The Probability of Casting a Decisive Vote : From IC to IAC through Ehrhart's Polynomials and Strong Mixing

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Abstract

The main purpose of this paper is to estimate the probability of casting a decisive vote for a class or random electorate models encompassing the celebrated IC and IAC models. The emphasis is on the impact of correlation across votes on the order of magnitude of this event. Our proof techniques use arguments from probability theory on one hand and the geometry of convex polytopes on the other hand.

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1 Introduction

The main purpose of this paper is to introduce a general model of a random electorate of N voters described by their preferences over two alternatives. Our model will admit, as special cases, the two most popular models in the literature on power measurement. The first, one called *Impartial Culture* (IC) is the basis of the celebrated Banzhaf power index (Banzhaf (1965, 1966, 1968)). It assumes that the preferences of the voters over the two alternatives are independent and equiprobable: correlation among the preferences of the voters is totally precluded. The second one, called *Impartial Anonymous Culture* (IAC) which has been pioneered independently in voting theory by Chamberlain and Rothschild (1981), Fishburn and Gehrlein (1976) and Kuga and Nagatani (1974) is the basis (as forcefully demonstrated by Straffin (1977, 1988)) of another celebrated power index due to Shapley and Shubik (Shapley and Shubik (1954), Straffin (1977, 1988)). The IAC model introduces correlation among voters and the specific distributional assumption which is considered implies that the real random variable defined as the number of voters supporting the first alternative is uniform over all feasible integers. From a computational perspective, this distributional property of the IAC model makes it very handy as compared to some other models and probably explains its success. Further, as noted convincingly by Chamberlain and Rothschild, the IAC model is more attractive than the IC model in the sense that the electoral predictions of the IAC models don't display a discontinuity in the neighborhood of the outcome of a tied election.

Given a random electorate λ, the power of a voter is defined as the probability of being pivotal[1] i.e. as the probability of the event[2] "There is a majority in favor of the first alternative iff that voter supports that alternative". Given that we will focus on a symmetric simple game (the ordinary majority game), if the model of random electorate λ is fully symmetric (i.e. if the preferences are interchangeable), then all voters will have the same power denoted $Piv(\lambda, N)$. Both the IC and the IAC models are symmetric. For the IC model, this defines the Banzhaf power index $Piv(IC, N)$ while for the IAC model this defines the Shapley-Shubik power index $Piv(IAC, N)$. It is well known that $Piv(IC, N)$ and $Piv(IAC, N)$ are respectively of order $\frac{1}{\sqrt{N}}$ and $\frac{1}{N}$.

The main purpose of this paper is to continue the exploration of the implications of *correlation* on the asymptotic behavior of the power index. Precisely, we will consider a general family of models of random electorate λ and study the asymptotic behavior of $Piv(\lambda, N)$

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[1] Good and Mayer (1975) refers to this as the *efficacy* of a vote.

[2] Of course, given a random electorate model (in particular in the case where there are more than two alternatives) we can calculate the probability of many other events and it becomes important to develop analytical representations of these probabilities (Huang and Chua (2000)).

with respect to N. Our motivation to do so is to depart from the IAC model which assumes that the correlation is the same for all pairs of voters in the population. It is likely that the intensity of the correlation between the votes of i and j will depend upon some characteristics of i and j suggesting that the correlation may vary from one pair to another. Most of the paper will however be based on a particular pattern of *heterogeneity*. Precisely, we will assume that the voters are partitioned into groups and that : correlation is positive and identical for any pair of voters belonging to the same group and null for any pair of voters belonging to two different groups. We will assume that within each group the correlation is defined as in the IAC model. This gives the IC and the IAC models as special cases : the IC model emerges when all the groups are singletons and the IAC model arises when there is a unique group which is then the entire population.

While particular, this model is general enough to cover many situations. We will offer a separate treatment of two polar cases. The first case is the case where there is a bound on the size of the groups; this bound does not depend upon the size of the population. This assumption is well suited to capture *local interactions* (within the family or the workplace for instance). The second case is the case where there is a fixed number of groups; this means that the size of the groups grows with the size of the population. This assumption is well suited to describe *large scale interactions* (special interest groups, geographical territories, electoral districts, countries if the population under scrutiny is multinational,...). After offering some general results, we proceed to the study of these two cases. The analysis of the two cases uses different techniques. When λ describes the local case, we conjecture that the use of some versions of the Central Limit Theorem allows to estimate $Piv(\lambda, N)$. Under the presumption that this local version holds, we show that it is of order $\frac{1}{\sqrt{N}}$ and we calculate explicitly $\underset{N\to\infty}{Lim}\ \sqrt{N}Piv(\lambda, N)$. In contrast, when λ describes the global case, our estimation of $Piv(\lambda, N)$ is based on different mathematical techniques. We approach the problem quite differently using combinatorial tools which amounts to study some polynomials known as Ehrhart's polynomials and the the volumes of some polytopes. We show that $Piv(\lambda, N)$ is of order $\frac{1}{N}$ and we calculate explicitly $\underset{N\to\infty}{Lim}\ NPiv(\lambda, N)$ in some specific cases. We also show through a sample of examples how our approach extends to situations where there is no a priori partition of the voters into well defined groups but instead more complicated correlation structures. We sketch the extension of our arguments to this general model.

Related Literature

We have already alluded to the pioneering contributions of Chamberlain and Rothschild, Fishburn and Gehrlein and Kuga and Nagatani. Our paper intersects different branches of the literature. On one hand, it aims to contribute to the systematic study of the implications of correlations on power measurement. Knowing the exact magnitude of the probability of being pivotal is interesting for itself but this information is also essential for the design of the optimal weights of representatives, as argued convincingly by Barbera and Jackson (2006). They discuss a block model which is quite similar to the model of partitions which is considered here except for the fact that instead of IAC, they assume perfect correlation within each block/group.

On the other hand, our model aims also to be a first step towards the analysis of the implications of correlations in a general model of random electorate. The general model of random electorate pioneered by Weber (1978, 1995) assumes independence but considers an arbitrary finite number of alternatives. Similarly, the model of Myerson and Weber (1993) and the general Poisson model developed by Myerson in a series of paper (1998, 2000) postulate independence. They both study the asymptotic properties of arbitrary voting rules when the strategic behavior of voters is taken into consideration. In our world of two alternatives, the difficulty attached to strategic voting disappears, which makes the analysis more simple. The probability of being pivotal between any two alternatives plays a critical role in their analysis of strategic voting.

2 The Model of a Random Electorate

A random electorate is a triple $\{\mathcal{N}, X, \lambda\}$ where $\mathcal{N}$ is a finite set of individuals (voters,...), X is a finite set of alternatives (candidates, parties,...) and λ is a probability distribution on $\mathcal{P}^{\mathcal{N}}$ where $\mathcal{P}$ is the set of linear orders over X. In the case where X consists of two alternatives say 0 and 1, the set $\mathcal{P}$ contains two preferences which will be coded 0 and 1 and $\mathcal{P}^{\mathcal{N}} = \{0,1\}^N$ where N denotes the cardinality of $\mathcal{N}$ i.e. the number of voters. The first popular random electorate model, called *Impartial Culture* (IC), is defined by $\lambda(P) = \frac{1}{2^N}$ for all profiles of preferences $P = (P_1, P_2, ..., P_N)$ in $\{0,1\}^N$. The IC model assumes that the preferences of the voters are independent Bernoulli random variables with a parameter p equal to $\frac{1}{2}$ (i.e. the electorate is not biased towards a particular candidate). In contrast, the second popular random electorate model, called *Impartial Anonymous Culture* (IAC) is defined by $\lambda(P) = \frac{1}{(N+1)\binom{N}{k}}$ for all profiles of preferences $P = (P_1, P_2, ..., P_N)$ in $\{0,1\}^N$ such that $\# N^0(P) = k$ where $N^0(P) \equiv \{i \in \mathcal{N} : P_i = 0\}$. In the IAC model, the events $E_k \equiv \left\{P \in \{0,1\}^N : \# \{i \in \mathcal{N} : P_i = 0\} = k\right\}$ for $k = 0, 1, ..., N$ are equally likely

A social choice mechanism is a mapping Ψ from $\{0,1\}^N$ into $[0,1]$ where $\Psi(P)$ denotes the probability of choosing candidate 0 when the profile of preferences is P. In this binary setting[3], we will not make any distinction between preferences and behavior. There is no room for strategic behavior here: if we interpret Ψ as a direct revelation game, then voting sincerely according to his/her preference is the unique dominant strategy. Further, we will focus[4] on the standard majority mechanism Maj defined as follows:

$$Maj(P) = \begin{cases} 0 \text{ if } \#N^0(P) < \frac{N}{2} \\ 1 \text{ if } \#N^0(P) > \frac{N}{2} \\ \frac{1}{2} \text{ if } \#N^0(P) = \frac{N}{2} \end{cases}$$

If N is odd, the third eventuality never arises and the mechanism is deterministic. If N is even, the third alternative arises when the electorate is split into two groups of equal size and the tie is broken fairly. The whole paper is about evaluating the probability of an event. We will say that voter $i \in \mathcal{N}$ is *pivotal* if either $\#N^0(P_{-i}) = \frac{N-1}{2}$ when N is odd or $\#N^0(P_{-i}) = \frac{N}{2}$ or $\#N^0(P_{-i}) = \frac{N-2}{2}$ when N is even. We denote by E_i this event and $Piv(\lambda, i)$ is the probability of E_i i.e. $Piv(i) = \lambda(E_i)$. There is a slight difference between the even and odd cases. In the odd case, the preference of i will be the social choice when i is pivotal. In contrast, in the even case, if i is pivotal and say on the 0 side, his preference will be for sure the social preference if $\#N^0(P_{-i}) = \frac{N}{2}$ and will be the social preference with probability $\frac{1}{2}$ if $\#N^0(P_{-i}) = \frac{N-2}{2}$. Up to this qualification, the two cases will be analyzed with similar methods.

Why should be interested in this event ? One strand of motivations has its roots in the power measurement literature where a lot of attention is paid to the fact of being influential: a player is defined as being powerful or influential if the probability that he is pivotal in the social choice mechanism is large. Scholars working in this area have calculated or estimated this probability for a large class of simple games including weighted majority games and compound games. A second strand of motivations, which does not apply here, appears when X contains more than two alternatives. With at least three alternatives any non trivial social mechanism will involve strategic voting. To determine his optimal strategic behavior a voter must estimate the probability of being pivotal for each possible pair of candidates. A voting equilibrium according to Myerson and Weber (1993) is a pair consisting of a profile of voting behaviors and a profile of pivot probabilities such that the voting behavioral responses are consistent with these probabilities and the beliefs about pivotal events are consistent with the voting behavior.

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[3] In this binary setting, a social choice mechanism is defined alternatively by a simple game (Taylor and Zwicker (1999)). A simple game is a monotonic family of coalitions $\mathcal{W}$. The mechanism is then defined as follows: $\Psi(P) = 0$ iff $\{i \in \mathcal{N} : P_i = 0\} \in \mathcal{W}$.

[4] In the last section, we will outline the difficulties in generalizing our formula to arbitrary simple games like those considered in the power measurement literature.

When the simple game is symmetric, if the probability measure is symmetric, then $Piv(\lambda, i)$ does not depend on i and will be denoted shortly by $Piv\,(\lambda)$. $Piv\,(\lambda)$ has been calculated for the two popular models of random electorate which have just been defined. For the IC model, $Piv\,(\lambda) = \binom{N-1}{\frac{N-1}{2}}\frac{1}{2^{N-1}}$ when N is odd and $Piv\,(\lambda) = \binom{N-1}{\frac{N-2}{2}}\frac{1}{2^{N-1}}$ when N is even. For the IAC model, $Piv\,(\lambda) = \frac{1}{N}$ for both cases. Using Strirling's formula, $N! \simeq \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$, we deduce that when N gets large $Piv\,(IC)$ behaves like $\sqrt{\frac{2}{\pi N}} \simeq \frac{0.797\,88}{\sqrt{N}}$.

In this paper, we assume that the electorate $\mathcal{N}$ is partitioned into K groups $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_K$ i.e. $\cup_{1\leq k\leq K}\mathcal{N}_k = \mathcal{N}$ and $\mathcal{N}_k \cap \mathcal{N}_{k'} = \varnothing$ for all k, k' such that $k \neq k'$. We will denote by N_k the size of group k: $\sum_{k=1}^{K} N_k = N$ and without loss of generality we assume that $N_1 \geq N_2 \geq ... \geq N_K$.We consider the following random electorate model.

