DUMMY PLAYERS AND THE QUOTA IN WEIGHTED VOTING GAMES [*]

Fabrice BARTHELEMY [†], Dominique LEPELLEY [‡] and Mathieu MARTIN [§]

Abstract. This paper studies the role of the quota on the occurrence of "dummy" players in weighted voting games. It is shown that the probability of having a player without voting power is very sensitive to the choice of the quota and the quota values that minimize this probability are derived.

JEL classification: **C7, D7**

Keywords: Cooperative game theory, weighted voting games, dummy player, probability of voting paradoxes.

[*]Financial support from the ANR SOLITER is gratefully acknowledged.

[†]THEMA, Université de Cergy-Pontoise, 33 boulevard du Port, 95011 Cergy-Pontoise Cedex, FRANCE. E-mail: fabrice.barthelemy@u-cergy.fr

[‡]CEMOI, Université de La Réunion, 97715 Saint-Denis Cedex 9, FRANCE. E-mail: dominique.lepelley@univ-reunion.fr

[§]THEMA, Université de Cergy-Pontoise, 33 boulevard du Port, 95011 Cergy-Pontoise Cedex, FRANCE. E-mail: mathieu.martin@u-cergy.fr

1 Introduction

In cooperative game theory, the power of a player in a voting game is defined as the probability to be decisive in the collective choice process. In a weighted voting game, if each player is given a weight that is both strictly positive and strictly lower than the quota, it could be expected that the voting power of every player is different from 0, that is that there is no *dummy* player. However, several (real) examples are given in the literature, showing that dummy players do exist. One of the most famous occurrences of a dummy player is offered by Luxembourg in the Council of Ministers of the EU between 1958 and 1973. Luxembourg held one vote, whereas the quota for a proposition to be approved was 12 out of 17. Since other member states held an even number of votes (4 for Germany, France and Italy, 2 for Belgium and The Netherlands), Luxembourg formally was never able to make any difference in the voting process and was a dummy. Such situations are obviously extremely undesirable but we should not worry about them if it could be shown that their occurrence is rare. Unfortunately, it is demonstrated in Barthélémy *et al.* (2011) that the theoretical probability of having a dummy player is far from being low, at least when the number of players is small. However, the study by Barthélémy *et al.* is restricted to *majority* voting games, in which 50% of the votes (weights) are sufficient for a proposition to be accepted. The purpose of the present study is to analyze more general weighted voting games and to investigate the impact of the quota value on the occurrence of a dummy player. We compute in this paper the probability to obtain a dummy player as a function of the quota and we determine the quota values which minimize and maximize this probability in weighted voting games.

To the best of our knowledge, the only related work is a paper by Zuckerman *et al.* (2012) who study the effects that a change of the quota may have on a given player's power. They provide, in particular, an efficient algorithm for determining whether there is a value of the quota that makes a given player a dummy. The results they obtain demonstrate that even small changes in the quota can have a significant effect on a player's power. These authors, however, adopt an algorithmic and computational perspective, clearly different from our probabilistic point of view.

The paper is built as follows: in the second section, our notation, definitions and assumptions are introduced. In the third section, we present analytical results for 3 and 4 players. Some numerical results obtained by computer enumerations and simulations for more than 4 players are given in section 4. The main teachings of our study are summarized and discussed in the fifth section.

2 Notation, definitions and assumptions

A voting game is a pair (N, W) where N is the set of n players (or voters) and W the set of winning coalitions, that is the set of groups of players which can enforce their decision.

In this paper, we consider the class of weighted voting games $[q; w_1, w_2, ..., w_n]$, where q is the quota needed to form a winning coalition and w_i is the number of votes (weight) of the ith player; we assume that q and w_i are integers. A coalition S is winning if and only if $\sum_{i \in S} w_i \geq q$. The total number of votes, $\sum_{i \in N} w_i$, is denoted by w. A particular case is the majority game where $q_{maj} = \frac{w}{2} + 1$ if w is even and $q_{maj} = \frac{w+1}{2}$ if w is odd. We assume that the game is proper, that is $q \geq q_{maj}$. When $q = w$, we get the unanimity rule: each player has a veto power and is not a dummy. Our study will focus on the weighted voting games such that $q < w$. Notice that we can also write $q = w - m$ with $m \in \{1, 2, ..., \frac{w}{2} - 1\}$ if w is even and $m \in \{1, 2, ..., \frac{w-1}{2}\}$ if w is odd.

It is supposed that each player has at least one vote and never more than $q-1$ votes (there is no dictator). We also assume, without loss of generality, that $q > w_1 \geq w_2 \geq ... \geq w_n \geq 1$. The relative quota is denoted by Q with $Q = q/w$.

A voter i is a dummy player in a voting game (N, W) if $S \in W$ implies $S \backslash \{i\} \in W$ for every $S \in W$. In words, player i is never decisive in every winning coalition: the coalition wins with or without him (her). In voting power theory (see Straffin, 1994, and Felsenthal and Machover, 1998, for a presentation), it means that this player has no power. The purpose of this paper is to compute the probability of obtaining a dummy player, given n, w and q (or Q), and to derive the quota which minimizes this probability (denoted by $\underline{q}$ or $\underline{Q}$) and the quota which maximizes this probability (denoted by $\overline{q}$ or $\overline{Q}$). The probability of having at least one dummy player is denoted by $P(w, n, q)$ or $P(w, n, Q)$ when w is finite and $P(n, Q)$ when w is infinite.

In order to compute $P(w, n, q)$ (or $P(w, n, Q)$ or $P(n, Q)$), we consider a particular probabilistic model called IAC (Impartial Anonymous Culture) which is one of the most often used in such problems where the likelihood of a voting event is to be calculated (see, for instance, Gehrlein and Lepelley, 2011). In our context, using this model is tantamount to assume that, n, w and q being given, all the possible distributions of the w_i's such that ($q > w_1 \geq w_2 \geq ... \geq w_n \geq 1$ and $\sum_{i \in N} w_i = w$) are equally likely to occur. To illustrate, suppose that $n = 3$ (three players), $w = 5$ (the total number of votes is equal to 5) and $q = 4$ (quota). It is easy to check that, under our constraints, the only possible distributions of the votes between the three players are $(3, 1, 1)$ and $(2, 2, 1)$. The IAC model assumes that each of these two distributions occurs with a probability equal to 1/2. Observe that, in the first distribution, the player 3 is not a dummy whereas he (she) has no voting power in the second one. We conclude that $P(5, 3, 4) = 1/2$.

3 Analytical representations

3.1 Preliminary results

We first present preliminary results concerning the general possibility of obtaining a dummy player in a weighted voting game. We know that there is no dummy player when $q = w$. We also know from Leech (2002) (see also Barthélémy *et al.*, 2011) that, in the three-player case, there is no dummy player when $q = q_{maj}$. Can we find some other values of q and n (the number of players) for which the "dummy paradox" never occurs? The following propositions give a negative answer to this question (as soon as w, the total number of votes, is not very small).

Proposition 1 *If w is sufficiently large and $n > 3$ then there exists a game $[q; w_1, w_2, ..., w_n]$ such that the player n is a dummy player for each value of $q \in \{q_{maj}, ..., w-2, w-1\}$.*

This result is a consequence of the following lemmas, the proofs of which are given in Appendix.

Lemma 1 *Consider the case where $m = \frac{w}{2} - 1$, that is the majority case with w is even. If $w \geq 6n - 12$, then there exists a game $[q; w_1, ..., w_n]$ whatever $n > 3$ for which the player n is a dummy.*

Lemma 2 *Consider the case where $m \geq 2n - 5$ and $m \neq \frac{w}{2} - 1$. If $w \geq 2n - 1$, then there exists a game $[q; w_1, ..., w_n]$ whatever $n > 3$ and m for which the player n is a dummy.*

Lemma 3 *Consider the case where $n - 2 \leq m < 2n - 5$. If $w > 5n - 10$, then there exists a game $[q; w_1, ..., w_n]$ whatever $n > 3$ and m for which the player n is a dummy.*

Lemma 4 *Consider the case where $0 < m \leq n-2$. If $w > (m+1)n - m^2$, then there exists a game $[q; w_1, ..., w_n]$ whatever $n > 3$ and m for which the player n is a dummy.*

Proposition 1 only deals with at least 4 players. Proposition 2 presents a similar result for the 3-player case for $q > q_{maj}$.

Proposition 2 *In the three-player case, there exists a game $[q; w_1, w_2, w_3]$ such that the player 3 is a dummy for each value of q, $q < w$ and $q \neq q_{maj}$.*

Proof. Consider the game $[q; q-2, w-q+1, 1]$. We have to show that $w_1 \geq w_2 \geq w_3$ (1) and the player 3 is not a dummy player *i.e.* $w_1 + w_2 \neq q - 1$ (2).

(1) $w_2 \geq w_3$ if $w - q + 1 \geq 1$ or $w \geq q$ which is always true. Furthermore, $w_1 \geq w_2$ if $q - 2 \geq w - q + 1$ or $q \geq \frac{w+3}{2}$ which is always true since $q \neq q_{maj}$.

(2) We have $w_1 + w_2 = w - 1$ which is different from $q - 1$ since $q < w$. QED

3.2 The case with 3 players

In this subsection, we derive probability representations for the 3-player case. These representations are based on the following lemma.

