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Describing all Nash Equilibria in Two Particular Cases of Singleton Congestion Games

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Abstract A crucial issue in the theory of congestion games is to be able to find possible solutions that correspond to Nash equilibria. However, according to the existing literature, the current methods do not find all of them. In this article, interested in a special class of congestion games, namely the congestion games with player-specific payoff functions, we analyze this missing aspect in this research field. Investigating two particular cases of these games, the symmetric case and the non-symmetric case with two resources, we propose a new method with which not only we describe all Nash equilibria but we also give their structure in a simple and direct way. Furthermore, we provide short and constructive proofs that establish the existence of at least one Nash equilibrium in this kind of games, without invoking either the notion of the potential function or the finite improvement property.

Keywords Singleton congestion games · Nash equilibria · Potential function · Finite improvement property.

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1 Introduction

Congestion games are a special class of noncooperative games that provide a natural framework for a wide range of economic and computer science applications such as resource allocation, routing and network design problems. Consider, for example, a traffic network where two players want to go from a point A to a point B. Suppose that node A is connected to node B via connection points C_1 and C_2, where C_1 is a little closer than C_2 (i.e. C_1 is more likely to be chosen by each player). However, both connection points get easily congested, meaning that the more players pass through a point the greater the delay of each player becomes. Thus, having both players going through the same connection point causes extra delay. A good outcome in this game would be for the two players to 'coordinate' and pass through different connection points. The question is, to determine whether such an outcome - corresponding to a Nash equilibrium - is achievable. And if so, what would the cost be for each player?

Rosenthal (1973), who was the first to analyze congestion games, showed by a potential function argument, that they always possess at least one pure-strategy Nash equilibrium. In his model, a set of players competes for a set of resources, and the payoff of each resource depends only on the number of players using it. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. This line of work has been continued by Monderer and Shapley (1996) who established a connection between potential games and congestion situations. An excellent survey of the related literature can be found in Voorneveld et al (2010). At this point, we also note that Konishi et al (1997) and Quint and Shubik (1994) considered that congestion games do not admit (in general) a potential function, but are likely to admit a Nash equilibrium in pure strategies.

In 1996, Milchtaich introduced a slightly different formulation of congestion games under the name of congestion games with player-specific payoff functions. This time, all players are restricted to the selection of a single resource and either they all share the same utility function - symmetric case - or each of them has his own payoff function - non-symmetric case. In these games, the specific payoff functions are decreasing as a function of increasing player numbers. A key game-theory property of this kind of games is that they always admit at least one Nash equilibrium.

So far, no technique has been developed to enable a straightforward identification of all Nash equilibria. In this article, interested in congestion games in the sense of Milchtaich, also called singleton congestion games, we analyze this missing aspect in the literature. Note that we use the term singleton congestion games to point out the fact that players' strategies are singleton and the payoff functions are decreasing and at the same time specific to each player.

More precisely, studying two distinct cases of these games, namely the symmetric case and the non-symmetric case with two resources, we:

1. prove that there exists at least one Nash equilibrium, by direct and constructive proofs, without using either the notion of the potential function or the finite improvement property, and
2. describe all Nash equilibria and give their structure using a simple and direct formula.

The direct characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic proprieties of a game such as the Price of Anarchy[1] (Koutsoupias and C. H. Papadimitriou, 1999). Identifying Nash equilibria straightaway can help to easily analyze their quality.

This paper is organized into five sections. Section 2 introduces congestion games with player-specific payoff functions, section 3 studies the symmetric case and section 4 establishes the non-symmetric case with two resources. The last section concludes the paper.

2 Congestion games with player-specific payoff functions

A game (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the utility function of player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i\in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies. A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a best-reply strategy: For each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy.

In a standard congestion game, defined by Rosenthal (1973), we are given a finite set $R = \{1, \ldots, m\}$ of m resources. A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \ldots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r\in\sigma_i} d_r(n_r(\sigma))$. Rosenthal shows that every congestion game possesses at least one Nash equilibrium by considering the exact potential function $P : S \to \mathbb{N}$ with $P(\sigma) = \sum_{r=1}^{m} \sum_{j=1}^{n_r(\sigma)} d_r(j)$, $\forall \sigma \in S$ [2]. Monderer and Shapley (1996) have continued to follow this line of

[1] When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

[2] Rosenthal's potential function shows that congestion games are potential. Monderer and Shapley (1996) proved that every potential game can be represented in a form of a congestion

work, who proved that the existence of an exact potential function implies the finite improvement property (FIP): That is, each path of single-handed (one player) profitable deviations is finite.