We assume that the preferences of any voter i from group $\mathcal{N}_k$ is the realization of a Bernoulli random variable with parameter p_k and that conditional on p_k, the preferences of any two voters in that group are independent. We assume that the coordinates of the vector $(p_1, p_2, ..., p_K)$ are the realizations of K independent random variables with a uniform distribution on $[0, 1]$. Let i be an arbitrary voter in $\mathcal{N}_k$. Consider first the case where N is odd. We obtain:

$$Piv(\lambda, k) = \sum_{\Pi\left(\frac{N-1}{2}, N_1, ..., N_{k-1}, N_k - 1, N_{k+1}, ..., N_K\right)} \binom{N_k - 1}{x_k}\left(\int_0^1 p_l^{x_k}\left(1 - p_k\right)\right)^{N_k - x_k - 1} dp_k\right)$$

$$\times \left[\prod_{l\neq k}\binom{N_l}{x_l}\left(\int_0^1 p_l^{x_l}\left(1 - p_l\right)\right)^{N_l - x_l - 1} dp_l\right)\right]$$

where $\Pi\,(\ M, R_1, ..., R_k, ..., R_K)$ denotes the set of decompositions of the integer M into K ordered integers under the constraint that the k^{th} integer does not exceed R_k. By using the formula:

$$\int_0^1 p^{t-1}\left(1 - p\right)\right)^{n-t} dp = \frac{(t-1)!(n-t)!}{n!}$$

we deduce:

$$Piv(\lambda, k) = \sum_{\Pi\left(\frac{N-1}{2}, N_1, ..., N_{k-1}, N_k - 1, N_{k+1}, ..., N_K\right)} \frac{1}{N_k}\left(\prod_{l\neq k}\frac{1}{N_l + 1}\right) =$$

$$\pi\left(\frac{N-1}{2}, N_1, ..., N_{k-1}, N_k - 1, N_{k+1}, ..., N_K\right)\frac{1}{N_k}\left(\prod_{l\neq k}\frac{1}{N_l + 1}\right) \qquad (1.a)$$

where $\pi\left(M, R_1, ..., R_k, ..., R_K\right)$ denotes the cardinality of $\Pi\left(\ M, R_1, ..., R_k, ..., R_K\right)$ i.e. the number of decompositions of the integer M into K ordered integers under the constraint that the k^{th} integer does not exceed R_k.

When N is even, we obtain along the same lines:

$$2Piv(\lambda, k) = \sum_{\Pi\left(\frac{N-2}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)} \binom{N_k-1}{x_k} \left(\int_0^1 p_l^{x_k}\left(1-p_k\right)\right)^{N_k-x_k-1} dp_k\right)$$

$$\times\left[\prod_{l\neq k}\binom{N_l}{x_l}\left(\int_0^1 p_l^{x_l}\left(1-p_l\right)\right)^{N_l-x_l-1} dp_l\right)\right]$$

$$+ \sum_{\Pi\left(\frac{N}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)} \binom{N_k-1}{x_k} \left(\int_0^1 p_l^{x_k}\left(1-p_k\right)\right)^{N_k-x_k-1} dp_k\right)$$

$$\times\left[\prod_{l\neq k}\binom{N_l}{x_l}\left(\int_0^1 p_l^{x_l}\left(1-p_l\right)\right)^{N_l-x_l-1} dp_l\right)\right]$$

and therefore:

$$Piv(\lambda, k) = \frac{1}{2}[\pi\left(\frac{N-2}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)$$

$$+\pi\left(\frac{N}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)]\frac{1}{N_k}\left(\prod_{l\neq k}\frac{1}{N_l+1}\right)$$

$$= \pi\left(\frac{N-2}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)\frac{1}{N_k}\left(\prod_{l\neq k}\frac{1}{N_l+1}\right) \qquad (1.b)$$

as $\pi\left(\frac{N-2}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right) = \pi\left(\frac{N}{2}, N_1, ..., N_{k-1}, N_k-1, N_{k+1}, ..., N_K\right)$. The factor $\frac{1}{2}$ corresponds to the fact that when i is pivotal, there is only a chance of $\frac{1}{2}$ to be effective i.e. a chance of $\frac{1}{2}$ that the tie is broken in his favor. The interest of the two formulas above lies in the fact that the calculation of the pivot probabilities is equivalent to a well defined *combinatorial problem* which amounts to count the number of possible decompositions of a given integer into K integers under some constraints. Note however that there are at most K cells i.e. K non zero integers in the decomposition. This means that the problem is different from the problem of counting the number of partitions of a given integer. Further, for each cell, there is an upper bound on the integer for that cell.

Let us check quickly that the IC and IAC models correspond to two extreme special cases of this general framework. The IC value is attached to the case where $K = N$ i.e. the partition structure consists of N singletons:

$$Piv(IC, k) = \pi\left(\frac{N-1}{2}, 1, ..., 1, N_k - 1, 1, ..., 1\right)\frac{1}{2^N} = \binom{N-1}{\frac{N-1}{2}}\frac{1}{2^{N-1}}$$

since $\pi\left(\frac{N-1}{2}, 1, ..., 1, 0, 1, ..., 1\right) = \binom{N-1}{\frac{N-1}{2}}$. The IAC value is attached to the case where $K = 1$ i.e. the partition structure consists of a single set: the set $\mathcal{N}$:

$$Piv(IAC, k) = \pi\left(\frac{N-1}{2}, N-1\right)\frac{1}{N} = \frac{1}{N}$$

since $\pi\left(\frac{N-1}{2}, N-1\right) = 1$.

An alternative approach to the counting problem is based on probability. Let X_{ik} denote the Bernoulli random variable describing the preference of voter i in group k and let S_k and S denote respectively the sums $\sum_{i\in\mathcal{N}_k} X_{ik}$ and $\sum_{k=1}^{K}\sum_{i\in\mathcal{N}_k} X_{ik} = \sum_{k=1}^{K} S_k$. With these notations, we can express the pivot probabilities as follows:

$$\begin{aligned} Piv(\lambda, k) &= \lambda\left(S_{-i} = \frac{N-1}{2}\right) \text{ when } N \text{ is odd and} \\ Piv(\lambda, k) &= \frac{1}{2}\left(\lambda\left(S_{-i} = \frac{N-2}{2}\right) + \lambda\left(S_{-i} = \frac{N}{2}\right)\right) \text{ when } N \text{ is even} \end{aligned}$$

This probabilistic approach will be very useful when we will focus on the asymptotic behavior of $Piv(\lambda, k)$ when N tends to infinity. We will have to be very careful in running asymptotic arguments about the behavior of the partition as N tends to infinity. Note that all the random variables X_{ik} are symmetric in the sense that $\Pr(X_{ik} = 0) = \Pr(X_{ik} = 1) = \frac{1}{2}$ since $\Pr(X_{ik} = 0) = \int_0^1 pdp = \frac{1}{2}$. We have $E\left[X_{ik}\right] = \int_0^1 pdp = \frac{1}{2}$ and $Var\left[X_{ik}\right] = \frac{1}{4}$. These variables do not need to be independent. Consider two random variables X_{ik} and X_{jk} i.e. the random variables describing the voting behavior of two voters from group k. We have:

$$\Pr(X_{ik} = 0, X_{jk} = 0) = \int_0^1 p^2 dp = \frac{1}{3} > \frac{1}{4}$$

The two variables are positively correlated: $Cov(X_{iR}, X_{jR}) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$; the coefficient of correlation ρ is then equal to $\frac{1}{3}$.

3 The case of Many Small Groups

In this section, we will focus on the case where there is an exogenous upper bound S on the size of the groups in the partition $(\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_K)$. This implies that as N gets large, then the number of groups increases.

To motivate the general result which will be presented hereafter, it is instructive to consider the case where $S = 2$. In any such partition structure , the groups are either singletons or pairs. We can think of this partition as describing a society where there are singles and couples but no other family types. Consider the case where N is even and all the groups are exactly of size 2. From (1.b), we deduce that:

$$Piv(\lambda, k) = Piv(\lambda) = \pi\left(\frac{N-2}{2}, 1, 2, ..., 2, 2\right)\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{N-2}{2}}.$$

We can check that:[5]

$$\pi\left(\frac{N-2}{2}, 1, 2, ..., 2, 2\right) = \sum_{k=0}^{\left\lfloor\frac{N-2}{4}\right\rfloor}\frac{\left(\frac{N-2}{2}\right)!}{(k!)^2\left(\left(\frac{N-2}{2}-2k\right)!\right)}\left(\frac{\frac{N}{2}-k}{k+1}\right).$$

Indeed, counting how many decompositions of $\frac{N-2}{2}$ into $\frac{N}{2}$ integers chosen in $\{0, 1, 2\}$ amounts first to choose how many pairs k we choose among $\frac{N-2}{2}$. The number of possibilities is $\frac{\left(\frac{N-2}{2}\right)!}{(k!)\left(\left(\frac{N-2}{2}-k\right)!\right)}$. This value of k cannot exceed $\left\lfloor\frac{N-2}{4}\right\rfloor$. To reach the integer $\frac{N-2}{2}$, we need $\frac{N-2}{2} - 2k$ singletons which can be chosen among $\frac{N}{2} - k$. The number of possibilities is $\frac{\left(\frac{N}{2}-k\right)!}{\left(\frac{N-2}{2}-2k\right)!(k+1)!} = \frac{\left(\frac{N}{2}-1-k\right)!}{\left(\frac{N-2}{2}-2k\right)!(k)!}\frac{\frac{N}{2}-k}{k+1}$. After collecting the terms, we obtain the expression reported above.

Calculating the above sum is not a straightforward combinatorial exercise and we will mostly focus on the asymptotic behavior of $Piv(\lambda)$.

We conjecture that:

$$\underset{N\to\infty}{Lim}\Phi(N) \equiv \sqrt{N}\left(\sum_{k=0}^{\left\lfloor\frac{N-2}{4}\right\rfloor}\frac{\left(\frac{N-2}{2}\right)!}{(k!)^2\left(\left(\frac{N-2}{2}-2k\right)!\right)} \times \frac{N-2k}{2k+2}\right) \times \frac{1}{2} \times \left(\frac{1}{3}\right)^{\frac{N-2}{2}} \quad \text{exists}$$

The following table contains some numerical values of $\Phi(N)$ which supports this conjecture:

N	102	202	1002	5002	100002
$\Psi(N)$	0.69015	0.69056	0.6909	0.69097	0.69098

Table 1: Values of $\Phi(N)$

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[5] $\lfloor x \rfloor$ denotes the integer part of x.

Interestingly, the function Φ seems to behave asymptotically as the function Ψ defined as follows:

$$\Psi(N) \equiv \sqrt{N} \left(\sum_{k=0}^{\lfloor \frac{N-2}{4} \rfloor} \frac{\left(\frac{N-2}{2}\right)!}{(k!)^2 \left(\left(\frac{N-2}{2} - 2k\right)!\right)} \right) \times \left(\frac{1}{3}\right)^{\frac{N-2}{2}}$$

The following table contains some numerical values[6] of $\Psi(N)$ which supports this suspicion:

N	102	202	1002	10002	100002
$\Psi(N)$	0.69525	0.69314	0.69143	0.69103	0.69099

Table 1 bis: Values of $\Psi(N)$

We now prove a generalized version of the conjecture. To proceed, we use a probabilistic approach. We assume that all the groups have a size smaller than S and we will be interested in societies where the set of voters is partitioned into groups of size s where s runs from 1 to S. We will consider societies where N gets indefinitely large but such that the proportion of the population in each type of group (described by its size) remains invariant in the population growth process. We will denote by γ^s the proportion of voters in a group of size s. We assume that $\gamma^s = \frac{sK^s}{K}$ where K^s is an integer for all $s = 1, ..., S$ and $K = \sum_{s=1}^{S} sK^s$. The initial society contains K^s groups of size s. For any integer R, its R^{th} replica has N voters where N is defined as follows:

$$N = N(R) = R \sum_{s=1}^{S} K^s s$$

For all R and all $i = 1, 2, ..., N(R)$, we arrange the random variables X_i^R describing the individual votes in the R^{th} replica in a *triangular array* defined as follows: the first RK^1 variables describe the vote of voters in groups of size 1, the next $2RK^2$ variables describe the votes of voters in groups of size 2 and so on.