Lemma 5 *In a 3-player weighted voting game, a dummy player exists if and only if*

$$w_1 + w_3 < q \text{ and } w_1 + w_2 \geq q.$$

Lemma 5 (the proof of which is given the Appendix) offers a very simple characterization of the vote distributions giving rise to a dummy player and makes possible the derivation of the following representation for $P(3, w, q)$.

Proposition 3 *For $w = 9$ modulo 6, the probability $P(3, w, q)$ is given as*

- *for $(w+1)/2 \leqslant q \leqslant (2w+2)/3$:*

$$P(3, w, q) = -\frac{3(w^2 + 2w(1-2q) + (2q-1)^2)}{2w^2 - 6wq + 3(q^2-1)} \quad for\ q\ odd,$$

$$P(3, w, q) = -\frac{3(w^2 + 2w(1-2q) + (2q-1)^2)}{2w^2 - 6wq + 3(q^2-2)} \quad for\ q\ even;$$

- *for $(2w+3)/3 \leqslant q \leqslant w$:*

$$P(3, w, q) = \frac{3(3w^2 + 2w(1-4q) + q(5q-2)}{2w^2 - 6wq + 3(q^2-1)} \quad for\ q\ odd,$$

$$P(3, w, q) = \frac{3(3w^2 + 2w(1-4q) + 5q^2 - 2q - 1)}{2w^2 - 6wq + 3(q^2-2)} \quad for\ q\ even.$$

Proof. We compute first the total number of vote distributions in the 3-player case. The parameters w and q being given, our constraints imply that a vote distribution is a vector of integers (w_1, w_2, w_3) such that:

$$w_1 \geq w_2,\ w_2 \geq w_3,\ w_3 \geq 1,\ n_1 < Q \text{ and } w_1 + w_2 + w_3 = w. \quad (1)$$

We know from Ehrhart's theory and its developments (see Lepelley *et al.* 2008) that the number of solutions of such a set of inequalities is a periodic quasi polynomial in w and q. This quasi polynomial can be obtained by using algorithms recently developed by, among others, Clauss and Barvinok (see, once again, Lepelley *et al.* for a presentation of these algorithms). In our case, the period that is associated to parameter w is equal to 6 and the period associated to q is equal to 2. It means that the polynomial slightly differs depending on whether q is odd or even, and on whether w, $w+1$, $w+2$, $w+3$, $w+4$ or $w+5$ is a multiple of 6. We only consider in what follows the polynomial that we obtain for $w = 9$ modulo 6, *i.e.* for $w + 3$ multiple of 6. This polynomial is the following:

$$-\frac{1}{6}w^2 + \frac{1}{2}qw - \frac{1}{4}q^2 + \frac{1}{2} \text{ if } q \text{ is even,}$$

$$-\frac{1}{6}w^2 + \frac{1}{2}qw - \frac{1}{4}q^2 + \frac{1}{4} \text{ if } q \text{ is odd.}$$

Consider now the number of vote distributions for which a dummy exists. From Lemma 5, these vote distributions have to verify (in addition to conditions (1)):

$$w_1 + w_3 < q \text{ and } w_1 + w_2 \geq q. \quad (2)$$

The polynomial giving the number of integer solutions of the sets of inequalities (1) and (2) differs depending on whether w is odd or even, but also on the value of q with respect to w. Two domains have to be distinguished.
For w odd, q odd or even and $(w+1)/2 \leq q \leq (2w+2)/3$ (domain 1), we obtain:

$$\frac{1}{4}w^2 - qw + \frac{1}{2}w + q^2 - q + \frac{1}{4}.$$

For w odd and $(2w+3)/3 \leq q \leq w$ (domain 2), we have:

$$-\frac{3}{4}w^2 + 2qw - \frac{1}{2}w - \frac{5}{4}q^2 + \frac{1}{2}q + \frac{1}{4} \text{ if } q \text{ is even,}$$

$$-\frac{3}{4}w^2 + 2qw - \frac{1}{2}w - \frac{5}{4}q^2 + \frac{1}{2}q \text{ if } q \text{ is odd.}$$

The desired representations are then obtained by dividing the number of vote distributions with a dummy player by the total number of vote distributions. QED

Remark 1. Very similar representations dealing with the cases where w is different from 9 modulo 6 can of course be obtained[1] and used to compute the probability of having a dummy player. Some computed values of $P(3, w, Q)$, with $Q = q/w$, are listed in Table 1. These results show that the probability of having a dummy in the 3-player case can reach very high values (close to 0.70) ; moreover, it turns out that $P(3, w, Q)$ first increases as the quota increases, then decreases for a quota higher than about 0.75.

Remark 2. Let $P(3, Q)$ be the limiting probability of having a dummy player in a 3-player weighted voting game when w tends to infinity. Simple representation for $P(3, Q)$ can easily be obtained from Proposition 3: replacing q by Qw and making w tend to infinity, we get

$$P(3, Q) = \frac{-3(2Q-1)^2}{3Q^2 - 6Q + 2} \quad for\ 1/2 < Q \leqslant 2/3,$$

1. These representations are available from the authors upon request.

$$P(3,Q) = \frac{3(Q-1)(5Q-3)}{3Q^2-6Q+2} \quad for\ 2/3 \leqslant Q \leqslant 1.$$

From this representation, we verify that $P(3,Q)$ is minimized and equal to 0 when Q tends to 1 (unanimity) or to 1/2 (majority) and it is easy to obtain that $P(3,Q)$ is maximized for $\overline{Q} = 0.7676$. We get $P(3,\overline{Q}) = 69.72\%$. Table 2 gives some calculated values of $P(3,Q)$.

Remark 3. Suppose that the values of Q are uniformly distributed on $[1/2,\ 1]$. It is easily obtained from our representation for $P(3,Q)$ that the *average* or *expected* probability of having a dummy player in 3-player weighted voting games is given as

$$E[P(3,Q)] = \int_{1/2}^{1} 2P(3,Q)dQ = 40.83\%.$$

Table 1. Probability of having a dummy with 3 players (in %)

	Value of Q								
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99
10	0	33.33	57.14	57.14	37.50	37.50	0	0	0
15	0	30.77	58.82	58.82	50.00	31.58	0	0	0
20	0	50.00	59.26	62.07	54.84	43.75	24.24	0	0
25	0	44.44	62.79	63.04	58.33	37.25	21.15	0	0
30	0	40.00	63.49	63.64	59.42	43.84	17.33	0	0
35	0	50.00	65.12	64.44	56.25	40.00	15.69	0	0
40	0	46.15	64.81	64.96	58.06	44.96	25.76	0	0
45	0	43.36	65.47	65.10	59.24	41.21	23.21	0	0
50	0	50.00	66.28	65.57	60.42	45.05	21.26	11.06	0
55	0	47.37	66.67	65.77	61.21	42.28	19.52	10.32	0
60	0	45.00	65.98	65.91	61.82	45.36	25.84	9.33	0
65	0	50.00	66.67	66.13	59.63	42.86	24.29	8.81	0
70	0	47.83	66.96	66.30	60.32	45.45	22.66	8.09	0
75	0	46.01	67.27	66.34	60.97	43.33	21.41	7.68	0
80	0	50.00	66.97	66.52	61.51	45.65	26.09	7.13	0
85	0	48.28	67.28	66.60	62.03	43.69	24.75	6.81	0
90	0	46.67	67.51	66.67	62.36	45.65	23.55	6.37	0
95	0	50.00	67.74	66.77	60.92	43.99	22.46	6.12	0
100	0	48.48	67.36	66.85	61.38	45.79	26.12	11.30	5.76
201	0	48.51	68.38	67.41	62.65	45.81	26.18	11.30	5.79
999	0	49.70	69.09	67.97	63.44	46.02	26.44	11.41	5.85

3.3 The case with 4 players

The 4-player case is more complex and we only propose limiting representations for $P(4, Q)$, assuming that w tends to infinity. We start from the following characterization result.

Lemma 6 *One (or two) dummy player(s) exist(s) in a 4-player weighted voting game if and only if* ($w_1+w_2+w_4 < q$ *and* $w_1+w_2+w_3 \geq q$) *or* ($w_1+w_3+w_4 < q$ *and* $w_1+w_2 \geq q$) *or* ($w_1+w_3+w_4 < q$ *and* $w_1+w_3 \geq q$ *and* $w_1+w_4 < q$) *or* $w_2+w_3 \geq q$. *If* ($w_1+w_3+w_4 < q$ *and* $w_1+w_2 \geq q$)*, and only in this case, two dummy players exist.*

Lemma 6 (the proof is in the Appendix) leads to the following proposition, where $P_2(4, Q)$ designates the limiting probability of having exactly two dummy players in a 4-player weighted voting game.

