	.06	.07	1.25	.03
Multiple 13	5.70%	8.55%	-1.50%	-2.47%	-2.65	102.58	.04	.06	1.21	.03

Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computation by the authors. TIPP strategies are presented from 01/02/1950 to 03/01/2011. Returns and volatility are annualized. The VaR of each column is an historic daily VaR associated to a 99% confidence level. See Eling and Schuhmacher (2007) and the Appendix for performance measure definitions.

Table 4: DARE and Unconditional Cushioned Portfolio Strategy Characteristics on a Five-year Basis based on Bootstrapped Simulated Series (on the Dow Jones Index from 1950 to 2011)

	Return	Volatility	VaR99%	Skewness	Kurtosis	Sharpe	Sortino	Omega	Kappa
Risky Asset	7.04%	14.49%	-2.33%	-1.23	39.30	.10	.05	1.10	.03
DARE Multiple	7.00%	2.31%	-2.37%	-.75	28.75	.13	.05	1.13	.03
Multiple 1	5.95%	1.22%	-.38%	-1.17	46.87	.13	.24	1.64	.12
Multiple 2	6.21%	4.70%	-.82%	-1.20	47.20	.12	.12	1.30	.06
Multiple 3	6.40%	7.23%	-1.30%	-1.24	52.46	.10	.08	1.20	.04
Multiple 4	6.49%	9.73%	-1.79%	-1.35	59.33	.09	.06	1.16	.03
Multiple 5	6.49%	11.83%	-2.22%	-1.86	93.25	.07	.05	1.14	.03
Multiple 6	6.52%	13.39%	-2.54%	-1.88	95.27	.07	.05	1.12	.02
Multiple 7	6.57%	14.49%	-2.76%	-1.70	82.47	.06	.05	1.12	.02
Multiple 8	6.59%	15.26%	-2.89%	-1.66	79.04	.06	.04	1.11	.02
Multiple 10	6.69%	16.29%	-3.10%	-1.51	77.43	.06	.04	1.11	.02
Multiple 11	6.75%	16.67%	-3.15%	-1.45	66.82	.07	.04	1.11	.02
Multiple 13	6.84%	17.27%	-3.25%	-1.45	65.06	.07	.04	1.11	.02

Source: *Bloomberg*, daily returns of the Dow Jones Index from 01/02/1950 to 03/01/2011; computation by the authors. The strategies characteristics are calculated using 1,500,000 simulations of daily returns, based on stationary bootstrap (Cf. Politis and Romano, 1994): artificial series are composed of Dow Jones random blocks of daily returns determined using a geometric probability law defined by a parameter of .9. Statistics presented here are the average of the statistics computed for each strategy over all simulations. Returns and volatilities are annualized. Each TIPP strategies use a five-year investment horizon. The VaR of each column is an historic daily VaR associated to a 99% confidence level. See Eling and Schuhmacher (2007) and the Appendix for performance measure definitions.

Appendix C: Complementary Results

In the *corpus* of the text, we present the first and second gap risk characterizations where the multiple can only be modified on a deterministic time basis - in the aforementioned case (i). Other alternatives may be considered in case (ii) and case (iii) presented below, depending on the event that drives the rebalancing.

Case (ii): first and second gap risks when rebalancing times correspond to jump times of the risky asset price

We assume now that the sequence of jump times $(T_l)_{l \in \mathbb{N}^+}$ of the risky asset S corresponds to the times at which the target multiple is modified. Therefore, the sequence $(\Theta_l)_{l \in \mathbb{N}^+}$ corresponds to the sequence $(T_l)_{l \in \mathbb{N}^+}$. However, the portfolio is rebalanced in continuous-time. In this case, we have the following condition on the target multiple.

Proposition 9. *The global quantile guarantee condition (defining the first gap risk) when the risky asset price follows a geometric Lévy process and the rebalancing time correspond to jumps, such that:*

$$\mathbb{P}[C_t \geq 0, \forall t \leq T] \geq 1 - \alpha, \quad (45)$$

is equivalent to the following condition on the conditional multiple \bar{m}_l :

$$\bar{m}_l \leq - [F^{-1}(-\alpha)]^{-1}, \quad (46)$$

where $F^{-1}(\cdot)$ denotes the α -quantile of the underlying return distribution of the risky asset.

Proof.

Condition (45) is equivalent to $\mathbb{P}_{T_l} [1 + \bar{m}_l (\Delta S_{T_{l+1}}/S_{T_{l+1}}) \geq 0 | C_{T_l} > 0] \geq 1 - \alpha$ and therefore, it is equivalent to the bound on \bar{m}_l . \square

In the same manner, we get the following result for the second gap risk stated in the next proposition.