We obtain

$$\sigma^2(R) \equiv Var(\sum_{i=1}^{N} X_i^R) = \sum_{i=1}^{N} Var(X_i^R) + \sum_{s=1}^{S} RK^s s(s-1) Cov(X_i^R, X_j^R)$$

where $Cov(X_i^R, X_j^R)$ denotes the covariance between when i and j belong to the same group. We have shown before that:

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[6] We show in appendix 4 that the largest value in $\Psi(N)$ behaves as $\frac{\sqrt{3}}{2\pi m}$.

$$Var(X_i^R) = \frac{1}{4} \text{ for all } i = 1, 2, ..., N$$

$$Cov(X_i^R, X_j^R) = \frac{1}{12} \text{ for all } i, j = 1, ..., N \text{ if } i \text{ and } j \text{ belong to the same group}$$

We obtain:

$$\sigma(R) = \sqrt{N} \left(\sqrt{\frac{1}{6} + \frac{\gamma^1}{12} + \frac{1}{12} \sum_{s=2}^{S} \gamma^s s} \right)$$

A random variable X_i^R is of type s if $R \sum_{l=1}^{s-1} lK^l < i \leq R \sum_{l=1}^{s} lK^l$. We pack the sRK^s random variables of type s into RK^s random variables $\left(Z_{ks}^R\right)_{1 \leq k \leq RK^s}$ where Z_{ks}^R is defined as follows :

$$Z_{ks}^R = r \text{ iff } \sum_{i=(k-1)s+1}^{ks} 1_{X_{is}^R} = r$$

This defines a new triangular array $\left(Z_{ks}^R\right)_{1 \leq s \leq S, 1 \leq k \leq RK^s}$ (indexed by R) where the random variables Z_{ks}^R are independent. Hereafter, we will refer to Z_{ks}^R as a random variable of type s. We note that all random variables are integer valued: the support of a random variable of type s is $\{0, 1, ..., s\}$ Let $1 \leq i \leq N(R)$ be a member of a group of type s and for each value of the row index R, consider the random variable S_i^R defined as follows:

$$S_i^R = \sum_{j=1, j \neq i}^{N(R)} X_j^R = \sum_{l=1, l \neq s}^{S} \sum_{k=1}^{K^l} Z_{kl}^R + \sum_{k=1}^{K^s - 1} Z_{ks}^R + W_i^R$$

where $W_i^R \equiv \sum_{j=2}^{s} 1_{X_{js}^R}$. The probability that i of type s is pivotal, $Piv(\lambda_R, s)$ is equal to the probability of the event $\left\{S_i^R = \frac{N-1}{2}\right\}$ if N is odd and to half the probability of the event $\left\{S_i^R = \frac{N-2}{2}\right\} \cup \left\{S_i^R = \frac{N}{2}\right\}$ if N is even.

We note that the span of the random variables Z_{kl}^R for $1 \leq l \leq S$ and $1 \leq k \leq K^l$ and W_i^R is equal to 1. Further, the distribution functions of these random variables belong to a finite set of cardinality at most S, are not degenerate and occur infinitely often in the sequence $\left(\left(Z_{kl}^R\right)_{1 \leq l \leq S, 1 \leq k \leq K^l} \cup \left\{W_i^R\right\}\right)^{R \geq 1}$. Let $\epsilon > 0$.

If N is odd, since $E\left[S_i^R\right] = \frac{N-1}{2}$, we deduce from Petrov's theorem that if R is large enough:

$$\left| \sigma(R) \, Piv(\lambda_R, s) - \frac{1}{\sqrt{2\pi}} \right| \leq \epsilon$$

Similarly, if N is even, since $E\left[S_i^R\right] = \frac{N-1}{2}$, we deduce from Petrov's theorem that if R is large enough:

$$\left|\sigma\left(S_i^R\right)\Pr\left(S_i^R = \frac{N-2}{2}\right) - \frac{e^{-\frac{1}{8\sigma(R)}}}{\sqrt{2\pi}}\right| \leq \epsilon$$

$$\left|\sigma\left(S_i^R\right)\Pr\left(S_i^R = \frac{N}{2}\right) - \frac{e^{-\frac{1}{8\sigma(R)}}}{\sqrt{2\pi}}\right| \leq \epsilon$$

Since $Piv(\lambda_R, s) = \frac{\Pr\left(S_i^R = \frac{N-2}{2}\right) + \Pr\left(S_i^R = \frac{N}{2}\right)}{2}$, $\frac{e^{-\frac{1}{8\sigma(R)}}}{\sqrt{2\pi}}$ tends to $\frac{1}{\sqrt{2\pi}}$ and $\frac{\sigma\left(S_i^R\right)}{\sigma(R)} = \frac{\sigma(R) - \frac{1}{12} - \frac{s}{6}}{\sigma(R)}$ tends to 1 when R tends to $+\infty$, we deduce that if R is large enough:

$$\left|\sigma\left(R\right)Piv(\lambda_R, s) - \frac{1}{\sqrt{2\pi}}\right| \leq \epsilon.$$

Proposition 1: Let λ_R be the random electorate defined above. For all $s = 1, 2, ..., S$

$$\underset{R\to\infty}{Lim}\sqrt{N}Piv(\lambda_R, s) = \frac{1}{\left(\sqrt{\frac{1}{6} + \frac{\gamma^1}{12} + \frac{1}{12}\sum_{s=2}^{S}\gamma^s s}\right)\sqrt{2\pi}}.$$

The random variable $S_i^R = S^R - X_i^R$ introduced in the proof of Proposition 1 counts the number of votes in favor of 1 in the population without individual i. Proposition 1 provides information on the asymptotic behavior of the probability of the event $\left\{S_i^R = \frac{N-1}{2}\right\}$ which plays a quite important role as $\frac{N-1}{2}$ is the first moment of S_i^R.

Proposition 1 provides information on the probability mass of the random variable S^R in the neighborhood of its mean. We could however under the presumption that a version of the central limit theorem holds in this setting derive more information on the behavior of the aggregate vote S^R. Berks's theorem, reported in Appendix 2[7], provides useful information on the asymptotic behavior of S^R. It is straightforward to check that conditions (i), (ii), (iii) and (iv) of Berk's theorem are verified here. Since then $\frac{S^R - \frac{N}{2}}{\sqrt{\sigma(R)}}$ converges to a unit Gaussian, we can estimate the probability of the event $\frac{S^R - \frac{N}{2}}{\sqrt{\sigma(R)}} \in [-x, x]$ for some fixed positive real number x as $\Phi(x) - \Phi(-x) = 1 - 2\Phi(-x)$. For instance for $x = 1.65$, $\Phi(x) - \Phi(-x) = 0.90$. therefore for R large enough the probability of the event $\frac{S^R - \frac{N}{2}}{\sqrt{\sigma(R)}} \in [-x, x]$ is approximatively equal to 90%. Equivalently the probability of the event $\frac{S^R}{N} \in \left[\frac{1}{2} - 1.65\frac{\sqrt{\sigma(R))}}{N}, \frac{1}{2} + 1.65\frac{\sqrt{\sigma(R)}}{N}\right]$

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[7] Note that Berk's theorem can accommodate correlation patterns much more general than those covered by the partitioning model in the sense that correlations need not be uniform within a block.

is close to 90%. Since here, $\sqrt{\sigma(R)} = \sqrt{N}\left(\sqrt{\frac{1}{6} + \frac{\gamma^1}{12} + \frac{1}{12}\sum_{s=2}^{S}\gamma^s s}\right) = c\sqrt{N}$, we deduce that there is a 90% chance for the percentage of people voting left to be in the interval $\left[\frac{1}{2} - 1.65\frac{c}{\sqrt{N}}, \frac{1}{2} + 1.65\frac{c}{\sqrt{N}}\right]$. If $N = 27225c^2$, then with probability almost equal to 90%, the percentage obtained by the left is in the interval $[0.49, 0.51]$.

Let us illustrate Proposition 1 through a sample of applications. Consider first the case of an electorate where all the groups have the same size s. This electorate is denoted λ_R^s . In such case we deduce from our result that:

$$Piv(\lambda_R^s) \simeq \frac{1}{\sqrt{N}} \times 2\sqrt{\frac{3}{2\pi(2+s)}}$$

The following table lists a sample of values of the probability of being pivotal for a sample of values of s.

s	1	2	3	4	5	...	10
$\sqrt{N}Piv(\lambda_R^s)$	0.798	0.691	0.618	0.564	0.522	...	0.399

Table 2: Probability of being pivotal as a function of s

We can also handle mixed situations i.e. random electorates λ where the sizes of the groups differ across voters. For instance ,when the random electorate λ is such and $\gamma^1 = 0.2$, $\gamma^2 = 0.3$, $\gamma^3 = 0.4$ and $\gamma^4 = 0.1$, we obtain : $Piv(\lambda_R) \simeq 0.658\,85$. We could interpret these groups as family groups : singles, couples without children voting, couples with one children voting, and so on.

The proof strategy of Proposition 1 based on Petrov's local central lmit theorem has exploited the fact that the individuals could be packed in a regular way. We could imagine a more general situation where the individuals could be arranged from left to right in such a way that two individuals distant from each other by more that some given number m (which may vary with the size of the population) vote independently. We could proceed as in the proof of Proposition 1 i.e. pack together m consecutive individual random variables. Even when m is fixed, we have no guarantee tha the number of distributions in the sequence of random variables which is constructed through this packing process is finite and we cannot therefore appply Petrov's theorem. To handle such more general situations, we need to appeal to a more general local central limit theorem like, for instance the version proved by Mc Donald (1979).

4 The Case of Few Large Groups

In this section we consider the polar case of a society divided into a finite (possibly large) number of groups. This means that as N gets larger and larger, the number of voters in each group gets larger and larger. We could apply the probabilistic approach which has been used in the preceding section. It was using extensively the observation that the sequence of Bernoulli random variables describing the votes of the citizens was exhibiting a property of m-dependence where m was independent of the size of the electorate. This approach cannot be used here as assumption (iv) on the growth of m in Berk's theorem is not satisfied when there is a finite number of groups. To circumvent this difficulty, we will approach the problem from a combinatorial angle and use the theory of Ehrart's polynomials and some of its developments[8].

4.1 Ehrhart theory and Barvinok's algorithm

For fixed values of K, the general problem of computing the number $\pi(M, R_1, ..., R_k, ..., R_K)$ can be phrased as counting the exact number of integer solutions of a system of linear inequalities with integer coefficients, where the variables are x_k $(k = 1, ..., K)$ and the parameters are M and R_k $(k = 1, ..., K)$. This system is :

$$\begin{cases} x_k \geq 0 \text{ for all } k = 1, ..., K \\ x_k \leq R_k \text{ for all } k = 1, ..., K \\ \sum_{k=1}^{K} x_k \leq M \\ \sum_{k=1}^{K} -x_k \leq -M \end{cases}$$

There is a well established mathematical theory for performing such a calculation, based on the use of (parametric) polytopes and Ehrhart polynomials. Lepelley et al. (2008) and Wilson and Pritchard (2007) were the first to introduce these tools in probability calculations under IAC hypothesis in voting theory. We refer to their papers for more details and we limit ourself, in this paragraph, to a short presentation of Ehrhart theorem and its extensions. Also, we only sketch the key idea of the algorithm we have used to compute Ehrhart polynomials[9].

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[8] There is a voluminous literature on this topic (Baldoni-Silva and Vergne (1999), Brion (1995), (1998), Brion and Vergne (1997), Cochet (2001) to cite few).