Proposition 4 *When w tends to infinity, the probability of obtaining at least one dummy player is*

$$P(4, Q) = \frac{2(-219Q^3 + 378Q^2 - 216Q + 41)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 1/2 < Q \leqslant 3/5,$$

$$P(4, Q) = \frac{2(156Q^3 - 297Q^2 + 189Q - 40)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 3/5 < Q \leqslant 2/3,$$

$$P(4, Q) = \frac{2(75Q^3 - 162Q^2 + 117Q - 28)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 2/3 < Q \leqslant 3/4,$$

$$P(4, Q) = \frac{2(-53Q^3 + 126Q^2 - 99Q + 26)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 3/4 < Q \leqslant 1;$$

and the probability of having exactly two dummies is given as:

$$P_2(4, Q) = \frac{6(2Q-1)^3}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 1/2 < Q \leqslant 2/3,$$

$$P_2(4, Q) = \frac{6(Q-1)^2(8Q-5)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 2/3 < Q \leqslant 1.$$

Proof. Our approach is similar to the one used to prove Proposition 3, with an additional difficulty since we have to distinguish four cases for evaluating the number of vote distributions giving rise to at least one dummy player, in accordance with Lemma 6. The representations we obtain (functions of both w and q) are unfortunately very complex and their practical use is limited. However, it is easy to deduce from these complicated expressions some limiting representations for the case where w tends to infinity (as already done in Remark 2): we have just to consider the higher degree term in w in the quasi polynomials we obtain. To illustrate, consider the total number of vote distributions in a weighted voting game with 4 players. This number corresponds to the number of integer solutions of the following set of (in)equalities:

$$w_1 \geq w_2,\ w_2 \geq w_3,\ w_3 \geq w_4,\ w_4 \geq 1,\ w_1 < q \text{ and } w_1 + w_2 + w_3 + w_4 = w.$$

For $(w+1)/2 \leq q < w$, w odd, the quasi polynomial we obtain is as follows:

$$-\frac{1}{48}w^3 + (\frac{1}{12}q - \frac{1}{48})w^2 + (-\frac{1}{12}q^2 + \frac{1}{12}q + \frac{1}{48})w + \frac{1}{36}q^3 + f(q),$$

where $f(q)$ is a degree-2 quasi polynomial in q. Let $Q = q/n$ and replace q by Qn in this representation. We obtain

$$-\frac{1}{48}w^3 + (\frac{1}{12}wQ - \frac{1}{48})w^2 + (-\frac{1}{12}w^2Q^2 + \frac{1}{12}wQ + \frac{1}{48})w + \frac{1}{36}w^3Q^3 + f(wQ),$$

where the degree of w in $f(wQ)$ is lower than 3. The coefficient of w^3 (higher degree term in w) is:

$$-\frac{1}{48} + \frac{1}{12}Q - \frac{1}{12}Q^2 + \frac{1}{36}Q^3 = \frac{4Q^3 - 12Q^2 + 12Q - 3}{144}. \qquad (3)$$

We obtain in the same way the coefficients of w^3 in the quasi polynomials associated to each of the four cases with a dummy player (notice that different domains have to be distinguished, depending on the value of Q), and the representation for $P(4, Q)$ is obtained by summing the four cases and dividing by (3). $P_2(4, Q)$ is obtained in a similar way by only considering the case where two dummies exist. QED.

Table 2 lists computed values of $P(4, Q)$ for various values of the relative quota Q. When the relative quota Q moves from 1/2 to 1, the probability of having at least one dummy player decreases over the range of values with $1/2 \leq Q \leq 0.54$, then increases over the range $0.54 \leq Q \leq \overline{Q} = 0.8621$ and decreases again and tends to 0 when Q tends to 1. Notice that for $Q = 0.54$, $P(4, Q) = 32.81\%$ (local minimum) and for $Q = \overline{Q}$, we obtain $P(4, Q) = 68.49\%$, a surprisingly high value. $P_2(4, Q)$, the probability of having two dummies, increases when Q moves from 1/2 to $\overline{Q} = 0.741$, and then monotonically decreases. Finally, assuming that Q is uniformly distributed on $[1/2,\ 1]$, we obtain that $E[P(4, Q)] = 50.97\%$ and $E[P_2(4, Q)] = 18.00\%$.

Figure 1 represents the limiting probabilities according to the quota Q for both the 3 and 4-player cases. Contrary to the 3-player case, the case with 4 players does not lead to a regular single-peaked function : as mentioned above, in addition to the global minimum reached for $Q = 1$, we obtain when $n = 4$ a local minimum which is not far from the majority case. We will see in the following section that the limiting probability curves for the cases with 5 players and more have the same shape than the one obtained for 4 players.

Table 2. Limiting Probability for having at least one dummy for 3 and 4 players (in%)

Quota Q	$P(3,Q)$	$P(4,Q)$	$P_2(4,Q)$
1/2	0	50.00	0
0.51	0.29	40.51	0.01
0.52	1.55	35.17	0.07
0.55	7.64	34.26	0.94
0.60	23.08	47.31	6.45
0.65	42.69	50.45	19.55
2/3	50.00	52.17	26.09
0.70	61.64	54.93	36.32
0.75	69.23	56.67	40.00
0.80	68.18	62.81	34.71
0.85	60.32	68.20	24.63
0.90	46.39	64.86	13.25
0.95	26.45	44.85	3.90
0.98	11.41	21.45	0.68
0.99	5.85	11.35	0.18
1	0	0	0

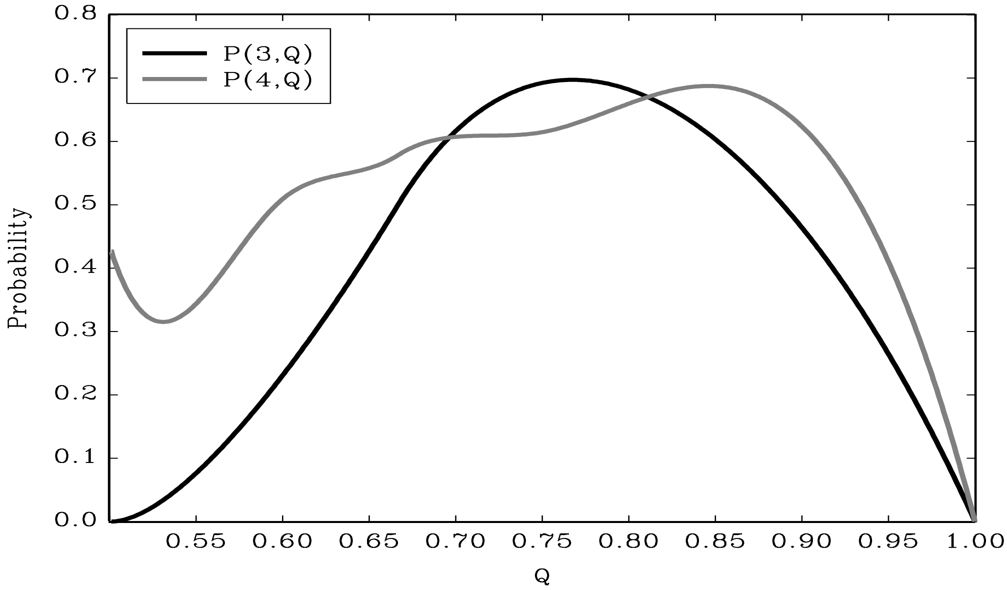

Figure 1 – Limiting probability of having at least one dummy player for 3 and 4 players

4 Numerical results for more than 4 players

We extend numerically the results presented in Table 1 (coming from Proposition 3) and those corresponding to Table 2 (computed from Remark 2 and Proposition 4). In order to obtain the desired probabilities for more than 4 players, two approaches are considered. The first one gives exact results whereas the second one is based on simulations and provides estimated probabilities.

Exact computations are done by considering the exhaustive list of all possible vectors of weights for a given number w of votes. For all these vectors $(w_1, \ldots, w_n)$, we check whether or not the last player is decisive (remember that $w_1 \geq w_2 \geq \ldots \geq w_n$). To do this, we use the classical Banzhaf power index [2] since a player, by construction, is a dummy if his (her) index is equal to zero. We compute this index using a generating functions approach which leads to exact values (this point is fundamental because we are looking for an index with a zero value, which prohibits the use of approximation methods). Finally, the exact probability of having at least one dummy player is the ratio between the number of times the last player is never decisive and the number of vectors $(w_1, \ldots, w_n)$ considered as admissible (in accordance with the assumed uniform distribution of weight vectors). Unfortunately, enumerating all these distributions is highly time consuming when the number of players and w become large (see, for example, Barthélémy *et al.*, 2011b). It is the reason why we also resort to simulations.

Our simulations are based on random vectors of weights. The estimated probability of having at least one dummy player is then obtained by dividing the number of times the last player is never decisive by the number of vectors $(w_1, \ldots, w_n)$ randomly generated. In order to simulate the probability of a dummy player, two steps have to be considered. First, we have to simulate a vector of weights for a given w and a given number of players n. This can be done by using for instance the `Rancom` algorithm proposed by Nijenhuis and Wilf (1978). Second, we have to check whether there is at least one dummy player in the weighted game associated to these weights. This is done as mentioned above by using the Banzhaf power index. Then repeating these two steps k times gives the estimated probability which corresponds to the proportion of weighted games leading to a dummy player.

We analyze first the results for a finite total number of votes w. In a second step, we will extend our study to the case where this total number of votes tends to infinity.

2. For a clear and simple presentation, see Straffin (1994).

4.1 Finite case

We compute both exact and simulated probabilities. Tables 3 to 10 give the probability of having at least one dummy player according to given values of the quota Q, for weighted games with 3 to 10 players.