Congestion games games with player-specific payoff functions, introduced by Milchtaich (1996), are defined by a tuple $\Gamma(N, R, (d_r)_{r\in R})$, where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a nonincreasing payoff function associated with resource r. The utility of player i for a profile σ is given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$ with all players being restricted to choose only one strategy and each of them has his own utility function. Since the utility of a player derived from selecting a single resource depends only on the number of the players making the same choice, the specific utility function of this player is simply a mapping: $u_i : R \times \{1, \ldots, n\} \to \mathbb{R}$, $(r, k) \mapsto u_i(r, k)$, where u_i decreases with k, $\forall k \in N$.

Theorem 1 (Milchtaich, 1996) *Every congestion game with palyer-specific payoff function possesses a Nash equilibrium in pure strategies.*

Milchtaich (1996) proved this theorem by using a variant of the finite improvement property. The main drawback of his method is that it finds only one Nash equilibrium. In the next sections, we develop our main results that enable to find *all* Nash equilibria in the symmetric case and the non-symmetric case with two resources of singleton congestion games.

3 The symmetric case

The aim of this section is to improve the study of symmetric singleton congestion games (hereafter mentioned SSCG) by providing a simple formula describing all Nash equilibria. Note that in such games all players share the same utility and each of them chooses only one strategy.

In order to state our result, we first need to simplify the analysis by moving to the ordinal representation of preferences, instead of the cardinal one. Indeed, we can replace, the values of the payoff functions (i.e. cardinal representation) by their ranks, a preference ordering representing the common utility function, without affecting the set of Nash equilibria.

Formally, a SSCG can be represented by a tuple $\Gamma(N, R, \precsim)$ where N is a set of n players, R a set of m resources and $\precsim$ a weak ordering on $R \times \{1, \ldots, n\}$. In this ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim (\sigma_i, \sigma^*_{-i})$ for all σ_i in R. We also note that, since players are anonymous, all strategy profiles that differ only by a permutation of players can be identified with the corresponding congestion vector. We refer to a congestion vector $\sigma^* = (n_1, \ldots, n_m)$ as a Nash equilibrium if, for all r, r' in R with $r \neq r'$, we have $(r, n_r) \succsim (r', n_{r'} + 1)$. Thus, no player can benefit from joining a group

game. Thus, classes of potential games and congestion games coincide. Hence, congestion games are essentially the only class of games for which one can show the existence of pure equilibria with an exact potential function.

of players sharing a different resource.

Our result is based on the following notion:

Definition 1 Let $\precsim$ be a (weak) ordering on $R \times \{1, \dots, n\}$. An *n-sequence* derived from $\precsim$ is a subset T of $R \times \{1, \dots, n\}$ such that:

- $|T| = n$.
- $\Big((r, k) \in T$ and $(r', k') \notin T\Big) \Rightarrow (r, k) \succsim (r', k')$.
- $(r, k) \in T \Rightarrow \Big((r, k') \in T, \forall k' < k\Big)$.

Thus, an $n-$sequence is a set of the most preferred n elements of $R \times \{1, \dots, n\}$.

Theorem 2 *Let $\Gamma(N, R, \precsim)$ be a monotone symmetric singleton congestion game, with $|N| = n$ and $|R| = m$. Then,*

1. *There is a unique Nash equilibrium per n-sequence. Let T be an n-sequence of $\precsim$. The corresponding Nash equilibrium is defined by $\sigma = \Big((1, \alpha_1), \dots, (m, \alpha_m)\Big)$, where α_r is the greater integer p satisfying $(r, p) \in T$.*
2. *When the players' preferences are expressed by a strictly decreasing ordering, the game Γ admits exactly one Nash equilibrium.*
3. *The number of Nash equilibria of the game Γ equals the number of all n-sequences extracted from $\precsim$.*