[9] For more details on Ehrhart theory we refer to Beck and Robins (2007) and for a general background

Consider a finite system of linear inequalities with integer coefficients: $Ax \leq b$, where x is in $\mathbb{R}^d$, A is an $m \times d$ integer matrix, b an integer vector with m components and m the number of independent linear inequalities. Let $\mathbf{P}$ be the set of all solutions of this system, $\mathbf{P}$ is called a rational polyhedron. If $\mathbf{P}$ is bounded, it is called a rational polytope. An extremal point of $\mathbf{P}$ is called a vertex, and $\mathbf{P}$ can be defined equivalently as the convex hull of its vertices. A simple case of parametric polytope is is the dilatation of a rational polytope $\mathbf{P}$ by a positive integer parameter n: $n\mathbf{P} = \{nx | x \in \mathbf{P}\}$. Let $L_{\mathbf{P}}$ be the function defined by $L_{\mathbf{P}}(n) = |n\mathbf{P} \cap \mathbb{Z}^d|$, giving the number of integer points inside the dilated polytope $n\mathbf{P}$. To describe the general form of this function, we need the two following notions. A rational periodic number, of period q, on the integer variable n is a function $U : \mathbb{Z} \to \mathbb{Q}$ such that $U(n) = U(n')$ whenever $n \equiv n' \mod q$. A quasi-polynomial (or Ehrhart polynomial) on n is a polynomial expression $f(n)$ on the variable n, $f(n) = \sum_{i=0}^{n} c_i(n)n^i$, where the coefficients $c_i(n)$ are rational periodic numbers on n. The period of a quasi-polynomial is the last common multiple (*lcm*) of the periods of its coefficients.

Theorem (Ehrhart (1962)): Let $\mathbf{P}$ be a rational polytope in $\mathbb{R}^d$. If $\mathbf{P}$ is d-dimensional, then[10]:

1. The function $L(\mathbf{P}, n)$ is given by a degree-d quasi-polynomial.
2. The coefficient of the leading term is independent of n and is equal to the volume of $\mathbf{P}$.
3. The period of the quasi-polynomial is a divisor of the *lcm* of the denominators of the vertices of $n\mathbf{P}$. When all the vertices of P have integral coordinates, $L_{\mathbf{P}}(n)$ is simply a polynomial.

The above result can be extended to more general situations with more than one parameter. Define a (linearly) parameterized polyhedron as the solution set of a system of linear inequalities where the constant terms in each constraint is an affine combination of a set of integer parameters: $\mathbf{P_p} = \{x \in \mathbb{R}^d | Ax \leq C\mathbf{p} + b\}$, where A and C are integer matrices, b is an integer vector and $\mathbf{p}$ a vector of r integer parameters. When $\mathbf{P_p}$ is bounded for each

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on algorithms computing Ehrhart polynomials, we recommend the technical report produced by Verdoolage et al. (2005).

[10] Note that $\mathbf{P}$ can be not full-dimensional; this is the case when the linear system describing $\mathbf{P}$ contains equalities. However there is no loss of generality with assuming $\mathbf{P}$ full-dimensional: If this is not the case, $\mathbf{P}$ can be transformed into another polytope which has the same number of integer points and is full-dimensional in a lower dimensional space (see Verdoolage et al. (2004), (2005)).

value of $\mathbf{p}$, it will be called a parametric polytope. The coordinates of the vertices of a parametric polytope are affine functions of the parameters. Each vertex only exists for a subset of the possible parameter values. Separate regions of the vector parameter space $\mathbb{N}^r$ where the vertices have stable expressions are called validity domains. Clauss and Loechner (1998) consider the enumerator function $E(\mathbf{P_p})$ that describes the number of integer points in a d-dimensional parametric polytope $\mathbf{P_p}$. They extended Ehrhart's result by showing that $E(\mathbf{P_p})$ can be described by a finite set of multivariate quasi-polynomials[11] of degree d in $\mathbf{p}$, each being valid on a different validity domain. They also proposed an algorithm for computing Ehrhart polynomials, based on the classical technique of interpolation. However, this method is seriously limited because the computation time is generally exponential and, in some cases, the algorithm can fail to produce a solution (Beyls (2004)).

To avoid these problems, an alternative approach for computing $E(\mathbf{P_p})$ was proposed by Verdoolaege et al. (2004). This method, known under the name of Parameterized Barvinok's algorithm, is essentially an adaptation of Barvinok' algorithm (Barvinok (1994), Barvinok and Pommersheim (1999)) to parametric polytopes. Barvinok's algorithm is a powerful tool that guarantees the polynomial-time counting of integer points inside rational polytopes (for fixed dimension)[12]. The key idea is to encode all the integer points inside a rational polyhedron $\mathbf{P}$ (not necessarily a polytope) into a multivariate generating function defined by:

$$f(\mathbf{P}, x) = \sum_{z \in P \cap \mathbb{Z}^d} x^z$$

where $x = (x_1, \ldots, x_d)$, $z = (z_1, \ldots, z_d)$ and $x^z = x_1^{z_1} \ldots x_d^{z_d}$. It is clear that, when $\mathbf{P}$ is a polytope, this sum is a (Laurent) polynomial and the number of integer points in $\mathbf{P}$ is equal to the number of monomials in the generating function. Thus, the number of integer points in $\mathbf{P}$ can be obtained by rewriting $f(\mathbf{P}, x)$ as a reasonably short function and then evaluating it at $x = (1, \ldots, 1)$. Barvinok's method uses a crucial identity of Brion (1995) to distribute the computation of $f(\mathbf{P}, x)$ on the vertices of $\mathbf{P}$ by considering the supporting cone at each vertex[13]. Indeed, Brion's theorem states that the generating function of a polytope is equal to the sum of the generating functions of the supporting cones at each vertex. The remainder of Barvinok's procedure uses an inclusion-exclusion method to replace the

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[11] A multivariate quasi-polynomial is a multivariate polynomial expression where the coefficients depend periodically on each variable.

[12] It was latter refined and implemented in the software LattE by De Loera et al. (2004)

[13] The supporting cone at a vertex is the polyhedron defined by the constraints that are saturated by the vertex, i.e., those for which equality holds for the vertex.

generating function of each supporting cone with a signed sum of polynomial number (in the size of the data) of unimodular cones[14]. The generating functions of these cones are simple and short rational functions that can be calculated explicitly. The function $f(\mathbf{P}, x)$ is then the sum of short rational functions. Note that the point $(1, \dots, 1)$ is a pole of all these functions, the evaluation of $f(\mathbf{P}, x)$ at this point is obtained by computing the residues[15].

Parameterized Barvinok's algorithm, which allows to compute Ehrhart polynomials analytically, keeps the overall structure of Barvinok's algorithm, but takes into account validity domains and handles periodic numbers. This technique always produces a solution in polynomial time, when the number of variables in the inequalities is fixed[16]. The results presented in subsections 4.4 and 4.5 (for $K = 5, 7, 9, 11$) have been obtained by applying this algorithm.

4.2 A Preliminary Result

It can be noticed that, when the number N_1 of voters in the largest group represents more than 50% of the total number of voters, then the probability of casting a decisive vote only depends, in each group, on the value of N_1. More precisely, we have the following general result (Recall that $\lfloor x \rfloor$ denotes the integer part of x).

Proposition 2: If $N_1 \geq \left\lfloor \frac{N}{2} \right\rfloor + 1$, then $Piv(\lambda, 1) = \frac{1}{N_1}$ and $Piv(\lambda, k) = \frac{1}{N_1+1}$ for $k = 2, 3, ..., K$.

Proof. Let x_k be the value of the k^{th} term in the decomposition of $\left\lfloor \frac{N-1}{2} \right\rfloor$: $x_1 + ... + x_k + ... + x_K = \left\lfloor \frac{N-1}{2} \right\rfloor$. If $N_1 \geq \left\lfloor \frac{N}{2} \right\rfloor + 1$, then $N_2 + N_3 + ... + N_k + ... + N_K \leq \left\lfloor \frac{N-1}{2} \right\rfloor$. Consequently, for $k = 2, 3, ..., K$, x_k can take any integer value between 0 and N_k (including 0 and N_k) and when $x_2, x_3, ..., x_K$ are set, the value of x_1 is given in a unique way by $x_1 = \left\lfloor \frac{N-1}{2} \right\rfloor - x_2 - x_3 - ... - x_K$. The number of possible decompositions is then given by

$$\pi(\left\lfloor \frac{N-1}{2} \right\rfloor, N_1, N_2, ..., N_k, ..., N_K) = (N_2 + 1)(N_3 + 1)...(N_K + 1)$$

and the result follows from relations (1.a) and (1.b). □

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[14] Let v ,u_1, . . . , u_t in $\mathbb{R}^d$. The (shifted) cone with apex v and generators u_1, . . . , u_t is the set C defined by $C = \{v + \sum_{i=1}^{t} \alpha_i u_i | \alpha_i \geq 0\}$. The cone C is called unimodular if its generators form a basis of the lattice $\mathbb{Z}^d$.

[15] For detailed explanation, see De Loera (2004) and Verdoolage et al. (2005).

[16] For a rigorous description of this algorithm and for implementation details, see Verdoolage et al. (2005).

4.3 The Case of Two Groups

Let us consider the case where $K = 2$ i.e. the situation where the voters are partitioned into two groups. This setting has been examined by various authors in the literature including Beck (1975), Kleiner (1980), Chamberlain and Rothschild (1981) and Le Breton et Lepelley (2010).

In such a case, if N is odd, then $N_1 > N_2$ as the two integers don't have the same parity. It is easily seen that:

$$\pi\left(\frac{N-1}{2}, N_1 - 1, N_2\right) = N_2 + 1 \text{ and } \pi\left(\frac{N-1}{2}, N_1, N_2 - 1\right) = N_2$$

and therefore:

$$Piv(\lambda, 1) = \frac{1}{N_1} \text{ and } Piv(\lambda, 2) = \frac{1}{N_1 + 1}$$

in accordance with Proposition 2.

4.4 Three groups of voters

In this section, we consider the case where the population is divided into three groups of voters i.e. $K = 3$: $N_1 \geq N_2 \geq N_3$ and $N_1 + N_2 + N_3 = \widehat{N} + 1$, with $\widehat{N}$ even.

The value of $\pi(\frac{\widehat{N}}{2}, N_1 - 1, N_2, N_3)$ is given by the number of integer solutions of the following set of (in)equalities, where x_k can be interpreted as the number of voters voting Left in group k, $k = 1, 2, 3$:

$$\begin{gathered} 0 \leq x_1 \leq N_1 - 1 \\ 0 \leq x_2 \leq N_2 \\ 0 \leq x_3 \leq N_3 \\ x_1 + x_2 + x_3 = \tfrac{\widehat{N}}{2} \end{gathered}$$

Given the last equality, $N_3 = N - N_1 - N_2$ and the above set of inequalities reduces to:

$$\begin{gathered} 0 \leq x_1 \leq N_1 - 1 \\ 0 \leq x_2 \leq N_2 \\ 0 \leq x_3 \leq N - N_1 - N_2 \\ x_1 + x_2 + x_3 = \tfrac{\widehat{N}}{2} \end{gathered}$$

where the parameters satisfy:

$$\begin{gathered} N_1 \geq N_2 \\ 2N_2 + N_1 - N - 1 \geq 0 \text{ and} \\ N_1 + N_2 \leq N + 1 \end{gathered}$$

A representation for the number of integer solutions of this set of inequalities with three variables and three parameters (N_1, N_2 and N) can be derived by using the multiparameter version of the Barvinok's algorithm (see Lepelley et al. (2008)). We obtain:

$$\pi(\frac{\widehat{N}}{2}, N_1 - 1, N_2, N_3) = (\widehat{N} - N_1 - N_2 + 2)(N_2 + 1) = (N_3 + 1)(N_2 + 1)$$

if $N_1 \geq \frac{\widehat{N}}{2} + 1$ and

$$\pi(\frac{\widehat{N}}{2}, N_1 - 1, N_2, N_3) = (-\widehat{N}^2 + 2\widehat{N}(2N_1 + 2N_2 - 1) - 4(N_1^2 + N_1(N_2 - 2) + N_2(N_2 - 1))/4$$

if $N_1 \leq \frac{\widehat{N}}{2}$.