The probabilities $P(n, w, Q)$ are not monotonic with respect to parameters Q, w and n. However, we observe that the probability tends to 0 when n increases, which is a particular illustration of the so-called Penrose's law. Penrose (1946, 1952) argues that when the number of players is large and the relative weights not too dissimilar, the ratio between the voting powers of any two voters, measured by their Banzhaf index, tends to the ratio between their weights. This result holds only when the quota is fixed to 1/2. Lindner and Machover (2003) have shown that, if the Penrose's law is not always true, "experience suggests that counter-examples are atypical". Using simulations, Chang *et al.* (2006) conclude that if the result holds only for a quota of 50% when the Banzhaf index is considered, the Penrose's law remains valid for all values of the quota when power is measured by the Shapley-Shubik index.[3] Let us notice that a dummy player with the Banzhaf index is a dummy player as well with the Shapley-Shubik index (and reciprocally). As there are no weights equal to zero by construction in our study, each player tends to get a positive power when n tends to infinity and the probability of having a dummy player tends to zero. Figure 2 illustrates this result for $w = 60$ and $Q = 2/3, 0.75, 0.90, 0.95$[4].

Tables 3 to 10 report as well the optimal probabilities $P(n, w, \underline{Q})$, $P(n, w, \overline{Q})$ (denoted P_{min} and P_{max}) and the corresponding quotas $\underline{Q}$ and $\overline{Q}$. For a given w, we compute all the quotas Q running from majority to unanimity. More precisely, we consider all the quotas from $Q = 0.50$ (corresponding to either $q = w/2$ or $q = (w + 1)/2$), to $Q = (w - 1)/w$ (the closest quota to unanimity, $q = w - 1$).

For instance, for $w = 10$, we calculate the probability of having at least a dummy player for $q = 5, 6, 7, 8$ and 9, leading to relative quotas equal to 0.5, 0.6, 0.7, 0.8 and 0.9. In this case, we get unanimity as soon as q is greater than 9 and this implies that for all $Q > 0.9$, $P(10, n, Q) = 0\%$. Similarly, when considering $w = 20$, the unanimity case is obtained as soon as $Q > 0.95$ (corresponding to $q > 19$). In Tables 3 to 10, the symbol '-' is represented for cases where the relative quota Q is equivalent to the unanimity case ($q = w$). Notice that we do not take into account the unanimity case in the computation of P_{min} (as mentioned above, unanimity leads to a zero probability).

As the number of different quotas may be high (50 possible quotas for $w = 100$), only se-

3. See Straffin (1994) for a presentation of this power index.
4. Obviously, other values of w or Q lead to the same kind of curves.

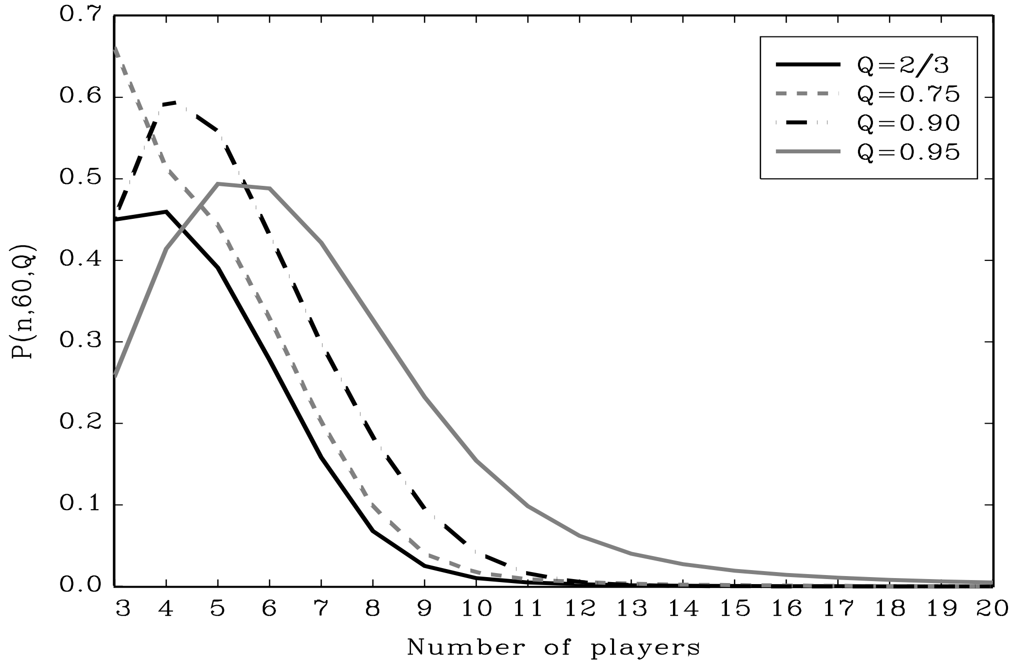

Figure 2 – Illustration of the Penrose's for $w = 60$

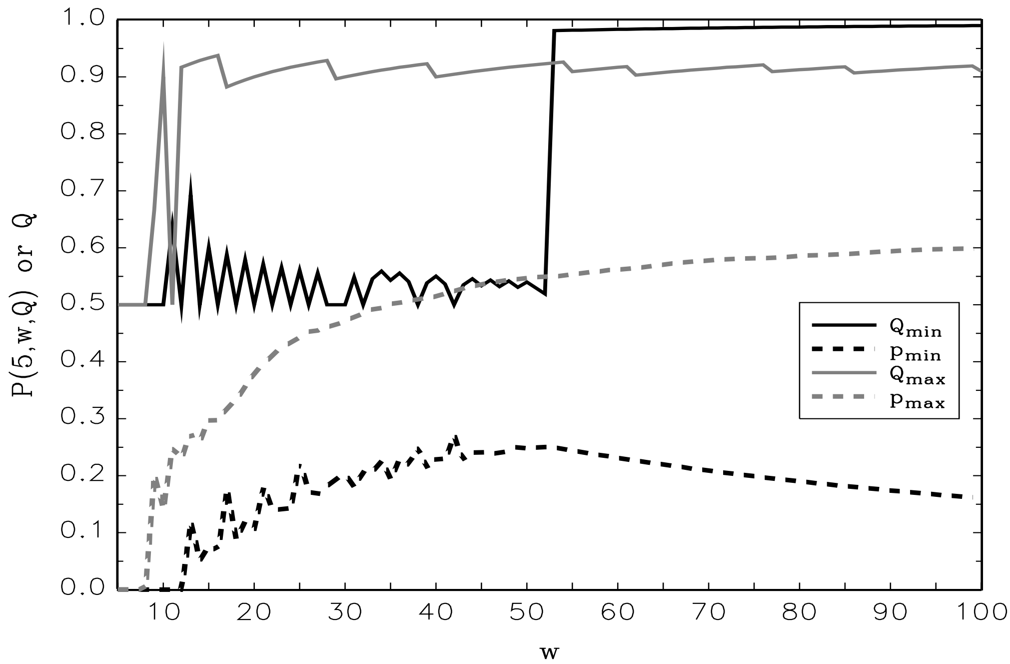

Figure 3 – 5-player case, smallest and highest probabilities of a dummy player according to w

lected values of Q are reported in Tables 3 to 10, with $Q \in \{0.5, 2/3, 0.75, 0.8, 5/6, 0.9, 0.95, 0.98, 0.99\}$. But P_{min} and P_{max} are computed using all the possible quotas. For instance, with $w = 15$

and $n = 5$, $P_{min} = 7.69\%$ with a quota $\underline{Q} = 0.60$ (not reported in Table 5). If more than one quota q lead to the smallest probability P_{min}, $\underline{q}$ is the smallest value, and $\underline{Q}$ is the corresponding relative value. For the above example, each $Q \in [0.60, 2/3[$ (corresponding to $[9/15, 10/15[$) leads to the same probability $P_{min} = 7.69\%$, and our convention gives $\underline{Q} = 0.60$.

Concerning $\underline{Q}$, a strange phenomenon is worth noticing. For small values of w, $\underline{Q}$ is close to the majority and when w increases, $\underline{Q}$ suddenly tends to unanimity. Figure 3 illustrates this phenomenon in the 5-player case. The value $\tilde{w}$ of the total number of votes for which $\underline{Q}$ becomes (almost) the unanimity instead of (almost) the majority increases with n, $n \geq 4$, the number of players. For instance $\tilde{w} = 35$ in the 4-player case, $\tilde{w} = 53$ in the 5-player case, $\tilde{w} = 87$ in the 6-player case and $\tilde{w} > 100$ when $n \geq 6$.

4.2 Infinite case

Limiting probabilities are simulated probabilities computed with high values of the total number of votes w. The logic behind this approach may be illustrated in the three-player case by comparing the $P(3, 999, Q)$ probabilities, given in Table 1, with the limiting ones $P(3, Q)$, given in Table 2. To get precise approximations, we take w equal to 9 999 [5].

Table 11 reports the estimated limiting probabilities from 3 to 15 players. Each row illustrates the Penrose's law : the probabilities tend to decrease when n increases, as previously mentioned for the finite case [6]. Figure 4 is given as an illustration of this remark with four values of Q.

Figure 5 presents the limiting probabilities plotted for 4, 6, 8, 10 and 15 players as a function of the quota Q. This Figure illustrates that fixing the quota at values higher than 50% (as 2/3 or even 3/4) does not lead to a smaller probability of having at least a dummy player. Moreover, in general, increasing the quota tends to increase this probability, except for values of Q very close to 1. This is particularly clear when the number of players is high.

5. We set $w = 99\ 999$ for games with more than 16 players. We have checked that the estimation is not statistically improved for higher values of w.