Proof Since the second and the third point are simple consequences of the first assertion, we have just the following statement. Let T be an n-sequence and let $\sigma^* = \Big((1, \alpha_1), \dots, (m, \alpha_m)\Big)$ be the $m-$components vector defined by $\alpha_r = max\{p : (r, p) \in T\}$. By definition of T and σ^*, we have $\sum_{r=1}^{m} \alpha_r = n$. Indeed, the sequence T exclusively consists of the following terms:

$$(1, \alpha_1), \dots, (1, 1), (2, \alpha_2), \dots, (2, 1), \dots, (m, \alpha_m), \dots, (m, 1).$$

Therefore, σ^* is a congestion vector. Furthermore, for all r, r' in R, $(r, \alpha_r) \succsim (r', \alpha_{r'} + 1)$ because $(r, \alpha_r) \in T$ and $(r', \alpha_{r'} + 1) \notin T$. Hence, σ^* is a Nash equilibrium. Reciprocally, let $\sigma^* = \Big((1, \alpha_1), \dots, (m, \alpha_m)\Big)$ be a Nash equilibrium. It is easy to see that $T = \{(1, \alpha_1), \dots, (1, 1), \dots, (m, \alpha_m), \dots, (m, 1)\}$ is an n-sequence. In fact, as σ^* is a congestion vector, we have $\sum_{r=1}^{m} \alpha_r = n$ and so $|T| = n$. On the other hand, by definition of T, $(r, k) \in T \Rightarrow \Big((r, k') \in T, \forall k' < k\Big)$. Finally, let $(r, k) \in T$ and $(r', k') \notin T$. By definition of T, we have $k \leq \alpha_r$ and $k' \geq \alpha_{r'} + 1$. Since σ^* is a Nash equilibrium, we have $(r, \alpha_r) \succeq (r', \alpha_{r'} + 1)$. By the monotonicity hypothesis, we obtain $(r, k) \succsim (r, \alpha_r) \succsim (r', \alpha_{r'} + 1) \succsim (r', k')$. $\square$

To illustrate the above theorem, we provide the following examples to show how we can easily characterize all Nash equilibria.

Example 1 Let $N = \{1, 2, 3, 4, 5\}$ and $R = \{a, b, c\}$. For simplicity, we will denote the couple (r, k) by kr. Suppose that the common ordinal utility function is given by the following strictly decreasing ordering:
$5b \prec 4b \prec 5c \prec 5a \prec 4c \prec 3c \prec 3b \prec 4a \prec 3a \prec 2a \prec \underbrace{2b \prec b \prec 2c \prec c \prec a}$.
We obtain $T = \{2b, b, 2c, c, a\}$. Selecting the greatest integer corresponding to each resource, we identify the unique Nash equilibrium: $\sigma^* = (a, 2b, 2c)$.

Example 2 Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{a, b, c, d\}$. Suppose the players' preferences are given by following weak ordering:
$7b \sim 7c \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5b \sim 5d \sim 4b \sim 7a \prec 5a \sim 4d \sim 6a \prec 4a \prec 3a \sim 3b \sim 2b \sim 3d \sim 2a \prec 2c \sim b \prec c \sim a \prec 2d \sim d$.
Here we find one Nash equilibrium per n-sequence:

For $T_1 = \{d, 2d, a, c, b, 2c, 2a\}$, $\sigma_1^* = (2a, b, 2c, 2d)$;
For $T_2 = \{d, 2d, a, c, b, 2c, 3d\}$, $\sigma_2^* = (a, b, 2c, 3d)$;
For $T_3 = \{d, 2d, a, c, b, 2c, 2b\}$, $\sigma_3^* = (a, 2b, 2c, 2d)$.

Hence, there are exactly three Nash equilibria in this game.

4 The non-symmetric case with two resources

In this section, we develop a method which simplifies the analysis of non-symmetric singleton congestion games in the case of two resources. As in the symmetric case, we replace the values of the payment functions (i.e. cardinal representation) by their ranks in a preference ordering representing the specific utility function of each player.