Representations for $\pi(\frac{\widehat{N}}{2}, N_1, N_2 - 1, N_3)$ and $\pi(\frac{\widehat{N}}{2}, N_1, N_2, N_3 - 1)$ can be derived in a similar way to obtain:

$$\pi(\frac{\widehat{N}}{2}, N_1, N_2 - 1, N_3) = (\widehat{N} - N_1 - N_2 + 2)N_2 = (N_3 + 1)N_2$$

if $N_1 \geq \frac{\widehat{N}}{2} + 1$ and

$$\pi(\frac{\widehat{N}}{2}, N_1, N_2 - 1, N_3) = (-\widehat{N}^2 + 2\widehat{N}(2N_1 + 2N_2 - 1) - 4(N_1^2 + N_1(N_2 - 1) + N_2(N_2 - 1))/4$$

if $N_1 \leq \frac{\widehat{N}}{2}$;

$$\pi(\frac{\widehat{N}}{2}, N_1, N_2, N_3 - 1) = (\widehat{N} - N_1 - N_2 + 1)(N_2 + 1) = N_3(N_2 + 1)$$

if $N_1 \geq \frac{\widehat{N}}{2} + 1$ and

$$\pi(\frac{\widehat{N}}{2}, N_1, N_2, N_3 - 1) = (-\widehat{N}^2 + 2\widehat{N}(2N_1 + 2N_2 + 1) - 4(N_1^2 + N_1N_2 + N_2^2 - 1))/4$$

if $N_1 \leq \frac{\widehat{N}}{2}$.

Observe that we recover the results we have mentioned for two groups by taking $N_3 = 0$. From the above results, we can now derive the probability of casting a decisive vote for a voter belonging to each of the three groups. We obtain :

$$Piv(\lambda, 1) = \frac{(N_3 + 1)(N_2 + 1)}{N_1(N_2 + 1)(N_3 + 1)} = \frac{1}{N_1}$$

$$Piv(\lambda, 2) = \frac{(N_3 + 1)N_2}{(N_1 + 1)N_2(N_3 + 1)} = \frac{1}{N_1 + 1}$$

$$Piv(\lambda, 3) = \frac{N_3(N_2+1)}{(N_1+1)(N_2+1)N_3} = \frac{1}{N_1+1}$$

if $N_1 \geq \frac{\widehat{N}}{2} + 1$ (in accordance with our preliminary result), and

$$Piv(\lambda, 1) = \frac{4N_1^2 + 4N_1(N_2 - \widehat{N} - 2) + 4N_2^2 - 4N_2(\widehat{N}+1) + \widehat{N}(\widehat{N}+2)}{4N_1(N_2+1)(N_1+N_2-\widehat{N}-2)} \tag{2}$$

$$Piv(\lambda, 2) = \frac{4N_1^2 + 4N_1(N_2 - \widehat{N} - 1) + 4N_2^2 - 4N_2(\widehat{N}+2) + \widehat{N}(\widehat{N}+2)}{4(N_1+1)N_2(N_1+N_2-\widehat{N}-2)} \tag{3}$$

$$Piv(\lambda, 3) = \frac{4N_1^2 + 4N_1(N_2 - \widehat{N}) + 4N_2^2 - 4N_2\widehat{N} + \widehat{N}^2 - 2\widehat{N} - 4)}{4(N_1+1)(N_2+1)(N_1+N_2-\widehat{N}-1)} \tag{4}$$

if $N_1 \leq \frac{\widehat{N}}{2}$.

In order to simplify the above three representations, let $\alpha_1 = N_1/\widehat{N}$ and $\alpha_2 = N_2/\widehat{N}$ denote the proportion of voters in the first and the second group. Replacing N_1 by $\alpha_1\widehat{N}$ and N_2 by $\alpha_2\widehat{N}$ and assuming that $\widehat{N}$ is large give, for $k = 1, 2, 3$ and $\alpha_1 \leq 0.50$:

$$Piv(\lambda, k) \simeq \frac{4\alpha_1^2 + 4\alpha_1\alpha_2 - 4\alpha_1 + 4\alpha_2^2 - 4\alpha_2 + 1}{4\alpha_1\alpha_2(\alpha_1+\alpha_2-1)} \times \frac{1}{N}.$$

Let $c_3(\alpha_1, \alpha_2) = \frac{4\alpha_1^2+4\alpha_1\alpha_2-4\alpha_1+4\alpha_2^2-4\alpha_2+1}{4\alpha_1\alpha_2(\alpha_1+\alpha_2-1)}$ if $\alpha_1 \leq 0.50$ and $c_3(\alpha_1, \alpha_2) = 1/\alpha_1$ if $\alpha_1 > 0.50$. We finally obtain that, for N large, the probability of casting a decisive vote for a voter belonging to an electorate divided in three groups is approximately equal to the Shapley-Shubik index multiplied by $c_3(\alpha_1, \alpha_2)$. We give in Table 3 some computed values of $c_3(\alpha_1, \alpha_2)$ for various values of α_1 and α_2.

α_1/α_2	1/3	0.35	0.40	0.45	0.50
1/3	2.250	-	-	-	-
0.35	2.248	2.245	-	-	-
0.40	2.219	2.214	2.188	-	-
0.45	2.145	2.143	2.130	2.099	-
0.50	2	2	2	2	2
> 0.50	$1/\alpha_1$	$1/\alpha_1$	$1/\alpha_1$	$1/\alpha_1$	$1/\alpha_1$

Table 3 : Values of $c_3(\alpha_1, \alpha_2)$

These values show that the probability of casting a decisive vote is maximum when $\alpha_1 = \alpha_2 = 1/3$, i.e. when each of the three groups has the same size.

4.5 The Symmetric Case

We consider here the case with $N_1 = N_2 = ... = N_K = \frac{\widehat{N}+1}{K}$ and we assume that $N = \widehat{N} + 1$ is a multiple of K, which implies that K is odd. In this symmetric case, the value of $\pi(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{K} - 1, \frac{\widehat{N}+1}{K}, ..., \frac{\widehat{N}+1}{K})$ is given as the number of integer solutions of the following set of (in)equalities:

$$\begin{gathered}
0 \leq x_1 \leq \tfrac{\widehat{N}+1}{K} - 1 \\
0 \leq x_2 \leq \tfrac{\widehat{N}+1}{K} \\
... \\
0 \leq x_K \leq \tfrac{\widehat{N}+1}{K} \\
x_1 + x_2 + ... + x_K = \tfrac{\widehat{N}}{2}
\end{gathered}$$

For specific small values of K, it is fairly easy to obtain close forms of and of the probability of being pivotal as a function of the parameter N. Let us consider the first values of K.

- $K = 3$. To compute $\pi\left(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{3} - 1, \frac{\widehat{N}+1}{3}, \frac{\widehat{N}+1}{3}\right)$, we proceed as follows. Let $\widehat{K} \equiv \frac{\widehat{N}+1}{3}$ and m be the number of voters taken from the smallest group. Of course: $0 \leq m \leq \widehat{K} - 1$. Given m, how many voters x_2 can we take in the second group ?

The smallest number $\underline{x}$ is solution of $m + \underline{x} + \widehat{K} = \frac{3\widehat{K}-1}{2}$ i.e. $\underline{x} = \frac{\widehat{K}-1}{2} - m$. This bound is derived when we chose the largest possible number (i.e. $\widehat{K}$) in the third group. This integer is larger than or equal to 0 when $m \leq \frac{-1}{2}$. On the other hand, the largest number $\overline{x}$ is solution of $m + \overline{x} + 0 = \frac{3\widehat{K}-1}{2}$ i.e. $\overline{x} = \frac{3\widehat{K}-1}{2} - m$. This integer is smaller than or equal to $\widehat{K}$ when $m \geq \frac{\widehat{K}-1}{2}$.

Case 1: $m \leq \frac{\widehat{K}-1}{2}$. In such case: $x_2 = \widehat{K} - \left(\frac{3\widehat{K}-1}{2} - m\right) + 1 = \frac{\widehat{K}+1}{2} + m + 1$

Case 2: $m \geq \frac{\widehat{K}-1}{2}$. In such case: $x_2 = \left(\frac{3\widehat{K}-1}{2} - m\right) - 0 + 1 = \frac{3\widehat{K}-1}{2} - m + 1$

From that, we deduce:

$$\begin{aligned}
\pi\left(\widehat{K} - 1, \widehat{K}, \widehat{K}\right) &= \sum_{m=0}^{\frac{\widehat{K}-1}{2}} \left(\frac{\widehat{K}+1}{2} + m + 1\right) + \sum_{m=\frac{\widehat{K}+1}{2}}^{\widehat{K}-1} \left(\frac{3\widehat{K}-1}{2} - m + 1\right) \\
&= \left(\frac{\widehat{K}+1}{2}\right)^2 + \frac{(\widehat{K}-1)(\widehat{K}+1)}{8} + \frac{\left(3\widehat{K}-1\right)\left(\widehat{K}-1\right)}{4} \\
&\quad - \frac{\widehat{K}(\widehat{K}-1)}{2} + \frac{(\widehat{K}-1)(\widehat{K}+1)}{8} + \widehat{K}
\end{aligned}$$

which simplifies to:

$$\pi\left(\widehat{K}-1,\widehat{K},\widehat{K}\right)=\frac{3\widehat{K}^2+4\widehat{K}+1}{4}$$

From that, we derive:

$$Piv(\lambda)=\frac{3\widehat{K}^2++4\widehat{K}+1}{4\widehat{K}\left(\widehat{K}+1\right)^2}$$

Changing to the variable $N=3\widehat{K}$, we obtain:

$$Piv(\lambda)=\frac{3\left(\frac{N}{3}\right)^2+4\frac{N}{3}+1}{4\left(\frac{N}{3}\right)\left(\frac{N}{3}+1\right)^2}=\frac{9N^2+36N+27}{4N^3+24N^2+36N}=\frac{9(N+1)}{4N(N+3)}$$

or equivalently

$$Piv(\lambda)=\frac{9(\widehat{N}+2)}{4(\widehat{N}+1)(\widehat{N}+4)} \tag{5}$$

for $\widehat{N}=2$ modulo 6 (recall that $\widehat{N}+1$ must be an odd multiple of 3). Notice that these results are consistent with the representations given in Section 2: Replacing N_1 and N_2 by $(N+1)/3$ in (2), (3) or (4) leads to (5).

Hence, we get for N large:

$$Piv(\lambda)\simeq c_3\frac{1}{N}$$

with $c_3=\frac{9}{4}=2.25$, in accordance with the result obtained in the preceding subsection for $\alpha_1=\alpha_2=1/3$.

- $K=5$. We obtain:

$$\pi(\frac{\widehat{N}}{2},\frac{\widehat{N}+1}{5}-1,\frac{\widehat{N}+1}{5},\frac{\widehat{N}+1}{5},\frac{\widehat{N}+1}{5},\frac{\widehat{N}+1}{5})=\frac{(\widehat{N}+2)(\widehat{N}+6)(23\widehat{N}^2+276\widehat{N}+928)}{24000}$$

and

$$Piv(\lambda)=\frac{25(\widehat{N}+2)(23\widehat{N}^2+276\widehat{N}+928)}{192(\widehat{N}+1)(\widehat{N}+6)^3}$$

for $\widehat{N}=4$ modulo 10. In this case, the limiting value of the probability of casting a decisive vote is given as:

$$Piv(\lambda)\simeq c_5\frac{1}{N}$$

with $c_5=\frac{575}{192}=2.995$.

- $K = 7$. The probability of casting a decisive vote is given as:

$$Piv(\lambda) = \frac{48(841\widehat{N}^6 + 35322\widehat{N}^5 + 616300\widehat{N}^4 + 3859680\widehat{N}^3 + 23167384\widehat{N}^2 + 67791768\widehat{N} + 66810120)}{11520(\widehat{N}+1)(\widehat{N}+8)^6}$$

for $\widehat{N} = 6$ modulo 14. And for N large:

$$Piv(\lambda) \simeq c_7 \frac{1}{N}$$

with $c_7 = \frac{41209}{11520} = 3.577$.

- $K = 9$ and $K = 11$. Although we have been able to obtain the complete polynomial associated with $\pi(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{9} - 1, \frac{\widehat{N}+1}{9}, ..., \frac{\widehat{N}+1}{9})$ and with $\pi(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{11} - 1, \frac{\widehat{N}+1}{11}, ..., \frac{\widehat{N}+1}{11})$, we only give here the values of c_9 and c_{11}:

$$c_9 = \frac{2337507}{573440} = 4.076$$

and

$$c_{11} = \frac{4199504287}{928972800} = 4.521.$$

For values of K higher than 11, the implementation of the Barvinok's algorithm demands a very long computation time that prevents from obtaining some numerical results.