6. Note however that for high values of Q, the convergence is not clear and more players are needed in order to recover Penrose's law.

Table 3. Probability of having at least one dummy for 3 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	33.33	57.14	57.14	37.50	37.50	-	-	-	0	(0.50)	57.14	(0.8000)
15	0	30.77	58.82	58.82	50.00	31.58	-	-	-	0	(0.50)	58.82	(0.8000)
20	0	50.00	59.26	62.07	54.84	43.75	24.24	-	-	0	(0.50)	62.07	(0.8000)
25	0	44.44	62.79	63.04	58.33	37.25	21.15	-	-	0	(0.50)	63.04	(0.8000)
30	0	40.00	63.49	63.64	59.42	43.84	17.33	-	-	0	(0.50)	63.64	(0.8000)
35	0	50.00	65.12	64.44	56.25	40.00	15.69	-	-	0	(0.50)	65.12	(0.7714)
40	0	46.15	64.81	64.96	58.06	44.96	25.76	-	-	0	(0.50)	65.49	(0.7750)
45	0	43.36	65.47	65.10	59.24	41.21	23.21	-	-	0	(0.50)	65.97	(0.7778)
50	0	50.00	66.28	65.57	60.42	45.05	21.26	11.06	-	0	(0.50)	66.29	(0.7800)
55	0	47.37	66.67	65.77	61.21	42.28	19.52	10.32	-	0	(0.50)	66.67	(0.7636)
60	0	45.00	65.98	65.91	61.82	45.36	25.84	9.333	-	0	(0.50)	66.93	(0.7667)
65	0	50.00	66.67	66.13	59.63	42.86	24.29	8.807	-	0	(0.50)	67.23	(0.7692)
70	0	47.83	66.96	66.30	60.32	45.45	22.66	8.088	-	0	(0.50)	67.44	(0.7714)
75	0	46.01	67.27	66.34	60.97	43.33	21.41	7.676	-	0	(0.50)	67.51	(0.7733)
80	0	50.00	66.97	66.52	61.51	45.65	26.09	7.129	-	0	(0.50)	67.70	(0.7750)
85	0	48.28	67.28	66.60	62.03	43.69	24.75	6.811	-	0	(0.50)	67.77	(0.7765)
90	0	46.67	67.51	66.67	62.36	45.65	23.55	6.370	-	0	(0.50)	67.83	(0.7778)
95	0	50.00	67.74	66.77	60.92	43.99	22.46	6.117	-	0	(0.50)	67.99	(0.7684)
100	0	48.48	67.36	66.85	61.38	45.79	26.12	11.30	5.762	0	(0.50)	68.05	(0.7700)

Table 4. Probability of having at least one dummy for 4 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	14.29	25.00	33.33	33.33	33.33	33.33	-	-	-	14.29	(0.5000)	33.33	(0.8000)
15	43.75	30.43	42.31	42.31	48.15	37.04	-	-	-	15.00	(0.6000)	48.15	(0.8667)
20	21.95	38.60	40.00	46.77	50.79	51.56	32.81	-	-	21.95	(0.5000)	51.56	(0.9000)
25	43.28	38.46	44.25	50.00	55.08	50.00	30.83	-	-	28.09	(0.5600)	55.46	(0.8800)
30	30.89	40.00	47.18	52.26	55.45	54.63	27.18	-	-	24.46	(0.5667)	58.33	(0.8667)
35	46.55	44.64	49.51	53.55	59.31	53.44	24.92	-	-	24.92	(0.5000)	59.31	(0.8286)
40	33.94	44.28	48.77	54.76	60.30	57.98	40.38	-	-	22.59	(0.9714)	60.30	(0.8500)
45	46.37	44.21	50.08	55.62	60.67	56.42	37.65	-	-	20.83	(0.9750)	61.65	(0.8667)
50	37.35	46.68	51.21	56.36	61.17	58.84	35.11	19.13	-	19.13	(0.9778)	61.83	(0.8600)
55	47.66	46.11	52.18	56.90	61.33	57.64	32.92	17.86	-	17.86	(0.9800)	62.64	(0.8545)
60	38.73	45.96	51.26	57.42	61.59	60.20	41.42	16.57	-	16.57	(0.9818)	63.25	(0.8667)
65	47.55	48.05	51.98	57.80	63.06	58.97	39.50	15.57	-	15.57	(0.9833)	63.57	(0.8615)
70	40.53	47.60	52.63	58.18	63.00	60.61	37.54	14.61	-	14.61	(0.9846)	63.92	(0.8571)
75	48.20	47.29	53.24	58.47	63.27	59.58	35.88	13.81	-	13.81	(0.9857)	64.30	(0.8667)
80	41.34	48.72	52.69	58.75	63.17	61.34	42.55	13.04	-	13.04	(0.9867)	64.39	(0.8625)
85	48.16	48.28	53.17	58.97	63.35	60.33	40.95	12.40	-	12.40	(0.9875)	64.75	(0.8588)
90	42.43	48.02	53.59	59.21	63.27	61.58	39.38	11.78	-	11.78	(0.9882)	64.97	(0.8667)
95	48.55	49.33	54.00	59.37	64.29	60.71	37.95	11.25	-	11.25	(0.9889)	65.12	(0.8632)
100	42.97	48.95	53.44	59.56	64.32	62.03	42.86	20.59	10.74	10.74	(0.9900)	65.30	(0.8600)

Table 5. Probability of having at least one dummy for 5 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	14.29	14.29	-	-	-	0	(0.5000)	14.29	(0.9000)
15	26.09	21.43	26.67	26.67	23.33	30.00	-	-	-	7.69	(0.6000)	30.00	(0.9333)
20	10.61	25.00	29.27	26.51	33.33	38.10	32.14	-	-	10.61	(0.5000)	38.10	(0.9000)
25	33.09	27.22	31.38	35.26	35.60	44.27	33.33	-	-	22.42	(0.5600)	44.27	(0.9200)
30	20.14	30.29	36.76	36.46	39.20	46.95	31.83	-	-	20.14	(0.5000)	46.95	(0.9000)
35	40.42	36.95	38.07	40.18	44.49	50.15	30.56	-	-	19.46	(0.5429)	50.15	(0.9186)
40	25.61	35.92	39.43	40.80	44.82	51.44	46.28	-	-	23.34	(0.5500)	51.44	(0.9000)
45	42.13	36.72	41.90	43.26	46.15	53.55	44.99	-	-	24.09	(0.5333)	53.55	(0.9111)
50	30.20	39.45	42.42	43.58	46.29	53.81	43.51	25.74	0	24.76	(0.5400)	54.67	(0.9200)
55	44.59	38.82	44.13	45.13	47.32	55.35	41.86	24.44	-	24.44	(0.9818)	55.35	(0.9091)
60	32.91	39.09	44.31	45.46	47.43	55.88	49.39	23.10	-	23.10	(0.9833)	57.14	(0.9167)
65	45.52	42.11	44.69	46.51	49.47	57.05	48.26	21.98	-	21.98	(0.9846)	57.05	(0.9077)
70	35.60	41.37	45.72	46.73	49.85	56.99	46.95	20.87	-	20.86	(0.9895)	57.79	(0.9143)
75	46.78	41.28	46.04	47.57	49.91	57.96	45.69	19.92	-	19.92	(0.9867)	58.15	(0.9200)
80	37.23	43.14	46.23	47.71	50.23	58.06	51.86	18.99	-	18.99	(0.9875)	58.53	(0.9125)
85	47.26	42.52	46.95	48.34	50.25	58.85	50.80	18.19	-	18.19	(0.9882)	58.90	(0.9176)
90	38.91	42.34	47.15	48.49	50.56	58.75	49.71	17.41	-	17.41	(0.9889)	59.40	(0.9111)
95	48.06	44.27	47.74	48.98	51.77	59.44	48.60	16.72	-	16.72	(0.9895)	59.73	(0.9158)
100	40.03	43.67	47.73	49.09	51.70	59.42	52.71	29.66	16.05	16.05	(0.9900)	59.89	(0.9100)