Let $\Gamma(N, R, (\precsim_i)_{i \in N})$ be a singleton congestion game, where N is a set of n players, $R = \{a, b\}$ a set of two resources and $\precsim_i$ a weak ordering on $R \times N$. In the ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim_i (\sigma_i, \sigma^*_{-i})$ for all σ_i in R. A congestion vector $\sigma^* = (n_{r_1}, n_{r_2})$ corresponds to a Nash equilibrium if $(r_1, n_{r_1}) \succsim_i (r_2, n_{r_2}+1)$, for all $r_1, r_2 \in R$, with $r_1 \neq r_2$. Thus, no player can benefit from joining a group of players sharing a different resource.

For the sake of clarity and without losing integrity, we distinguish two cases. In the first case, each player has a strict order of preferences while in the second one each order of preferences may include ties.

Case 1: Strict order of preferences

To develop our approach, we need the following notation: For all players $i \in N$, we note $(a, 0) \succ_i (b, n+1)$ (or $0 \cdot a \succ_i (n+1) \cdot b$ by adopting a

simplified notation) when $(a,1) \prec_i (b,n)$. Similarly, we note $(b,0) \succ_i (a,n+1)$ (or $0 \cdot b \succ_i (n+1) \cdot a$) when $(b,1) \prec_i (a,n)$.
Thus, we define the following integers:

$$p_i = \max \{p \in \{0,1,\ldots,n\} : (a,p) \succ_i (b,n+1-p)\}$$

$$q_i = \max \{q \in \{0,1,\ldots,n\} : (b,q) \succ_i (a,n+1-q)\}$$

The integer p_i denotes the maximum size of a group choosing the alternative a in a given strategy profile, in which the player i can belong. Beyond this size, the player i will choose the resource b. Indeed, by definition we have $p_i \cdot a \succ_i (n+1-p_i) \cdot b$ and $(p_i+1) \cdot a \prec_i (n-p_i) \cdot b$. The integer q_i is interpreted in the same way; we replace a by b. We mention that $p_i + q_i = n$, $\forall i \in N$.
Using the list of integers p_i and q_i, $\forall i \in N$, we define two other integers that will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game:

$$n(a) = \max \{p \in \{0,1,\ldots,n\} :\mid \{i \in N : p_i \geq p\} \mid \geq p\}$$

$$n(b) = \max \{q \in \{0,1,\ldots,n\} :\mid \{i \in N : q_i \geq q\} \mid \geq q\}$$

We point out that $n(a)$ (resp. $n(b)$) represents the maximum size of a group of players that can choose the resource a (resp. b) without any member of this group having interest in deviating from his strategy. In this case, we necessarily have $n(a) + n(b) = n$, with the corresponding congestion vector being $v = (n(a), n(b))$.

In order to describe all Nash equilibria, we introduce the three following sets that allow us to identify the alternatives that correspond to each player:

$$A(G) = \{i \in N : p_i > n(a)\}, \quad B(G) = \{i \in N : p_i < n(a)\},$$
$$\text{and} \quad C(G) = \{i \in N : p_i = n(a)\}$$

Here, N is the disjoint union of these three sets, each of which may be empty and $|C(G)| \succeq na - |A(G)|$.

Case 2: Order of preferences with ties

This time we denote:

$$p_i = \max \{p \in \{0,1,\ldots,n\} : (a,p) \succsim_i (b,n+1-p)\}$$

$$q_i = \max \{q \in \{0,1,\ldots,n\} : (b,q) \succsim_i (a,n+1-q)\}$$

where p_i and q_i, $\forall i \in N$, have the same meaning as in the above case. However, we do not necessarily have $p_i + q_i = n$ because of the possible presence of ties. Hence, $p_i + q_i \geq n$, for all $i \in N$. It is therefore possible to have $p_i + q_i > n$ for some players i. This point is important because in this case, the possibility exists of having more than one congestion vector corresponding to a Nash equilibrium.

Using the list of integers p_i and q_i, we define $n(a)$ and $n(b)$ as in case 1. The latter ones will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game. In this case we have the inequality $n(a) + n(b) \geq n$ and the corresponding congestion vector is $v = (\alpha, \beta)$, where $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$.