Proposition 3: Let K be an odd number ($K \geq 3$).
Let $\varphi(K) = \lim\limits_{N \to +\infty} [\frac{1}{N^{K-1}} \pi(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{K} - 1, \frac{\widehat{N}+1}{K}, ..., \frac{\widehat{N}+1}{K})]$. Then, for each fixed value of K, we have:

$$\varphi(K) = \frac{1}{(K-1)!} \sum_{m=0}^{\frac{K-1}{2}} (-1)^m \binom{K}{m} (\frac{K-2m}{2K})^{K-1}.$$

Proof. By definition, $\pi(\frac{\widehat{N}}{2}, \frac{\widehat{N}+1}{K} - 1, \frac{\widehat{N}+1}{K}, ..., \frac{\widehat{N}+1}{K})$ is the number of integer solutions of the following parametric linear system:

$$\begin{cases} 0 \leq x_1 \leq \dfrac{N}{K} - 1 \\ 0 \leq x_k \leq \dfrac{N}{K} \text{ for all } k = 2, ..., K \\ \displaystyle\sum_{k=1}^{K} x_k = \dfrac{N-1}{2} \end{cases}$$

We know by Ehrhart's theorem that this number is a quasi-polynomial of degree $K - 1$ on the variable N. Hence, $\varphi(K)$ is equal to the leading coefficient of this quasi-polynomial.

The additive constants in the second member of the constraints do not affect this coefficient, $\varphi(K)$ is also the leading coefficient of the quasi-polynomial computing the number of integer solutions of the system

$$\begin{cases} 0 \leq x_k \leq \dfrac{N}{K} \text{ for all } k = 1, 2, ..., K \\ \displaystyle\sum_{k=1}^{K} x_k = \dfrac{N}{2} \end{cases}$$

The system represents the dilatation by the factor N of the rational $(K-1)-$dimensional polytope $\mathbf{Q}$ defined by:

$$\begin{cases} 0 \leq x_k \leq \dfrac{1}{K} \text{ for all } k = 1, ..., K \\ \displaystyle\sum_{k=1}^{K} x_k = \dfrac{1}{2} \end{cases}$$

By the second assertion of Ehrhart's theorem, and by definition of $\varphi(K)$, we know that $\varphi(K)$ is equal to the relative volume of $\mathbf{Q}$, which is the (normalized) volume in $\mathbb{R}^{K-1}$ of the full-dimensional polytope $\mathbf{P}$ defined by:

$$\begin{cases} 0 \leq x_k \leq \dfrac{1}{K} \text{ for all } k = 1, ..., K-1 \\ \dfrac{K-2}{2K} \leq \displaystyle\sum_{k=1}^{K-1} x_k \leq \dfrac{1}{2} \end{cases}$$

Let Vol($\mathbf{P}$) be the volume of $\mathbf{P}$. To compute this volume, we consider some particular subsets of $\mathbb{R}^{K-1}$. Let Δ and Δ' be the $K-1$-dimensional simplices defined by:

$$\Delta = \{x \in \mathbb{R}^{K-1} : x_k \geq 0 \text{ for all } k = 1, \dots, K-1 \text{ and } x_1 + \ldots + x_{K-1} \leq 1/2\}$$

$$\Delta' = \{x \in \mathbb{R}^{K-1} : x_k \geq 0 \text{ for all } k = 1, \dots, K-1 \text{ and } x_1 + \ldots + x_{K-1} \leq (K-2)/2K\}$$

It is easy to see that Vol($\mathbf{P}$) = Vol(A) − Vol(B), where:

$$A = \{x \in \Delta : x_k \leq 1/K, \ \forall k = 1, \dots, K-1\}$$

$$B = \{x \in \Delta' : x_k \leq 1/K, \forall k = 1, \dots, K-1\}$$

We only show how to compute Vol(A), the same method will be applied to obtain Vol(B). For each i in $\{1, \dots, K-1\}$ let $\Delta_i = \{x \in \Delta : x_i \geq 1/K\}$. More generally, for each non empty subset S of $\{1, \dots, K-1\}$, we define Δ_S by $\Delta_S = \cap_{i \in S} \Delta_i$. Note that $\Delta_S = \emptyset$ for $\mid S \mid > \frac{K-1}{2}$.

For S such that $\#S \leq \frac{K-1}{2}$, let $\#S = m$ and let t_u be the translation of vector u, where u is the vector of $\mathbb{R}^{K-1}$ defined by $u_i = -\frac{1}{K}$ if $i \in S$ and $u_i = 0$ if not. It is obvious that $t_u(\Delta_S) = \Delta(m)$, where $\Delta(m) = \{x \in \mathbb{R}^{K-1} : x_k \geq 0$ for all $k = 1, \ldots, K-1$ and $x_1 + \ldots + x_{K-1} \leq (K-2m)/2K\}$. Since translations conserve volumes, and applying the formula giving the volume of a simplex, we obtain:

$$\text{Vol}(\Delta_S) = \text{Vol}(\Delta(m)) = \frac{1}{(K-1)!}(\frac{K-2m}{2K})^{K-1}$$

On the other hand, we can write $\text{Vol}(A) = \text{Vol}(\Delta) - \text{Vol}(\cup_{i=1}^{K-1}\Delta_i)$. Applying the inclusion-exclusion principle, we get:

$$\begin{aligned}\text{Vol}(\cup_{i=1}^{K-1}\Delta_i) &= \sum_{m=1}^{\frac{K-1}{2}}(-1)^{m-1}\sum_{S,|S|=m}\text{Vol}(\Delta_S)\\ &= \sum_{m=1}^{\frac{K-1}{2}}(-1)^{m}\binom{K-1}{m}\frac{1}{(K-1)!}(\frac{K-2m}{2K})^{K-1}\end{aligned}$$

Since $\text{Vol}(\Delta) = \frac{1}{(K-1)!}\left(\frac{1}{2}\right)^{K-1}$, we obtain:

$$\text{Vol}(A) = \frac{1}{(K-1)!}\sum_{m=0}^{\frac{K-1}{2}}(-1)^{m}\binom{K-1}{m}(\frac{K-2m}{2K})^{K-1}$$

Now, $\text{Vol}(B)$ can be computed in a similar way and we can easily establish that:

$$\text{Vol}(B) = \frac{1}{(K-1)!}\sum_{m=0}^{\frac{K-3}{2}}(-1)^{m}\binom{K-1}{m}(\frac{K-2-2m}{2K})^{K-1}.$$

Finally, the following simple calculus gives the result:

$$\begin{aligned}\text{Vol}(\mathbf{P}) =& \frac{1}{(K-1)!}(\sum_{m=0}^{\frac{K-1}{2}}(-1)^{m}\binom{K-1}{m}(\frac{K-2m}{2K})^{K-1} - \sum_{m=0}^{\frac{K-3}{2}}(-1)^{m}\binom{K-1}{m}(\frac{K-2-2m}{2K})^{K-1})\\ =& \frac{1}{(K-1)!}((\frac{1}{2})^{K-1} + \sum_{m=1}^{\frac{K-1}{2}}(-1)^{m}[\binom{K-1}{m} + \binom{K-1}{m-1}](\frac{K-2m}{2K})^{K-1}\\ =& \frac{1}{(K-1)!}((\frac{1}{2})^{K-1} + \sum_{m=1}^{\frac{K-1}{2}}(-1)^{m}\binom{K}{m}(\frac{K-2m}{2K})^{K-1}\\ =& \frac{1}{(K-1)!}\sum_{m=0}^{\frac{K-1}{2}}(-1)^{m}\binom{K}{m}(\frac{K-2m}{2K})^{K-1}\ . \ \square\end{aligned}$$

Using the analytical expression obtained in the previous proposition, we can extend the calculation of $c_K = K^K \varphi(K)$ to larger values of K. The following table gives the exact value of c_K for $K = 5$ to 49 (K odd).

K	5	7	9	11	13	15	17	19	21	23	25	27
c_K	2.995	3.577	4.076	4.521	4.925	5.298	5.647	5.976	6.288	6.584	6.870	7.143
K	29	31	33	35	37	39	41	43	45	47	49	
c_K	7.408	7.657	7.903	8.141	8.372	8.597	8.817	9.031	9.240	9.444	9.644	

Table 4 : Exact values of c_K

Notice that the limiting result obtained in this subsection can be easily extended to the case where N is even and the population is divided into K groups of size $\frac{N}{K}$. The integer K can be odd or even and the unique assumption is that N is an even multiple of K. Let $\psi(K) = \lim\limits_{N \to +\infty} [\frac{1}{N^{K-1}} \pi(\frac{N-2}{2}, \frac{N}{K} - 1, \frac{N}{K}, ..., \frac{N}{K})]$. With slight modifications in the proof of Proposition 3, we obtain:

$$\psi(K) = \frac{1}{(K-1)!} \sum_{m=0}^{\frac{K-1}{2}} (-1)^m \binom{K}{m} (\frac{K-2m}{2K})^{K-1}$$

if K is odd, and

$$\psi(K) = \frac{1}{(K-1)!} \sum_{m=0}^{\frac{K}{2}} (-1)^m \binom{K}{m} (\frac{K-2m}{2K})^{K-1}$$

if K is even.

4.6 A Probabilistic Argument

To study the asymptotic behavior of the the above expression i.e. to understand how c_K behaves when K tends to ∞, we will develop a probabilistic argument. To this end we will consider the probability of being pivotal from the perspective of a small group of size ϵN where $\epsilon > 0$ is fixed instead of a group of size 1 as done until now. Such a group is pivotal iff :

$$\frac{N}{2} - \frac{\epsilon N}{2} \leq S_N \leq \frac{N}{2} + \frac{\epsilon N}{2}$$

where:

$$S_N = \sum_{k=1}^{K} S_N^k \text{ where } S_N^k = \sum_{i=1}^{N_k} X_i^k \text{ and } N_k = \frac{N}{K}$$

The random variables $S_N^1, S_N^2, ..., S_N^K$ are independent and identically distributed. Following the argument used in Proposition 4 of Chamberlain and Rothschild (1981), we deduce that for all $k = 1, ..., K$, $\frac{S_N^k}{N_k}$ converges weakly to the uniform law on the interval $[0, 1]$ when $N_k \to \infty$. This implies that $\frac{S_N}{N}$ converges weakly to $Z = \frac{1}{K}\sum_{k=1}^K U^k$ where the random variables U_k are independent and identically distributed. Their common distribution is the uniform distribution on $[0, 1]$. From the central limit theorem, we deduce that if K is large then:

$$\frac{\sum_{k=1}^K U_k}{K} - \frac{1}{2} \simeq N(0, \frac{1}{\sqrt{12K}})$$

since$\sqrt{\frac{1}{12}}$ is the standard deviation of the uniform variable on $[0, 1]$. We deduce that the probability of a group of relative size ϵ to be pivotal denoted $Piv(\epsilon, N)$ is approximatively equal to

$$\begin{aligned}\Pr\left\{-\frac{\epsilon}{2} \leq \frac{\sum_{k=1}^K U_k}{K} - \frac{1}{2} \leq \frac{\epsilon}{2}\right\} &\simeq \Pr\left\{-\frac{\epsilon}{2} \leq N(0, \frac{1}{\sqrt{12K)}} \leq \frac{\epsilon}{2}\right\} \\ &\simeq \epsilon\sqrt{\frac{6K}{\pi}}\end{aligned}$$

and therefore:

$$c_K \simeq \sqrt{\frac{6K}{\pi}} \text{ when } K \text{ is large}$$

We have tabulated few values of $\sqrt{\frac{6K}{\pi}}$ below:

K	3	5	7	9	11	...	51	...	99
$\sqrt{\frac{6K}{\pi}}$	2.3937	3.0902	3.6564	4.1459	4.5835	...	9.8693	...	13.7505

Table 5 : Approximate values of c_K

5 Correlations and Further Applications

In this paper, we have mostly focused on a specific pattern of correlation that we have called the IAC partitioning model. It is important to recall that this model is specific on two grounds. First, it is based on a partition of the individuals such that individuals belonging to two different groups in that partition have independent preferences. Second, it has been assumed that in each group the correlations among the preferences in the group were resulting from the IAC model. The main purpose of this last section is to show that the techniques developed in our paper can be used to accommodate many other types of quantitative and qualitative correlations. We will also show the difficulties in extending some of the results to arbitrary correlation patterns.