Table 6. Probability of having at least one dummy for 6 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	0	0	-	-	-	0	(0.5000)	0	(0.5000)
15	18.18	8.00	11.54	11.54	3.846	19.23	-	-	-	0	(0.6000)	19.23	(0.9333)
20	2.56	10.23	16.85	12.22	16.67	18.89	25.56	-	-	2.56	(0.5000)	25.56	(0.9500)
25	22.28	12.28	15.88	23.08	18.72	28.94	29.79	-	-	11.11	(0.5600)	29.79	(0.9600)
30	10.02	17.15	23.48	21.89	26.37	32.33	30.83	-	-	10.02	(0.5000)	35.90	(0.9333)
35	31.74	24.81	24.86	27.83	30.87	38.13	31.50	-	-	11.58	(0.5429)	40.21	(0.9429)
40	14.87	22.97	26.43	28.22	30.06	36.50	42.67	-	-	14.87	(0.5000)	42.67	(0.9500)
45	34.07	24.77	29.83	31.46	33.66	40.77	43.95	-	-	16.30	(0.5333)	44.97	(0.9333)
50	19.31	27.40	29.99	31.71	33.58	41.28	44.41	29.52	-	18.46	(0.5400)	46.84	(0.9400)
55	37.19	26.51	32.58	34.26	35.82	44.27	44.34	28.64	-	17.69	(0.5273)	48.19	(0.9455)
60	22.32	27.77	32.87	34.24	35.80	43.10	48.83	27.63	-	19.86	(0.5167)	48.96	(0.9333)
65	38.62	31.05	33.31	36.05	37.50	45.62	49.19	26.71	-	20.37	(0.5231)	50.41	(0.9385)
70	25.41	30.23	34.83	36.21	38.93	45.47	49.14	25.75	-	21.04	(0.5286)	51.42	(0.9429)
75	40.28	30.62	35.13	37.48	38.81	47.46	48.93	24.86	-	21.79	(0.5200)	52.06	(0.9467)
80	27.39	32.59	35.55	37.59	39.92	46.45	52.40	23.99	-	22.30	(0.5250)	52.59	(0.9375)
85	41.04	31.80	36.49	38.66	39.88	48.18	52.51	23.18	-	23.18	(0.9882)	53.31	(0.9412)
90	29.43	32.02	36.75	38.70	40.73	47.91	52.43	22.39	-	22.39	(0.9889)	53.75	(0.9444)
95	42.15	34.23	37.65	39.56	41.69	49.37	52.21	21.66	-	21.66	(0.9895)	54.29	(0.9368)
100	30.91	33.48	37.71	39.64	41.61	48.52	54.15	36.79	20.96	20.96	(0.9900)	54.88	(0.9400)

Table 7. Probability of having at least one dummy for 7 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	0	0	-	-	-	0	(0.5000)	0	(0.5000)
15	5.26	4.76	4.76	4.76	0	9.52	-	-	-	0	(0.6000)	9.52	(0.9333)
20	0	2.47	8.54	2.44	6.10	6.10	17.07	-	-	0	(0.5000)	17.07	(0.9500)
25	8.26	3.69	6.07	12.90	7.26	14.52	23.39	-	-	3.69	(0.6800)	23.39	(0.9600)
30	3.25	6.93	12.18	8.75	15.37	18.61	26.38	-	-	3.25	(0.5000)	26.38	(0.9667)
35	18.53	13.80	12.33	16.12	19.75	24.65	28.60	-	-	4.83	(0.5429)	28.60	(0.9714)
40	5.78	10.50	13.57	15.36	16.81	21.80	33.02	-	-	5.78	(0.5000)	33.02	(0.9500)
45	21.35	13.15	17.47	18.70	22.00	26.40	36.48	30.20	-	8.65	(0.5333)	36.48	(0.9556)
50	8.58	14.87	16.61	18.91	21.30	29.14	38.94	30.06	-	8.58	(0.5000)	38.94	(0.9600)
55	25.1	13.70	19.94	22.17	23.86	32.19	40.62	29.65	-	9.36	(0.5273)	40.62	(0.9636)
60	10.93	15.83	20.14	21.43	23.70	29.77	42.16	29.18	-	10.93	(0.5000)	42.16	(0.9500)
65	27.04	18.88	20.55	23.94	25.80	32.56	43.90	28.60	-	11.68	(0.5231)	43.90	(0.9538)
70	13.50	17.82	22.42	24.01	27.89	33.78	45.17	28.01	-	12.63	(0.5286)	45.17	(0.9571)
75	29.17	18.92	22.59	25.56	27.25	35.86	46.13	27.36	-	12.56	(0.5200)	46.13	(0.9600)
80	15.26	20.65	23.21	25.61	28.91	34.45	46.08	27.36	-	14.07	(0.5250)	46.78	(0.9625)
85	30.28	19.58	24.36	27.09	28.76	36.30	47.36	26.73	-	15.02	(0.5176)	47.36	(0.9529)
90	17.17	20.39	24.62	26.96	29.83	36.76	48.39	26.09	-	15.01	(0.5222)	48.39	(0.9556)
95	31.73	22.64	25.82	28.18	31.13	38.29	49.19	25.46	-	16.21	(0.5157)	49.19	(0.9579)
100	18.69	21.61	25.92	28.22	30.94	37.22	49.10	40.86	24.84	15.87	(0.5200)	49.79	(0.9600)

Table 8. Probability of having at least one dummy for 8 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	0	0	-	-	-	0	(0.5000)	0	(0.5000)
15	7.14	6.67	6.67	6.67	0	6.67	-	-	-	0	(0.6000)	7.14	(0.5000)
20	0	1.43	2.86	0	2.86	1.43	10	-	-	0	(0.5000)	10.00	(0.9500)
25	1.89	1.31	1.74	5.65	1.74	5.22	16.52	-	-	0.90	(0.5600)	16.52	(0.9600)
30	1.01	2.85	5.81	3.135	7.52	9.40	20.53	-	-	0.97	(0.6000)	20.53	(0.9667)
35	9.24	7.59	4.92	8.45	11.07	14.01	23.84	-	-	1.88	(0.5429)	23.84	(0.9714)
40	1.82	4.23	5.56	6.54	7.14	10.97	22.02	-	-	1.82	(0.5000)	25.91	(0.9750)
45	10.50	5.77	8.55	9.02	12.48	14.63	26.43	-	-	3.05	(0.5556)	27.51	(0.9778)
50	2.99	6.38	6.79	8.92	10.94	19.13	30.10	28.43	-	2.99	(0.5000)	30.10	(0.9600)
55	13.73	5.09	9.95	11.94	13.18	21.87	33.04	29.07	-	3.97	(0.5273)	33.04	(0.9636)
60	4.83	6.79	9.72	10.41	12.86	18.23	32.70	29.34	-	4.13	(0.5000)	35.39	(0.9667)
65	15.27	9.22	9.89	13.14	14.83	20.85	35.32	29.46	-	5.53	(0.5385)	37.22	(0.9692)
70	5.54	7.94	11.67	13.00	17.23	23.45	37.50	29.37	-	5.54	(0.5286)	38.63	(0.9714)
75	17.46	9.32	11.51	14.44	15.90	25.47	39.40	29.20	-	5.71	(0.5200)	39.68	(0.9733)
80	6.43	10.37	12.12	14.29	17.97	23.70	36.27	28.92	-	6.43	(0.5000)	40.95	(0.9625)
85	18.54	9.01	13.25	15.98	17.67	25.52	38.33	28.60	-	7.27	(0.5176)	42.26	(0.9647)
90*	7.434	10.69	13.52	15.38	17.87	26.19	39.68	28.14	-				
95*	20.32	11.93	14.91	17.14	19.50	27.62	42.63	28.73	-				
100*	9.154	11.15	14.62	17.04	20.17	27.32	40.83	41.51	27.49				

* simulated values with 10 000 replications and $w = 9\ 999$

Table 9. Probability of having at least one dummy for 9 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	0	0	0	-	-	-	(0.5000)	0	(0.5000)
15	0	0	0	0	0	0	0	-	-	-	(0.5000)	0	(0.5000)
20	0	0	0	0	1.85	0	5.56	-	-	0	(0.5000)	5.56	(0.9500)
25	0	1.00	0	2.99	0.50	1.49	10.95	-	-	0	(0.5000)	10.95	(0.9600)
30	0.35	1.18	2.84	0.67	3.34	4.68	14.88	-	-	0	(0.6000)	14.88	(0.9667)
35	4.34	4.67	2.07	4.58	5.68	7.49	18.59	-	-	0.52	(0.6571)	18.59	(0.9714)
40	0.59	1.93	2.15	2.54	2.37	4.49	12.90	-	-	0.59	(0.5000)	21.29	(0.9750)
45	4.34	2.73	4.35	4.23	6.65	6.91	17.08	-	-	1.04	(0.5556)	23.52	(0.9778)
50	0.92	2.60	2.39	3.60	4.69	11.51	20.91	25.14	-	0.91	(0.5000)	25.14	(0.9800)
55	6.24	1.93	4.66	5.88	6.31	13.79	24.33	26.42	-	1.57	(0.5636)	26.42	(0.9818)
60	1.93	2.51	3.99	3.98	5.73	9.49	23.24	27.32	-	1.38	(0.5000)	27.32	(0.9833)
65	6.78	4.15	3.93	6.25	7.03	11.62	26.10	27.99	-	2.08	(0.5692)	29.88	(0.9692)
70	1.99	3.10	5.12	5.82	9.16	14.92	28.67	28.42	-	0.99	(0.5000)	32.06	(0.9714)
75	8.36	4.03	4.64	6.72	7.37	16.73	31.02	28.70	-	2.30	(0.5200)	33.89	(0.9733)
80	2.27	4.38	5.01	6.36	9.46	14.50	25.96	28.83	-	2.27	(0.5000)	35.42	(0.9750)
85*	8.76	3.21	5.69	7.58	8.63	16.04	28.53	29.06	-				
90*	2.70	3.78	5.69	7.21	9.80	17.48	30.10	28.99	-				
95*	9.81	5.12	6.85	8.44	10.99	18.62	32.81	27.73	-				
100*	2.85	4.22	6.96	8.60	10.99	16.66	31.46	38.87	28.69				

* simulated values with 10 000 replications and $w = 9\ 999$

Table 10. Probability of having at least one dummy for 10 players (in%)