Proposition 10. *The conditional and unconditional expectations of a loss (defining the second gap risk denoted $GR_{loc,2,T_l}$), when stock prices follow a Lévy process and the rebalancing correspond to jump times, are given by (with the previous notations):*

$$\begin{cases} CEL_{T_l} = C_{T_l} / \tilde{F}_{T_l} \left[\lambda \int_{-1}^{-1/\bar{m}_l} (1 + \bar{m}_l x) F(dx) \right] \left\{ F^{-1}(-1/\bar{m}_l) [\lambda - r - \bar{m}_l(\mu - r)] \right\}^{-1} \\ \text{and} \\ UEL_{T_l} = C_{T_l} / \tilde{F}_{T_l} \left[\lambda \int_{-1}^{-1/\bar{m}_l} (1 + \bar{m}_l x) F(dx) \right]. \end{cases} \quad (47)$$

Proof.

We have:

$$C_{T_{l+1}} = C_{T_l} \exp \left\{ \left[r + \bar{m}_l(\mu - r) - \frac{1}{2} \bar{m}_l^2 \sigma^2 \right] (T_{l+1} - T_l) + \bar{m}_l (W_{T_{l+1}} - W_{T_l}) \right\} \\ \times [1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l})].$$

Due to Lévy property, we get (with the previous notations):

$$\mathbb{E}_{T_l} [(C_{T_{l+1}} / C_{T_l}) | T_l < T_{NG} \leq T_{l+1} \cap C_{T_l} > 0] \\ = \mathbb{E} \left[\exp \left\{ \left[r + \bar{m}_l(\mu - r) - \frac{1}{2} \bar{m}_l^2 \sigma^2 \right] (T_{l+1} - T_l) + \bar{m}_l (W_{T_{l+1}} - W_{T_l}) \right\} \right. \\ \left. \times \mathbb{E} \left\{ [1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l})] \mathbb{1}_{[1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l})] < 0} \mid 1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l}) < 0 \right\} \right].$$

Note that we have:

$$\mathbb{E} \left\{ \exp \left\{ \left[r + \bar{m}_l(\mu - r) - \frac{1}{2} \bar{m}_l^2 \sigma^2 \right] (T_{l+1} - T_l) + \bar{m}_l (W_{T_{l+1}} - W_{T_l}) \right\} \right\} \\ = \mathbb{E} \left\{ \exp \left\{ [r + \bar{m}_l(\mu - r)] (T_{l+1} - T_l) \right\} \right\}.$$

Assuming $[r + \bar{m}_l(\mu - r)] < \lambda$, we get:

$$\mathbb{E} \left\{ \exp \left\{ [r + \bar{m}_l(\mu - r)] (T_{l+1} - T_l) \right\} \right\} = \lambda [\lambda - r - \bar{m}_l(\mu - r)]^{-1}.$$

We also have:

$$\mathbb{E} \left[[1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l})] \mathbb{1}_{\{[1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_{l+1}})] < 0\}} \mid [1 + \bar{m}_l (\Delta S_{T_{l+1}} / S_{T_l})] < 0 \right] \\ = \left[\int_{-1}^{-1/\bar{m}_l} (1 + \bar{m}_l x) F(dx) \right] \times [F^{-1}(-1/\bar{m}_l)]^{-1}.$$

Thus:

$$\mathbb{E}_{T_l} [(C_{t_{l+1}} / C_{t_l}) | (T_l < T_{NG} \leq T_{l+1}) \cap (C_{T_l} > 0)] = \{F^{-1}(-1/\bar{m}_l) [\lambda - r - \bar{m}_l(\mu - r)]\}^{-1}.$$

□

Case (iii): gap risks with both multiple changes and rebalancing happen in discrete-time

Here, both the target multiple and the portfolio strategy can only be modified at dates $t_0 < t_1 < \dots < T$ (Case 3). We still assume that the risky asset is a geometric Lévy process (see relation 4 where $\mu(\cdot, \cdot)$, $\sigma(\cdot, \cdot)$ and δ are constant).