5.1 Partitioning

In this section, we keep the partitioning assumption but we depart from the IAC setting, which implies that the covariance between the votes of two voters from the same group is equal to $\frac{1}{3}$. We consider instead the case where the covariance between the votes of two voters is arbitrary and denoted ρ: $Cov(X_{iR}, X_{jR})$, the covariance between the votes of i and j when they belong to the same group is then equal to $\frac{\rho}{4}$. As before, we obtain:

$$\underset{R\to\infty}{Lim}\sqrt{N}Piv(\lambda_R, s) = \frac{1}{\left(\sqrt{\frac{1-\rho+\rho\gamma^1}{4}} + \frac{\rho}{4}\sum_{s=2}^{S}\gamma^s s\right)\sqrt{2\pi}}$$

In the case of perfect correlation i.e. $\rho = 1$, we obtain for instance:

$$\underset{R\to\infty}{Lim}\sqrt{N}Piv(\lambda_R, s) = 0.564\,19 \text{ when } s = 2$$

and

$$\underset{R\to\infty}{Lim}\sqrt{N}Piv(\lambda_R, s) = 0.460\,66 \text{ when } s = 3$$

Note that in the case of perfect correlation, the above results follow immediately from a redefinition of the random variables. For instance, in the case where $s = 2$, we would pack the N random variables into $\frac{N}{2}$ packs of size 2. The aggregate vote in a pack is a discrete random variable with support $\{0, 2\}$ instead of $\{0, 1\}$. It has mean 1 and variance 1. A voter or a pair of voters from the same block is pivotal iff in the remaining blocks, the number of blocks voting left is equal to the number of blocks voting right. It is as if we were in the standard case with a variance σ of 1 instead of a variance of $\frac{1}{4}$ and $\frac{N}{2}$ instead of N. We know that the probability of being pivotal behaves as $\frac{1}{\sigma\sqrt{2\pi}\sqrt{\frac{N}{2}}} = \frac{1}{\sqrt{N}\sqrt{\pi}} = \frac{0.564\,19}{\sqrt{N}}$.

Note finally that we could run the same computations without assuming that the covariances are constant within each group. An interesting situation of that kind appears in the Le Breton and Lepelley (2011) study of the French electoral law of June 29 1820. This electoral law, known as the law of double vote is a law which has been used in France to elect the deputies. France was divided into a number of electoral districts (the so called French "départements") and each district sent a number of deputies to the chamber. Each district was divided itself into subdistricts (the so called "arrondissements"). Each arrondissement elected one deputy and to be voter in an arrondissement, your amount of tax must be above some fixed level (called the "cens"). In addition, the voters in the top quartile of the income distribution of the voters in the département form an additional electoral college and elected D deputies. These "rich" voters had a double vote: they voted in their arrondissement and also in the electoral college constituted at the level of the département. This explains the name which was given to this law. It was decided that $\frac{3}{5}$ of the deputies was elected by the arrondissements and $\frac{2}{5}$ by the voters in the top colleges. Le Breton and Lepelley (2011) study a symmetric version of that problem where there are K départements, with A arrondissements in each département and $4r+1$ voters in each arrondissement where r is an odd integer denoting the number of voters with two votes in that arrondissement. The size N of the chamber is therefore $K(A+D)$. They assumed that A was an odd integer and that D was an even integer. A good approximation of the French data at that time is given by $K=86$, $A=3$ and $D=2$ leading to $N=430$: 258 being elected in arrondissements and 172 elected by the top colleges. Hereafter, we will limit however our attention to the case where K is odd. In the case where $A=3$ and $D=2$, the $5K$ deputies are partitioned into groups of size 5. These legislators have in common to be elected from the same territory. Even if we assume that the preferences of the $A(4r+1)$ districts are independent, the preferences of the deputies are not independent because some voters have a double vote. Let $\left(S_j^1, S_j^2, S_j^3, S_j^4, S_j^5\right)$ be the profile of the five votes in the j^{th} département where the first three coordinates denote the votes in the three arrondissements and the last two the votes in the top college. When r is large this random vector is approximatively Gaussian with (after normalization) the matrix of variances-covariances:

$$\Omega = \begin{pmatrix} \frac{\sqrt{4r+1}}{2} & 0 & 0 & \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} \\ 0 & \frac{\sqrt{4r+1}}{2} & 0 & \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} \\ 0 & 0 & \frac{\sqrt{4r+1}}{2} & \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} \\ \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} & \frac{\sqrt{3r}}{2} & \frac{\sqrt{3r}}{2} \\ \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} & \frac{\sqrt{r}}{2} & \frac{\sqrt{3r}}{2} & \frac{\sqrt{3r}}{2} \end{pmatrix}$$

The variance of S_N is equal to:

$$3K\left(\frac{4r+1}{4}\right)+2K\left(\frac{3r}{4}\right)+K(6\left(\frac{r}{4}\right)+2\left(\frac{3r}{4}\right))\simeq K\left[3r+\frac{3r}{2}+\frac{3r}{2}+\frac{3r}{2}\right]=\frac{15Kr}{2}$$

We also note that the coefficient of correlation ρ between any of the first three variables and any of the last two ones is equal to $\sqrt{\frac{1}{12}}$ Now we consider the 5-dimensional vector of Bernoulli variables $\left(X_j^1, X_j^2, X_j^3, X_j^4, X_j^5\right)$ where $X_j^l=1$ if $S_j^l \geq 2r+1$ for $l=1,2,3$ and

$X_j^l = 1$ if $S_j^l \geq \frac{Ar+1}{2}$ for $l = 4, 5$. Based on the Gaussian orthant probabilities, the matrix of variances-covariances of this vector is:

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} \\ \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} + \frac{\arcsin\rho}{2\pi} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

In the specific case where $\rho = \sqrt{\frac{1}{12}}$, we obtain that $\frac{1}{4} + \frac{\arcsin\rho}{2\pi} = 0.29849$. By using the argument above we would deduce that the probability for a deputy to be pivotal if both r and K are large integers is approximatively equal to:

$$\frac{1}{\left(\frac{1}{4} + \frac{1}{\sqrt{5}}\left[\frac{1}{2} + 6 \times 0.29849\right]\right)\sqrt{2\pi N}} \simeq \frac{0.313\,01}{\sqrt{N}}$$

In contrast, note that if the $N = 5K$ were all independent, then, as we know from table 2, the probability of being pivotal for a deputy will be approximatively equal to $\frac{0.798}{\sqrt{N}}$. The correlations introduced by the law of double vote has reduced by a factor of 2 the probability of being pivotal !

5.2 Networks and Local Interactions

In the above illustration, the correlation takes place or not once some partitioning of the voters has been considered. In this dichotomous setting correlations among the preferences of voters or deputies exist iff the considered voters or deputies belong to the same group. A more general approach distinct from the one considered in this paper does not assume any a priori partitioning of the population of voters into groups. We could simply assume that the votes of two different voters are correlated and denote by λ_{ij} the probability that i and j both vote for 1 (which is equal (in our neutral world) to the probability that i and j both vote for 0). Of course $0 \leq \lambda_{ij} \leq \frac{1}{2}$. If $\lambda_{ij} = 0$ (respectively $\lambda_{ij} = \frac{1}{2}$), then the covariance ρ_{ij} between the votes of i and j is equal to $\frac{1}{4}$ (respectively $-\frac{1}{4}$). A nice general model along these lines is the following. Suppose that the N voters are ordered from left to right on the integer line (located in the positions $1, 2, ..., N$) and that the joint probability describing the votes of these N voters is stationary[17]. We may interpret this one dimensional ranking as their ranking on a one dimensional ideological (left-right) axis. The more distant are voters i and j on this axis, the more unlikely they are going to have the same preferences.

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[17] The few atypical notions of probability theory used in this paper are described in the appendix.

Moving directly to the asymptotic setting, let us describe the profile of votes as a sequence $X = (X_i)_{i \in N}$ of Bernoulli random variables on the probability space $(\Omega, \mathcal{F}, \Pr)$ where as before $X_i = 1$ iff voter i votes left. Let α_n be the n^{th} mixing coefficient.

Since $E\left[X_i^{12}\right] < \infty$, we deduce from Billingsley's (1995) theorem 27.4[18] that if $\alpha_n = O(n^{-5})$, then:

$$\frac{Var\left[S_n - \frac{n}{2}\right]}{n} \text{ tends to } \sigma \equiv \frac{1}{4} + 2\sum_{k=1}^{\infty} Cov\left(X_1, X_{1+k}\right)$$

and if $\sigma > 0$, $\frac{S_n - \frac{n}{2}}{\sigma\sqrt{n}}$ converges weakly to a unit Gaussian.

A nice application of this approach, taken from Billingsley (1995), is the following. Let $(Y_n)_{n \geq 1}$ be a Markov chain with a finite state space and positive transition probabilities $(p_{ij})_{1 \leq i,j \leq s}$ and suppose that $X_n = f(Y_n)$ where f is some real function on the state space. If the initial probabilities $(p_i)_{1 \leq i \leq s}$ are the stationary ones, then clearly $(X_n)_{n \geq 1}$ is stationary. By Billingsley's theorem 8.9. we deduce that $\left|p_{ij}^{(n)} - p_j\right| \leq \rho^n$ where $\rho = (1 - s\eta) < 1$ with $\eta \equiv \underset{1 \leq i,j \leq s}{Min}\, p_{ij}$. We have:

$$\Pr\left[Y_1 = i_1, ..., Y_k = i_k, Y_{k+n} = j_0, ..., Y_{k+n+l} = j_l\right] = p_{i_1} p_{i_1 i_2} ... p_{i_{k-1} i_k} p_{i_k j_0}^{(n)} p_{j_0 j_1} ... p_{j_{l-1} j_l}$$

which differs from $\Pr\left[Y_1 = i_1, ..., Y_k = i_k\right] \times \left[Y_{k+n} = j_0, ..., Y_{k+n+l} = j_l\right]$ by at most:

$$p_{i_1} p_{i_1 i_2} ... p_{i_{k-1} i_k} \rho^n p_{j_0 j_1} ... p_{j_{l-1} j_l}.$$

It follows that for sets of the form $A = \{(Y_1, ..., Y_k) \in H\}$ and $B = \{(Y_{k+n}, , ..., Y_{k+n+l}) \in H'\}$, we obtain:

$$\left|\Pr(A \cap B) - \Pr(A)\Pr(B)\right| \leq s\rho^n$$

Since these sets generate respectively the $\sigma-$fields $\mathcal{F}_0^k$ and $\mathcal{F}_{k+n}^{+\infty}$, we deduce that $(X_n)_{n \geq 1}$ is strongly mixing. One illustration consists in taking $s = 2$ (preference for the left or preference for the right) and assuming that the right neighbor of a leftist (rightist) voter is a leftist (rightist) voter with probability λ and a rightist (leftist) voter with probability $1 - \lambda$ where $\lambda \in [0, 1]$. Here $\rho = 1 - 2Min(\lambda, 1 - \lambda) = 1 - 2(1 - \lambda) = 2\lambda - 1$ if $\lambda \geq \frac{1}{2}$. The stationary distribution is $\left(\frac{1}{2}, \frac{1}{2}\right)$. Some straightforward calculations give:

$$Cov\left(X_1, X_{1+k}\right) = \frac{1}{4}\left(2\lambda - 1\right)^k$$

---

[18] There are many more central limit theorems where the variables are not assumed to be independent (Bergstrom (1970), Bernstein (1927), Diananda (1955), Hoffding and Robins (1948), Merlevede and Peligrad (2000), Orey (1958) to cite few).

and therefore:

$$\sigma = \frac{1}{4}\frac{\lambda}{1-\lambda}$$

From Nagaev (1957) local central limit theorem for Markov's chains, we deduce:

$$\underset{N\to\infty}{Lim}\sqrt{N}Piv(N) = \sqrt{\frac{2(1-\lambda)}{\lambda\pi}}$$

Without any surprise, since λ measures how correlated are the votes of two neighbors, the probability of being pivotal decreases with λ.