	Value of Q												
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99	P_{min}	$\underline{Q}$	P_{max}	$\overline{Q}$
10	0	0	0	0	0	0	0	-	-	0	(0.5000)	0	(0.5000)
15	0	0	0	0	0	0	0	-	-	0	(0.5000)	0	(0.5000)
20	0	0	0	0	0	0	2.38	-	-	0	(0.5000)	2.38	(0.9500)
25	0	0	0	1.83	0	0	6.71	-	-	0	(0.5000)	6.71	(0.9600)
30	0	0.57	1.13	0	1.70	2.64	10.19	-	-	0	(0.5000)	10.19	(0.9667)
35	2.59	2.96	1.10	2.41	3.16	4.19	13.81	-	-	0.13	(0.6571)	13.81	(0.9714)
40	0.29	1.09	1.00	1.09	0.61	1.64	6.77	-	-	0.20	(0.5550)	16.66	(0.9750)
45	1.85	1.62	2.42	2.26	3.56	2.81	9.97	-	-	0.38	(0.5556)	19.19	(0.9778)
50	0.30	1.19	0.91	1.50	1.95	6.51	13.29	21.20	-	0.30	(0.5000)	21.20	(0.9800)
55	2.95	0.79	2.43	3.06	3.06	8.14	16.46	22.92	-	0.56	(0.5636)	22.92	(0.9818)
60	0.84	1.01	1.71	1.40	2.35	4.17	15.43	24.27	-	0.48	(0.5000)	24.27	(0.9833)
65	2.82	2.04	1.57	3.01	2.95	5.54	18.02	25.38	-	0.72	(0.6308)	25.38	(0.9846)
65*	2.85	1.95	1.79	2.97	2.82	5.87	18.17	25.45	-				
70*	0.67	1.40	2.25	2.52	4.46	9.02	20.74	25.87	-				
75*	3.45	2.07	1.82	2.80	2.55	10.42	23.38	26.79	-				
80*	0.96	1.88	1.76	2.61	4.30	7.74	17.18	27.09	-				
85*	3.45	1.25	2.30	3.42	3.93	8.79	18.94	27.96	-				
90*	0.92	1.44	2.26	2.78	4.34	9.82	21.90	27.66	-				
95*	4.87	2.33	3.30	3.42	5.04	10.68	22.68	28.44	-				
100*	1.01	1.49	2.41	3.17	5.07	9.67	24.83	35.55	27.64				

* simulated values with 10 000 replications and $w = 9\ 999$

Table 11. Simulated Limiting Probability of having at least one dummy for 3 to 15 players, $w = 9\ 999$, 10 000 simulations (in%)

	Number of players												
Quota Q	3	4	5	6	7	8	9	10	11	12	13	14	15
0.50	0	49.34	53.20	49.54	42.54	34.13	26.16	17.71	11.43	7.00	4.13	1.99	0.68
0.51	0.51	39.91	40.01	34.30	27.27	21.01	15.72	10.15	6.64	3.48	1.95	0.63	0.13
0.52	1.62	35.03	36.05	33.68	28.97	21.42	16.16	10.44	6.35	3.57	1.80	0.59	0.16
0.55	8.48	34.12	41.20	37.04	30.40	22.69	16.72	10.42	6.20	3.45	1.66	0.59	0.18
0.60	23.27	47.16	46.39	39.97	31.96	23.76	16.60	10.89	6.25	3.61	1.87	0.82	0.12
0.65	43.18	51.19	48.62	42.55	33.46	25.04	18.16	11.23	7.17	4.01	2.07	0.77	0.20
2/3	50.08	52.85	49.78	42.22	34.80	25.39	17.86	11.78	6.92	4.01	2.19	0.85	0.25
0.70	62.12	55.67	52.85	44.26	35.14	26.27	19.54	11.86	7.72	4.48	2.15	0.89	0.32
0.75	68.81	56.43	53.86	45.90	37.47	28.71	20.87	13.96	9.27	5.57	3.17	1.60	0.72
0.80	68.03	63.21	55.58	48.67	40.73	31.54	24.35	17.08	11.83	7.60	4.77	2.53	1.17
0.85	68.38	68.27	58.57	50.64	43.83	35.72	28.75	21.83	15.93	11.03	6.98	4.64	2.73
0.90	46 08	65.00	65.59	57.47	49.49	43.55	36.23	29.28	23.35	17.57	13.13	9.92	6.96
0.95	26.58	44.93	58.37	62.52	61.63	55.29	50.30	43.73	38.62	33.45	28.73	23.39	19.79
0.98	11.08	21.56	32.61	42.10	50.58	35.54	58.57	59.28	57.09	53.90	50.32	45.76	42.89
0.99	5.50	11.40	18.54	25.89	33.14	39.61	46.69	51.43	54.52	56.44	58.16	57.86	57.63

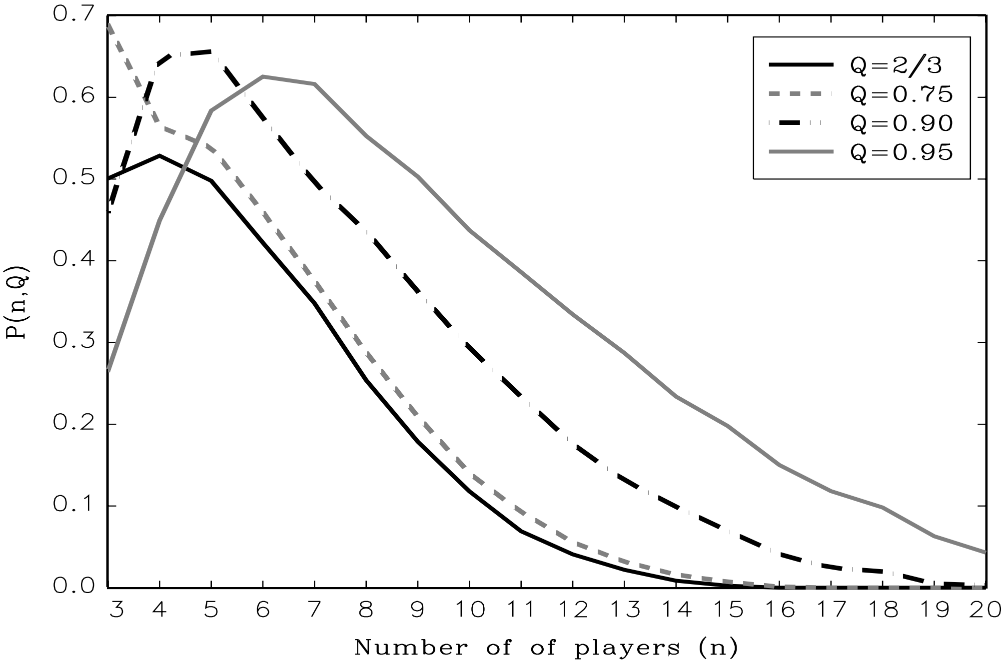

Figure 4 – Penrose's law when w tends to infinity

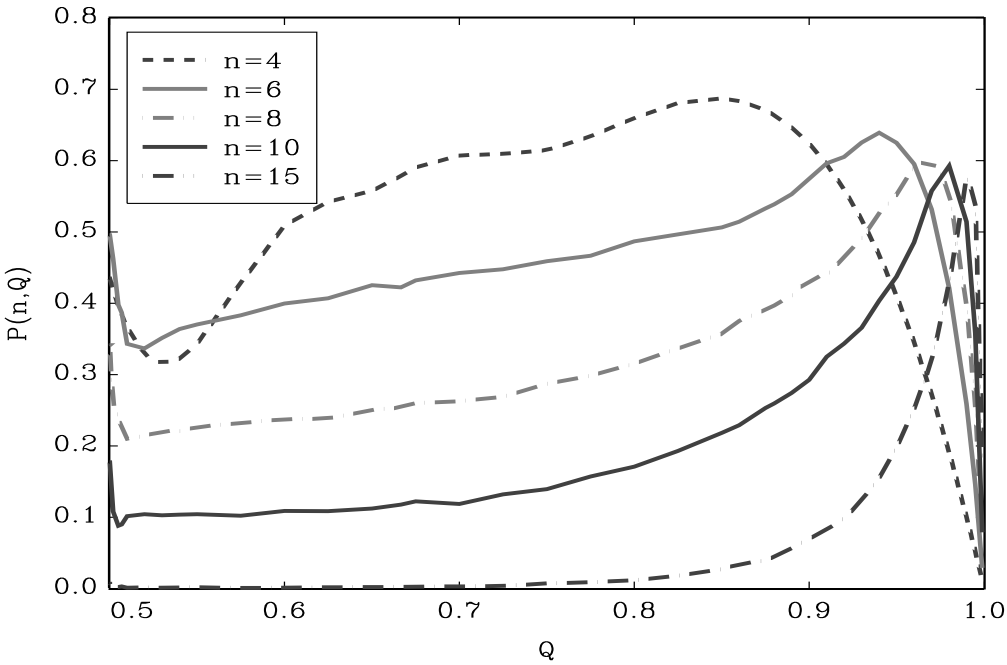

Figure 5 – Limiting probability of having at least one dummy player for 4 to 15 players

5 Conclusion and final remark

The main teachings that emerge from our results can be summarize as follows.

1) Barthélémy *et al.* (2011) have shown that the probability of having a dummy player is surprisingly high in majority voting games ($Q = 50\%$). Our results demonstrate that increasing the quota to 2/3 or 3/4 *increases* the probability that the "dummy paradox" occurs. In the three-player case, this (limiting) probability is maximized for $Q = 0.77$ and the corresponding quota values for $n > 3$ increase with n and stand between 0.80 and 0.99.