In order to describe all Nash equilibria, we introduce the three following sets that allow us to identify the alternatives that correspond to each player:

$$A(G, v) = \{i \in N : p_i \geq \alpha \text{ and } q_i < \beta\}, \quad B(G, v) = \{i \in N : p_i < \alpha \text{ and } q_i \geq \beta\},$$

$$\text{and } C(G, v) = \{i \in N : p_i \geq \alpha \text{ and } q_i \geq \beta\}$$

N is the disjoint union of these three sets and that each of these sets may be empty. We do not examine the case in which $p_i < \alpha$ and $q_i < \beta$, as $p_i + q_i \geq n$ and $\alpha + \beta = n$.

We are now ready to provide our result with which all Nash equilibria are described and a comprehensive list of all of them is established in the special case of two resources. Our approach is such that we are not making use of the potential function or the finite improvement property invoked by Rosenthal and Milchtaich respectively.

Theorem 3 *Let $R = \{a, b\}$ and $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game where all preference orderings are strict.*

1. *G admits at least one Nash equilibrium. All equilibria correspond to the same congestion vector : $v = (n(a), n(b))$.*
2. *Each Nash equilibrium of G, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset D (possibly empty) of $C(G)$, of cardinal $n(a) - |A(G)|$, such that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G) \cup D$ and $\sigma_i^* = b$ if $i \in B(G) \cup (C \setminus D)$.*
3. *The game admits exactly $C_{|C(G)|}^{n(a)-|A(G)|}$ Nash equilibria. In particular, if $n(a) = |A(G)|$ the game admits a single Nash equilibrium.*

Proof 1) By definition of $n(a)$, there are at least $n(a)$ players $i \in N$ such that $p_i \geq n(a)$. Therefore, we choose $n(a)$ players satisfying this condition including all players for whom $p_i > n(a)$. Denote by A the set of these players. For all players who are in $B = N \backslash A$, we must have $p_i \leq n(a)$ and therefore $q_i \geq n(b)$. It is easy, returning to the definition of p_i and q_i, to verify that the profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ defined by $\sigma_i^* = a$ if $i \in A$ and $\sigma_i^* = b$ if $i \in B$ is a Nash equilibrium. Let σ^* be a Nash equilibrium of G and let (α, β) be the congestion vector associated with σ^*. Suppose that $\alpha > n(a)$. As σ^* is a Nash equilibrium, there exist α players such that $p_i \geq \alpha$, which contradicts the maximality of $n(a)$. We must therefore have $\alpha \leq n(a)$. Similarly, we show that $\beta \leq n(b)$. As $\alpha + \beta = n$ and $n(a) + n(b) = n$, we necessarily have $\alpha = n(a)$ and $\beta = n(b)$.

2) Let D be a subset (possibly empty) of $C(G)$, of cardinal $n(a)-|A(G)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile defined by: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G) \cup D$ and $\sigma_i^* = b$ if $i \in B(G) \cup (C(G) \setminus D)$. The profile σ^* is a Nash equilibrium. Indeed, let $i \in A(G) \cup D$. By definition of $A(G)$ and D, we have $p_i \geq n(a)$. By definition of p_i and the assumption of monotonicity, we get: $n(a) \cdot a \succsim_i (n(b)+1) \cdot b$. Similarly, we show that for all i in $B(G) \cup (C(G) \setminus D$, $n(b) \cdot b \succsim_i (n(a)+1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G. It is known from (1) that the congestion vector associated with σ^* is $(n(a), n(b))$. We must have $\sigma_i^* = a$ if $i \in A(G)$ and $\sigma_i^* = b$ if $i \in B(G)$. We just have to consider $D = \{i \in N : \sigma_i^* = a \text{ and } i \notin A(G)\}$.
3) The result is obtained by a simple calculation from (2). □

Example 3 Let $N = \{1, 2, 3, 4, 5, 6\}$ be a number of players and $R = \{a, b\}$ two alternatives. Suppose that the players' preferences are given by the following strict orderings:

$$\text{Player}_1: \quad 6b \prec 5b \prec 4b \prec 6a \prec 5a \prec 3b \prec 4a \prec 3a \prec 2b \prec 2a \prec b \prec a$$

$$\text{Player}_2: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 3b \prec 2a \prec 2b \prec a \prec b$$

$$\text{Player}_3: \quad 6b \prec 6a \prec 5a \prec 4a \prec 3a \prec 2a \prec 5b \prec 4b \prec 3b \prec 2b \prec a \prec b$$

$$\text{Player}_4: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec b \prec a$$

$$\text{Player}_5: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 2a \prec a \prec 3b \prec 2b \prec b$$

$$\text{Player}_6: \quad 6b \prec 5b \prec 4b \prec 6a \prec 5a \prec 4a \prec 3a \prec 2a \prec 3b \prec 2b \prec a \prec b$$