5.3 Blocks of Increasing Size

In this paper, we have considered the polar cases of many groups of small fixed size and few groups whose sizes grow linearly with the size of the population. We could consider of course intermediate situations where the society of N voters is partitioned into $K(N) = \frac{N}{S(N)}$ groups of size $S(N) \leq N$ where the function S is neither constant, neither linear in N. For instance, we could consider for $S(N)$:[19] $S(N) = \log(N)$ or $S(N) = \lfloor N^{\theta} \rfloor$ with $0 < \theta < 1$. If we want to use Berck's theorem, we need to sort out the implications of condition (iv) which asks for $\underset{N\to\infty}{Lim}\frac{S(N)^{2+\frac{2}{\delta}}}{N} = 0$. Since in our setting, δ can be chosen arbitrarily large, it amounts to ask that $S(N)^{2+\epsilon}$ tend less rapidly to ∞ than N for some arbitrarily small $\epsilon > 0$. For instance, $S(N) = \log(N)$ and $S(N) = N^{\frac{1}{4}}$ verify this condition. In such case to meet conditions (ii) and (iii) of Berk's theorem, we cannot assume that the covariances are the same within each group. We could assume for instance that the covariance between i and j in the same group is non zero iff $j = i + 1$. Note however that in such setting, $Piv(N)$ still behaves asymptotically like $\frac{1}{\sqrt{N}}$.

Berk's assumption can be relaxed to permit more general dependence structures as for instance in Romano and Wolf (2000) presented in appendix 3. The force of their result comes from the relaxation of conditions (ii) and (iii) of Berk's theorem through the introduction of a parameter $\gamma < 1$. Now, the variance of a block can behave as $m^{1+\gamma}$. Under the presumption that a local version of such central limit theorem holds, we now obtain:

$$Piv(N) \simeq \sqrt{\frac{12}{(2N + NS(N))}} \simeq \sqrt{\frac{6}{NS(N)}}$$

For instance when $S(N) = N^{\frac{1}{4}}$, the order of magnitude of the probability of being pivotal is $\frac{1}{N^{\frac{5}{8}}}$ instead of the standard $\frac{1}{\sqrt{N}}$.

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[19] $\lfloor x \rfloor$ denotes the integer part of x.

6 Concluding Remarks

In this paper, we have studied the impact of correlation across preferences and votes on the probability of being pivotal. The analysis has been conducted under a number of assumptions and we think that it would be of interest to examine how far we can go without being too much specific. One key assumption is the the neutrality among the two alternatives. We have assumed that the two alternatives were similar ex ante. One interesting generalization could consist in assuming that there is a partition of the population into groups where in each group the preferences are as here correlated but also possibly biased towards one candidate. The bias could of course vary from group to another. In such a setting a group could be defined as a subset of individuals displaying some homogeneity defined through a vector of characteristics.

We are not aware of an ambitious attempt to generalize the current theory to a setting that would allow for differences across alternatives. To the best of our knowledge, the only[20] model along these lines is due to Beck (1975). He considers a population divided into two groups of equal size. In the first group, the votes are independent and people vote left with probability $p > \frac{1}{2}$. In the second group, votes are also independent and people vote left with probability $1 - p$. Beck estimates numerically the probability for a voter to be pivotal for several values of the parameter p. Modulo a simple adjustment of the proof of Proposition 1, we obtain an asymptotic exact value of the probability of being pivotal in Beck's model. Precisely, we obtain :

$$\underset{N\to\infty}{Lim}\sqrt{N}Piv(N) = \frac{1}{\sqrt{2\pi p(1-p)}}$$

When $p = \frac{1}{2}$, we obtain the traditional constant $\sqrt{\frac{2}{\pi}} = 0.797\,88$. When $p = \frac{3}{4}$, we obtain $\frac{1}{\sqrt{2\pi\times\frac{3}{16}}} = 0.921\,32$ and when $p = \frac{4}{5}$, we obtain $\frac{1}{\sqrt{2\pi\times\frac{4}{25}}} = 0.997\,36$. Moving towards polarization increases drastically the probability of being pivotal !

Another promising direction of research is the analysis of simple games which are not symmetric. In such case, the probability of being pivotal varies across the voters. It would also be of interest to use the models of random electorate developed in our paper to settings with more than two alternatives.

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[20] See also Berg (1990) for another illustration.

7 Appendix

Suppose that for each k, $y_1^k, y_2^k, ..., y_{n(k)}^k$ are independent random variables on the probability space $(\Omega, \mathcal{F}, \Pr)$; the probability space for the sequence may change with n. Such a collection is called a triangular array of random variables.

For any two fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, consider the following measure of dependence[21] (Rosenblatt (1956)):

$$\alpha(\mathcal{A}, \mathcal{B}) = \underset{A \in \mathcal{A}, B \in \mathcal{B}}{Sup} \left|\Pr(A \cap B) - \Pr(A)\Pr(B)\right|$$

For a sequence of random variables $y_1^k, y_2^k, ..., y_{n(k)}^k$, let α_n be a number such that:

$$\alpha_n = \underset{A \in \mathcal{A}, B \in \mathcal{B}}{Sup} \left|\Pr(A \cap B) - \Pr(A)\Pr(B)\right| \text{ for } n \in \mathbb{N}^*$$

for $A \in \sigma\left(y_1^k, y_2^k, ..., y_j^k\right), B \in \sigma\left(y_{j+n}^k, y_{j+1+n}^k, ...\right)$ and $j \geq 1, n \geq 1$. The triangular array is called *strongly mixing* or $\alpha-$mixing if $\underset{n \to \infty}{Lim} \alpha_n = 0$.

The triangular array $\left(y_{n(k)}^k\right)_{k \geq 1}$ is $m-$*dependent* if $\left(y_1^k, y_2^k, ..., y_j^k\right)$ and $\left(y_{j+n}^k, y_{j+1+n}^k, ..., y_{j+n+l}^k\right)$ are independent whenever $n > m$. If the distribution of the random vector, $\left(y_n^k, y_{n+1}^k, ..., y_{n+j}^k\right)$ does not depend on n, the triangular array is said to be *stationary.*

7.1 Petrov's Local Central Limit Theorem[22]

Let k be an arbitrary fixed positive integer. A sequence of random variables $(y_n)_{n \geq 1}$ is said to be a $k-$sequence if the number of different distribution functions in the sequence of the distribution functions corresponding to $(y_n)_{n \geq 1}$ is equal to k. Consider a $k-$sequence of independent integer-valued random variables $(y_n)_{n \geq 1}$ each having finite variance. We denote by $F^1, ..., F^l$ the l distributions which are non-degenerate and occur infinitely often in the sequence $(F^i)_{1 \leq i \leq k}$. We denote by H^r the maximal span of F^r for $r = 1, ..., l$. Let $S_n = \sum_{j=1}^n y_j, S_n = \sum_{j=1}^n E(y_j)$, $B_n = \sum_{j=1}^n E(y_j - E(y_j))^2$ and $\Pr_n(N) = \Pr(S_n = N)$. Then:

If g.c.d. $\left(H^1, H^2, ..., H^l\right) = 1$, then $\underset{N}{Sup} \left|\sqrt{B_n} \Pr_n(N) - \frac{1}{\sqrt{2\pi}} e^{-\frac{(N-M_n)^2}{2B_n}}\right| \underset{n \to \infty}{\to} 0$

---

[21] There are many other measures of dependence which have been used in this literature. We refer the reader to Bradley (2005) for a comparison of these measures and their use in limit theorems.

[22] Other local versions of the conventional (variables are assumed to be independent but non necessarily identically distributed) central limit theorem have been proved (Davis and Mc Donald (1995), Gamkrelidze (1964), MC Donald (1979), Mukhin (1991)). To the best of my knowledge, no such result exists in the general dependent case. We conjecture that Berk's theorem and Billingsley's central limit theorems which are used in this paper admit a local version.

7.2 Berk's Theorem

For each $k = 1, 2, \ldots$ let $n = n(k)$ and $m = m(k)$ be specified and suppose that $y_1^k, y_2^k, \ldots, y_n^k$ is an $m-$dependent sequence of random variables with zero means. Assume the following conditions hold. For some $\delta > 0$ and some constants M and K:

(i) For some $\delta > 0$, $E\left|y_i^k\right|^{2+\delta} \leq M$ for all i and all k.

(ii) $Var\left(y_{i+1}^k + \ldots + y_j^k\right) \leq (j-i)\, K$ for all i, j, and k.

(iii) $\underset{k\to\infty}{Lim} \frac{Var\left(y_1^k + \ldots + y_n^k\right)}{n}$ exists and is nonzero. Denote v the limit.

(iv) $\underset{k\to\infty}{Lim} \frac{m^{2+\frac{2}{\delta}}}{n} = 0$

Then $\frac{y_1^k + \ldots + y_n^k}{n}$ is asymptotically normal with mean 0 and variance v.

7.3 Romano and Wolf's Theorem

For each $k = 1, 2, \ldots$ let $n = n(k)$ and $m = m(k)$ be specified and suppose that $y_1^k, y_2^k, \ldots, y_n^k$ is an $m-$dependent sequence of random variables with zero means. Assume the following conditions hold. For some $\delta > 0$, some $-1 \leq \gamma < 1$ and some sequences $(\Delta_k), (K_k)$ and (L_k):

(i) $E\left|y_i^k\right|^{2+\delta} \leq \Delta_k$ for all i and all k.

(ii) $\frac{Var\left(y_{i+1}^k + \ldots + y_{j+i-1}^k\right)}{(j-i)^{1+\gamma}} \leq K_k$ for all i and j such that $j - i \geq m$.

(iii) $B_k \equiv \frac{Var\left(y_1^k + \ldots + y_n^k\right)}{n(k) m^{\gamma}} \geq L_k$

(iv) $\frac{K_k}{L_k} = 0(1)$

(v) $\frac{\Delta_k}{L_k^{\frac{2+\delta}{2}}} = O(1)$

(vi) $\underset{k\to\infty}{Lim} \frac{m^{1+(1-\gamma)(1+\frac{2}{\delta})}}{n(k)} = 0$

Then $\frac{y_1^k + \ldots + y_n^k}{B_k}$ is asymptotically normal with mean 0 and variance 1.

This theorem extends Berk's theorem in two ways. Berk essentially proves this theorem for the special case $\gamma = 0$. Condition (ii) of his theorem corresponds to condition (ii) of this theorem with γ replaced by 0. The greater generality of this theorem allows to accommodate stronger dependence structures. For example, it can handle the situation of $B_k \sim n(k)^{1+\gamma}$ for positive γ smaller than 1. Note that for $\gamma > 0$ the condition (vi) becomes weaker than the corresponding condition in Berk's theorem. Further, unlike Berk's theorem, this theorem permits the bounding constants to depend on the row index k.

7.4 On the Coefficients in $\Psi(N)$

Let us rewrite $\Psi(N) = \left(\sum_{k=0}^{\frac{N-2}{4}} \frac{\left(\frac{N-2}{2}\right)!}{(k!)^2\left(\left(\frac{N-2}{2}-2k\right)!\right)}\right) \times \left(\frac{1}{3}\right)^{\frac{N-2}{2}}$ as $\sum_{x=0}^{\frac{m}{2}} \frac{f(x)}{3^m}$ where:

$$f(x) = \frac{(m)!}{(x!)^2\left((m-2x)!\right)} \text{ with } m = \frac{N-2}{4}$$

The function f is increasing i.e. $f(x) \geq f(x-1)$ iff

$$(m-2x+2)(m-2x+1) \geq x^2$$

which simplifies to :

$$3x^2 - 2x(2m+3) + (m+1)(m+2) \geq 0$$

The discriminant Δ of this quadratic function is equal to $4m^2$. Therefore, the roots are:

$$\frac{m}{3} + 1 \text{ and } m+1$$

Therefore, the function f is increasing between 0 and $\frac{m}{3}$ and decreasing after. Let us compute$f(\frac{m}{3}) = \frac{(m)!}{\left(\frac{m}{3}!\right)^3}$. From the Stirling's formula, we deduce:

$$m! \simeq \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \text{ and } \left(\frac{m}{3}!\right)^3 \simeq \left(\frac{2\pi m}{3}\right)^{\frac{3}{2}}\left(\frac{m}{3e}\right)^m$$

and then:

$$f(\frac{m}{3}) \simeq \frac{3^{m+\frac{1}{2}}}{2\pi m}$$

which implies that the largest term in the sum $\Psi(N)$ behaves as:

$$\frac{\sqrt{3}}{2\pi m}.$$

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