When the risky asset price follows a geometric Lévy process, the cushion value at time t_{l+1} is given by:

$$C_{t_{l+1}}/C_{t_l} = (1 - \bar{m}_l) \{\exp[r(t_{l+1} - t_l)]\} \\ + \bar{m}_l \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) (t_{l+1} - t_l) + \sigma (W_{t_{l+1}} - W_{t_l}) \right] \times \prod_{t_l < T_k \leq t_{l+1}} [1 + \Delta S_{T_k}/S_{T_{k-1}}].$$

Denote:

$$X_{t_l} = \frac{W_{t_{l+1}} - W_{t_l}}{\sqrt{(t_{l+1} - t_l)}},$$

and:

$$h_{\bar{m}_l, n, t_l, t_{l+1}}(u_1, \dots, u_n) = \frac{\ln \left\{ (\bar{m}_l - 1) \exp[r(t_{l+1} - t_l)] [\bar{m}_l \prod_{i=1}^n (1 + u_i)]^{-1} \right\} - (\mu - \frac{1}{2} \sigma^2) (t_{l+1} - t_l)}{\sigma \sqrt{(t_{l+1} - t_l)}}.$$

In this framework, the $GR_{loc,1,t_l}$ condition corresponds to the following proposition.

Proposition 11. *When both the target multiple and the portfolio strategy can only be modified at deterministic times, the quantile condition $GR_{loc,1,t_l}$ can be expressed as:*

$$\sum_{n=0}^{\infty} \exp[-\lambda(t_{l+1} - t_l)] (n!)^{-1} [\lambda(t_{l+1} - t_l)]^n \int \dots \int \{1 - \Phi[h_{\bar{m}_l, n, t_l, t_{l+1}}(u_1, \dots, u_n)]\} F^{*n}(du_1, \dots, du_n) \geq 1 - \alpha, \quad (48)$$

where $F^{*n}(\cdot)$ denotes the n -convolution product of the distribution of the n risky asset relative jumps.

Proof.

Using the Lévy property and by conditioning by the number of jumps during the period $]t_l, : t_{l+1}]$, we deduce the result. \square

Examine now the $GR_{loc,2,t_l}$ conditions.

First denoting by $H(\bar{m}_l, t_l, t_{l+1})$ the probability to keep a non-negative cushion on time period $]t_l, : t_{l+1}]$, such as:

$$H(\bar{m}_l, t_l, t_{l+1}) = \sum_{n=0}^{\infty} \exp[-\lambda(t_{l+1} - t_l)] (n!)^{-1} [\lambda(t_{l+1} - t_l)]^n \int \dots \int \Phi [h_{\bar{m}_l, n, t_l, t_{l+1}}(u_1, \dots, u_n)] F^{*n}(du_1, \dots, du_n).$$

Also denote:

$$G(\bar{m}_l, t_l, t_{l+1}) = \sum_{n=0}^{\infty} \exp[-\lambda(t_{l+1} - t_l)] (n!)^{-1} [\lambda(t_{l+1} - t_l)]^n \int \dots \int \Phi [h_{\bar{m}_l, n, t_l, t_{l+1}}(u_1, \dots, u_n) - \sigma \sqrt{(t_{l+1} - t_l)}] F^{*n}(du_1, \dots, du_n).$$

Proposition 12. *The conditional and unconditional expectations of a loss (defining the second gap risks), when stock prices follow a Lévy process and when both the target multiple and the portfolio strategy can only be modified at deterministic times, are given by (with the previous notations):*

$$\begin{cases} CEL_{t_l} = C_{t_l} \left\{ (1 - \bar{m}_l) \exp[r(t_{l+1} - t_l)] + \bar{m}_l \exp[\mu(t_{l+1} - t_l)] G(\bar{m}_l, t_l, t_{l+1}) [H(\bar{m}_l, t_l, t_{l+1})]^{-1} \right\} \\ \text{and} \\ UEL_{t_l} = C_{t_l} \left\{ (1 - \bar{m}_l) \exp[r(t_{l+1} - t_l)] + \bar{m}_l \exp[\mu(t_{l+1} - t_l)] G(\bar{m}_l, t_l, t_{l+1}) \right\}. \end{cases} \quad (49)$$

Proof.

If we denote:

$$Z_{t_l} = \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) (t_{l+1} - t_l) + \sigma \sqrt{(t_{l+1} - t_l)} X_{t_l} \right] \prod_{t_l < T_k \leq t_{l+1}} (1 + \Delta S_{T_k} / S_{T_k-}).$$

Then, we have:

$$\begin{aligned} & \mathbb{E}_{t_l} [C_{t_{l+1}} | C_{t_{l+1}} < 0 \cap C_{t_l} > 0] \\ &= C_{t_l} \left\{ (1 - \bar{m}_l) \exp[r(t_{l+1} - t_l)] + \bar{m}_l \frac{\mathbb{E}_{t_l} \{ Z_{t_l} \mathbb{1}_{\{Z_{t_l} < \exp[r(t_{l+1} - t_l)](\bar{m}_l - 1/\bar{m}_l)\}} \}}{H(\bar{m}_l, t_l v)} \right\} \\ &= C_{t_l} \left[(1 - \bar{m}_l) \exp[r(t_{l+1} - t_l)] + \bar{m}_l \frac{\exp[\mu(t_{l+1} - t_l)] G(\bar{m}_l, t_l v)}{H(\bar{m}_l, t_l v)} \right]. \end{aligned}$$