We have omitted the indices of players in the order of preferences. For each player i, we search the integer p_i which is the greatest p such that $p \cdot a \succ_i (n+1-p) \cdot b$ and $(n-p) \cdot b \succ_i (p+1) \cdot a$.

$$\begin{array}{lll} p_1 = 4: & 4a \succ_1 3b & \text{and} \quad 2b \succ_1 5a \\ p_2 = 3: & 3a \succ_2 4b & \text{and} \quad 3b \succ_2 4a \\ p_3 = 1: & a \succ_3 6b & \text{and} \quad 5b \succ_3 2a \\ p_4 = 3: & 3a \succ_4 4b & \text{and} \quad 3b \succ_4 4a \\ p_5 = 3: & 3a \succ_5 4b & \text{and} \quad 3b \succ_5 4a \\ p_6 = 3: & 3a \succ_5 4b & \text{and} \quad 3b \succ_5 4a \end{array}$$

So, we can verify that $n(a) = 3$ and $n(b) = 3$. The only congestion vector corresponding to a Nash equilibrium is the vector $(3a, 3b)$. Furthermore, we have $A(G) = \{1\}$, $B(G) = \{3\}$ and $C(G) = \{2, 4, 5, 6\}$. By theorem 1, we know that there are exactly $C_4^2 = 6$ different Nash equilibria. All these equilibria are common: $\sigma^* = (a)$ if $i \in A(G)$ and $\sigma^* = (b)$ if $i \in B(G)$. Each of these equilibria is characterized by a subset D of $C(G)$ with $|D| = 2$ and $\sigma_i^* = a$ if $i \in D$. The list of the Nash equilibria of this game is:

(a, a, b, a, b, b), (a, a, b, b, a, b), (a, a, b, b, b, a),
(a, b, b, a, b, a), (a, b, b, a, a, b), (a, b, b, b, a, a).

Remark 1 With the above example, we show that, contrary to the approaches already proposed in the literature, our method gives directly the exact number of Nash equilibria, before even knowing which they are. Hence, we avoid a repeating procedure to find all Nash equilibria, like the one given by Milchtaich.

Theorem 4 *Let $R = \{a, b\}$ and $G(N, R, (\precsim)_{i \in N})$ be a singleton congestion game where the order of preferences my include ties.*

1. *Each congestion vector $v = (\alpha, \beta)$ such that $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$, corresponds to (at least) one Nash equilibrium of G.*
2. *Each of the Nash equilibrium of G corresponding to the vector v, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset D (possibly empty) $C(G, v)$, of cardinal $\alpha - |A(G, v)|$, to ensure that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G, v) \cup D$ and $\sigma_i^* = b$ if $i \in B(G, v) \cup (C(G, v) \setminus D)$.*

Proof It suffices to prove (2), because (1) is obtained as a consequence of (2). Let $v = (\alpha, \beta)$ be a congestion vector such that $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$. Let D be a subset (possibly empty) of $C(G, v)$, of cardinal $\alpha - |A(G, v)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a strategy profile such that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G, v) \cup D$ and $\sigma_i^* = b$ if $i \in B(G, v) \cup (C(G, v) \setminus D)$. σ^* is a Nash equilibrium. Indeed, let $i \in A(G, v) \cup D$. By definition of $A(G, v)$ and of D, we have $p_i \geq \alpha$. By definition of p_i and by the assumption of monotonicity, we obtain: $\alpha \cdot a \succsim_i (\beta + 1) \cdot b$. Similarly, we show that for all i in $B(G, v) \cup (C(G, v) \setminus D$, $\beta \cdot b \succsim_i (\alpha + 1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G and let $v = (\alpha, \beta)$ be the congestion vector associated with this equilibrium. We have $\alpha \leq n(a)$, otherwise there exist α players i with $p_i \geq \alpha > n(a)$. This is impossible by definition of $n(a)$. Similarly, we show that $\beta \leq n(b)$. By definition of a congestion vector, we also have $\alpha + \beta = n$. As σ^* is a Nash equilibrium, for any $i \in N$, we must have: $\sigma_i^* = a$ if $i \in A(G, v)$ and $\sigma_i^* = b$ if $i \in B(G, v)$. We just need to consider $D = \{i \in N : \sigma_i^* = a \text{ and } i \notin A(G, v)\}$ and to note that the case $p_i < \alpha$ and $q_i < \beta$ is not possible. □