□

Appendix D: Definition and General Properties of the Performance Measures

Performance measurement is one of the most studied subjects in financial literature. A large variety of new measures has appeared constantly in scientific journals as well as in practitioners' publications. The most complete and significant studies of performance measures, so far, have been written by Aftalion and Poncet (2003), Le Sourd (2007), Bacon (2000 and 2008), Cogneau and Hübner (2009a and 2009b) and Caporin *et al.* (2013a). Among all these performance measures, one of the most prominent is the Sharpe ratio such as:

$$S_p = [E(r_p) - r] \times (\sigma_{r_p})^{-1}, \quad (50)$$

where $E(\cdot)$ is the expectation operator, r_p is the studied investor's portfolio return, r is the risk-free rate and σ_{r_p} is the total risk of the managed portfolio.

However, this measure has been severely criticized, especially when the normality of the returns of underlying investments is not respected. In the case of protected portfolios, the Gaussian hypothesis is clearly not fulfilled (*per* construction since the left parts of resulting distributions are distorted). In this case, one may want to rely on *extra performance criteria*, in order to complement the results related to a quadratic utility investor and to strengthen the final conclusion regarding the economic value of proposed strategies. In case of obvious non-linearities or non-Gaussianities, several measures have been used in the literature for gauging performance of funds or active strategies (*e.g.* Eling and Schuhmacher, 2007; Annaert *et al.* 2009; Brandouy *et al.*, 2010; Bertrand and Prigent, 2011). Caporin *et al.* (2013b) show that some main performance measures can be written as:

$$\begin{aligned}
PM_p &= \mathcal{H}_p(r_p, \tau_1, \tau_2, \tau_3, \tau_4, o_1, o_2, k_1, k_2) \\
&= [-E(|\tau_1 - r_p|^{o_1} | \tau_1 - r_p < \tau_3)]^{(k_1)^{-1}} \\
&\quad \times \left\{ [-E(|\tau_2 - r_p|^{o_2} | r_p - \tau_2 < \tau_4)]^{(k_2)^{-1}} \right\}^{-1}, \tag{51}
\end{aligned}$$

where τ_1 corresponds to a threshold (a reserve return \underline{r} , a MAR, the null return or the risk-free rate $r...$) for computing gains and τ_2 for calculating losses or risk, $\tau_3 = VaR_{-r_p, a_1}$ is another threshold – related to a given confidence level a_1 – specifying the right part of the support of the return density under study (*i.e.* gains), $\tau_4 = VaR_{r_p, a_2}$ is a last threshold – linked to another given confidence level a_2 – associated with the left part (*i.e.* losses), o_1 and o_2 are intensification constants reflecting the investor’s attitude towards gains and losses, k_1 and k_2 are normalizing constants (with $k_1 > 0$ and $k_2 > 0$).

This notation can be easily applied to several relative performance measures used in the article. For instance since the *Omega* (Keating and Shadwick, 2002), *Kappa 3* (Kaplan and Knowles, 2004) and the Sharpe-*Omega* (Kazemi *et al.*, 2004) ratios that are defined such as:

$$\begin{cases} \Omega(\tau) = GHPM_{r_p, \tau, \tau, 1} \times (GLPM_{r_p, \tau, \tau, 1})^{-1} \\ \kappa_3(\tau) = [E(r_p) - \tau] \times (GLPM_{r_p, \tau, \tau, 3})^{-\frac{1}{3}} \\ S_\Omega(\tau) = [E(r_p) - \tau] \times (GLPM_{r_p, \tau, \tau, 1})^{-1}, \end{cases}$$

with:

$$\begin{cases} GHPM_{r_p, \tau_1, \tau_3, o_1} = \int_{-\infty}^{\tau_3} |\tau_1 - r_p|^{o_1} f(-r_p) d(-r_p) \\ GLPM_{r_p, \tau_2, \tau_4, o_2} = \int_{-\infty}^{\tau_4} |\tau_2 - r_p|^{o_2} f(r_p) d(r_p). \end{cases}$$