Example 4 Let $N = \{1, 2, 3, 4, 5\}$ be a number of players and $R = \{a, b\}$ two alternatives. Suppose that the players' preferences are given by the following weak orderings:

$$\text{Player}_1: \quad 5a \prec 5b \prec 4b \prec 4a \prec 3b \sim 3a \sim 2a \prec 2b \sim a \prec b$$

$$\text{Player}_2: \quad 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b$$

$$\text{Player}_3: \quad 5a \prec 5b \prec 4b \prec 4a \sim 3b \sim 3a \sim 2b \prec 2a \prec a \prec b$$

$$\text{Player}_4: \quad 5b \prec 4b \prec 5a \prec 4a \sim 3b \sim 3a \sim 2b \prec 2a \prec b \prec a$$

$$\text{Player}_5: \quad 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b$$

It is easy to see that:
$p_1 = 3$, $q_1 = 3$, $p_2 = 5$, $q_2 = 5$, $p_3 = 4$, $q_3 = 3$, $p_4 = 4$, $q_4 = 3$, $p_5 = 5$, $q_5 = 5$.
Hence, $n(a) = 4$ and $n(b) = 3$. By theorem 2, the possible congestion vectors are: $v_1 = (4a, b)$, $v_2 = (3a, 2b)$, $v_3 = (2a, 3b)$.
Since $v_1 = (4a, b)$, we have $A(G, v_1) = \emptyset$, $B(G, v_1) = \{1\}$ and $C(G, v_1) = \{2, 3, 4, 5\}$. Thus, there exists a unique equilibrium corresponding to v_1, which is the profile (b, a, a, a, a).
Similarly, as $v_2 = (3a, 2b)$ we get $A(G, v_2) = \emptyset$, $B(G, v_2) = \emptyset$ and $C(G, v_3) = \{1, 2, 3, 4, 5\}$. The Nash equilibria corresponding to v_2 are:

$$(b, b, a, a, a),\ (b, a, b, a, a),\ (b, a, a, b, a),\ (b, a, a, a, b),\ (a, b, a, a, b),$$
$$(a, a, b, a, b),\ (a, a, a, b, b),\ (a, b, a, b, a),\ (a, b, b, a, a),\ (a, a, b, b, a).$$

Finally, for $v_3 = (2a, 3b)$ we have $A(G, v_3) = \emptyset$, $B(G, v_3) = \emptyset$ and $C(G, v_3) = \{1, 2, 3, 4, 5\}$. The Nash equilibria corresponding to v_3 are:

$$(b, b, b, a, a),\ (b, b, a, b, a),\ (b, b, a, a, b),\ (b, a, a, b, b),\ (a, b, b, b, a),$$
$$(a, a, b, b, b),\ (b, a, b, b, a),\ (b, a, b, a, b),\ (a, b, b, a, b),\ (a, b, a, b, b).$$

Remark 2 If players' preferences are represented by a strict order, there is a single congestion vector; otherwise we can find at least one congestion vector.

5 Concluding remarks

In this paper we have proposed a new approach which enables one to find all Nash equilibria of a given symmetric singleton congestion game and a non-symmetric one with two strategies. While we do not deal with the question of the computational complexity, we believe that our method can contribute to the algorithmic analysis of this class of games. For instance, it can help to improve the time complexity of computing optimal Nash equilibria or calculate the price of anarchy. In our future research, we hope to extend our approach to the general case of non-symmetric congestion games with player-specific payoff functions.

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