Thus, the Sharpe, the *Omega*, the *Kappa 3* and the Sharpe-*Omega* can be rewritten such as:

$$\left\{ \begin{array}{l} S_p = \mathcal{H} [r_p, r, E(r_p), +\infty, +\infty, 1, 2, 1, 2] \\ \Omega(\tau) = \mathcal{H}_p(r_p, \tau, \tau, \tau, \tau, 1, 1, 1, 1) \\ \kappa_3(\tau) = \mathcal{H}_p[r_p, \tau, E(r_p), +\infty, \tau, 1, 3, 1, 3] \\ S_\Omega(\tau) = \mathcal{H}_p[r_p, \tau, E(r_p), +\infty, \tau, 1, 1, 1, 1]. \end{array} \right.$$

Appendix E: Definition and General Properties of the Expectiles

As shown by Taylor (2008a, 2008b) the notion of τ -expectile is connected to the notion of expected shortfall.

Recall that the quantile of a random variable X at a given probability level α is the solution of the following minimization problem:

$$\text{Min}_x \mathbb{E} [(\alpha - \mathbb{I}_{\{X < x\}})(X - x)]. \quad (52)$$

Relation (52) shows that quantile regression is a convenient estimation tool in a conditional quantile model.

In the same manner, the τ - expectile of X , denoted by μ_τ is defined as the solution of another minimization problem which reads:

$$\text{Min}_x \mathbb{E} [|\tau - \mathbb{I}_{\{X < x\}}| (X - x)^2]. \quad (53)$$

Thus, the estimation of parameter β of a conditional model for expectile μ_τ would be based on asymmetric least squares (ALS) regression, as considered by Newey and Powell (1987) who proposed to solve (for given observations x_i):

$$\text{Min}_\beta \sum_i \left[\left| \tau - \mathbb{I}_{\{x_i < \mu_{i,\tau}\}} \right| (x_i - \mu_{i,\tau})^2 \right]. \quad (54)$$

Problem (54) corresponds to Problem (38) when $X = r_t^S$.

Additionally, we can get:

1) First, as proposed by Efron (1991), an estimation of the α - quantile by using the expectile μ_τ for which the proportion of in-sample observations x_i smaller than μ_τ is equal to α . Indeed, for each τ -expectile, there is a corresponding α -quantile (see Jones, 1994; Abdous and Rémillard, 1995; Yao and Tong, 1996);

2) Secondly (see Newey and Powell, 1987), an estimation of expected shortfalls $ES_{1-\alpha}$ using expectiles μ_τ . Indeed, the solution of Problem (54) when the expectile μ_τ is one-dimensional is also the solution of the following equation:

$$\frac{1 - 2\tau}{\tau} \mathbb{E} [(X - \mu_\tau) \mathbb{I}_{\{X < \mu_\tau\}}] = \mu_\tau - \mathbb{E}[X]. \quad (55)$$

From Equation (55), we have:

$$\mathbb{E}[X|X < \mu_\tau] = \left[1 + \frac{\tau}{(1-2\tau)F_X(\mu_\tau)}\right] \mu_\tau - \left[\frac{\tau}{(1-2\tau)F_X(\mu_\tau)}\right] \mathbb{E}[X].$$

Then, letting $F_X(\mu_\tau) = \alpha$ (implying $\mu_\tau = q(\alpha)$, the α -quantile of X), we can deduce:

1) The value of τ :

$$\tau = \frac{\alpha q(\alpha) - \alpha \mathbb{E}[X|X < q(\alpha)]}{\mathbb{E}[X] - 2\alpha \mathbb{E}[X|X < q(\alpha)] - (1-2\alpha)q(\alpha)}.$$

Assuming that X has a pdf denoted $f(\cdot)$, we get $\alpha \mathbb{E}[X|X < q(\alpha)] = \int_{-\infty}^{q(\alpha)} x f(x) dx$. Therefore, we have:

$$\tau = \frac{\alpha q(\alpha) - \int_{-\infty}^{q(\alpha)} x f(x) dx}{\mathbb{E}[X] - 2 \int_{-\infty}^{q(\alpha)} x f(x) dx - (1-2\alpha)q(\alpha)}. \quad (56)$$

Letting $X = r_t^S$, we deduce relation (39).

2) The value of the expected shortfall $ES_{1-\alpha}$:

$$ES_{1-\alpha} = \left[1 + \frac{\tau}{(1-2\tau)\alpha}\right] \mu_\tau - \left[\frac{\tau}{(1-2\tau)\alpha}\right] \mathbb{E}[X].$$

If for example $\mathbb{E}[X] = 0$, we get:

$$ES_{1-\alpha} = \left(1 + \frac{\tau}{(1-2\tau)\alpha}\right) \mu_\tau. \quad (57)$$

If $X = r_t^S$, we deduce relation (40).