2) In order to minimize the probability of having dummy players, it is advisable to choose a quota between 0.50 and 0.55. If the choice is restricted to "standard" quotas (1/2, 2/3, 3/4...), taking $Q = 0.50$ appears to be the right solution. The choice of a quota close to 1, that could be suggested by the observation that there is no dummy player for $Q = 1$, would be a serious mistake: for Q close to 1, the risk of a dummy player would be more likely maximized rather than minimized.

3) The probability of having a dummy first increases then decreases with the number of players, *whatever the quota value*, suggesting that the Penrose's law holds for $Q > 1/2$. It is worth noticing, however, that this convergence towards 0 is much slower with high values of Q than with small values. To illustrate, with 15 players, the probability of a dummy player is less than 0.25% for $0.5 < Q < 0.66$ and is still about 43% for $Q = 0.98$.

Final Remark. Our results suppose that the player 1, the "biggest" player, holds a weight that may be (almost) as high as the quota: $w_1 < Q$. This assumptions is of course disputable and it seems of interest to study the extent to which it impacts our results. In order to clarify this question, we have considered the more constrained -but perhaps more realistic- situation where the number of votes of the biggest player may not be higher or equal to half of the total number of votes: $w_1 < w/2$. Let $P^*(n, Q)$ be the probability of having at least one dummy player under this stronger constraint when w tends to infinity. Using exactly the same approach as in Section 3, we obtain the following results for the 3-player case:

$$P^*(3, Q) = 6(2Q - 1)^2 \quad for\ 1/2 < Q \leqslant 2/3,$$

$$P^*(3, Q) = -6(14Q^2 - 20Q + 7) \quad for\ 2/3 \leqslant Q \leqslant 3/4,$$

$$P^*(3, Q) = 12(Q - 1)^2 \quad for\ 3/4 \leqslant Q \leqslant 1.$$

The comparison of the computed values of $P^*(3, Q)$ (not reported here) with the values listed in Table 2 shows that $P^*(3, Q)$ is lower than $P(3, Q)$ for values of Q close to 1/2 or 1 ; however, we observe that for intermediate values of Q, $P^*(3, Q)$ is clearly higher than $P(3, Q)$. On the average, we obtain that $E[P^*(n, Q)] = 1/3 < E[P3, Q)] = 40.83\%$: introducing the additional constraint that $w_1 < w/2$ tends to reduce the risk of having a dummy player but this risk still remains high.

In the 4-player case, the results are the following:

$$P^*(4,Q) = 2(2-3Q)(126Q^2 - 132Q + 35) \quad for\ 1/2 < Q \leqslant 3/5,$$

$$P^*(4,Q) = 2(372Q^3 - 702Q^2 + 441Q - 92) \quad for\ 3/5 < Q \leqslant 2/3,$$

$$P^*(4,Q) = 2(48Q^3 - 108Q^2 + 81Q - 20) \quad for\ 2/3 < Q \leqslant 3/4,$$

$$P^*(4,Q) = -(512Q^3 - 1224Q^2 + 972Q - 257) \quad for\ 3/4 < Q \leqslant 5/6,$$

$$P^*(4,Q) = 2(Q-1)(68Q^2 - 130Q + 59) \quad for\ 5/6 < Q \leqslant 1.$$

From computed values of $P^*(4,Q)$, we conclude in the same way as for the 3-player case: requiring that w_1 must not be higher or equal to $w/2$ reduces the probability of having a dummy for some values of Q but increases this probability for other values. We obtain that the expected probability is given as $E[P^*(4,Q)] = 47.08\%$, a value slightly lower than $E[P(4,Q)] = 50.97\%$.

Finally, we have estimated *via* simulations the values of $P^*(n,Q)$ for $n = 5,6,7,...,15$. It turns out that these values converge towards those of $P(n,Q)$ given in Tables 5 from 10, confirming the rather limited impact of our assumption ($w_1 < Q$) on the probability of having a dummy player for $n > 3$.

References

Barthélémy F., D. Lepelley and M. Martin (2011), On the likelihood of dummy players in weighted majority games, to be published in *Social Choice and Welfare*.

Barthélémy F., M. Martin and B. Tchantcho (2011b), Some conjectures on the two main power indices, Working paper 2011-14, THEMA.

Chang P.-L., V. C H Chua and M. Machover (2006), L.S. Penrose's limit theorem: Tests by simulation, *Mathematical social sciences* 51, 90-106.

Felsenthal D, and M. Machover (1998), *The Measurement of Voting Power. Theory and Practice, Problems and Paradoxes*, Edward Elgar Publishing, UK.

Gehrlein W.V., and D. Lepelley (2011), *Voting Paradoxes and Group Coherence*, Springer.

Leech D. (2002), Power indices as an aid to institutional design: the generalized apportionment problem, Warwick Economic Research Papers, n°648.

Lepelley D, A. Louichi and H. Smaoui (2008), On Ehrhart polynomials and probability calculations in voting theory, *Social Choice and Welfare* 30, 363-383.

Lindner I. and M. Machover (1998), L.S. Penrose's limit theorem: proof of some special cases, *Mathematical Social Sciences* 47, 37-49.

Nijenhuis A. and Wilf H. (1978), Combinatorial Algorithms for Computers and Calculators, 2nd ed. New York: Academic Press,
(http ://www.math.upenn.edu/∼wilf/website/CombinatorialAlgorithms.pdf).

Penrose L.S. (1946), The elementary statistics of majority voting, *Journal of the Royal Statistical Society* 109, 53-57.
Penrose L.S. (1952), *On the objective study of crowd behavior*, London : H.K. Lewis and Co.
Straffin P.D. (1994), Power and stability in politics, Chapter 32, Aumann. R.J and S. Hart, eds, *Handbook of Game Theory*, vol 2, North-Holland, Amsterdam, The Netherlands.
Zuckerman M., P. Faliszewski, Y. Bachrach and E. Elkind (2012), Manipulating the quota in weighted voting games, *Artificial Intelligence* 180-181, 1-19.

6 Appendix

Proof of Lemma 1. Consider the following game $[\frac{w}{2}+1; \frac{w}{2}-2, \frac{w-6n+26+a}{4}, \frac{w-6n+26-a}{4}, 3, 3, ..., 3, 1]$ with $a = 0$ if n is odd (even) and $\frac{w}{2}$ is even (odd) and $a = 2$ if n is odd (even) and $\frac{w}{2}$ is odd (even).

We have to show that $q > w_1 \geq w_2 \geq ... \geq w_n$ (1) and since $w_n = 1$ we have to show that there is no coalition $S \subseteq N/\{n\}$ such that $\sum_{i\in S} w_i = q-1$ (2). Indeed, this is the only situation for which the player n, with one vote, is a dummy player.

(1) clearly $q > w_1$ and $w_2 \geq w_3$ thus we have just to show that $w_1 \geq w_2$ and $w_3 \geq 3$. We have $w_1 \geq w_2$ if $\frac{w}{2} - 2 \geq \frac{w-6n+26+a}{4}$ that is to say $w \geq 34 + a - 6n$. Furthermore, $w_3 \geq 3$ if $\frac{w-6n+26-a}{4} \geq 3$, that is to say $w \geq 6n - 14 + a$. Since $n > 3$, the different inequalities are implied by $w \geq 6n - 12$.

(2) Consider a coalition $S \subseteq N/\{n\}$ with $1 \in S$. Since $w_3 \geq 3$ we have $\sum_{i\in S} w_i \geq q$ except if $S = \{1\}$. Since $w_1 = q - 3$, it is never possible to obtain a total weight equal to $q - 1$ with the player 1 belonging to S. Assume now that the player 1 does not belong to S, we have $\sum_{i=2}^{n-1} w_i = q$. Since $w_i \geq 3$, $i = 2, 3, ..., n-1$, it is not possible to obtain a total weight equal to $q - 1$. QED

Proof of Lemma 2 Consider the following game $[q; w - m - 2, m - 2n + 7, 2, 2, ..., 2, 1]$ We have to show that $q > w_1 \geq w_2 \geq ... \geq w_n$ (1) and since $w_n = 1$ we have to show that there is no coalition $S \subseteq N/\{n\}$ such that $\sum_{i\in S} w_i = q - 1$ (2).

(1) we have just to show that $w_1 \geq w_2$ and $w_2 \geq 2$. $w_1 \geq w_2$ if $w - m - 2 \geq m - 2n + 7$ or $m \leq \frac{w+2n-9}{2}$. Since $m \geq 2n - 5$, we must have $2n - 5 \leq \frac{w+2n-9}{2}$ or $w \geq 2n - 1$. This is true by hypothesis. Furthermore, $w_2 \geq 2$ if $m - 2n + 7 \geq 2$ or $m \geq 2n - 5$ which is true by hypothesis.

(2) Consider a coalition $S \subseteq N/\{n\}$ with $1 \in S$. Since $w_2 \geq 2$ we have $\sum_{i\in S} w_i \geq q$ except if $S = \{1\}$. Since $w_1 = q - 2$, it is never possible to obtain a total weight equal to $q - 1$ with the player 1 belonging to S. Assume now that the player 1 does not belong to S, thus we have $\sum_{i=2}^{n-1} w_i = w - q + 1$. But $w - q + 1 = q - 1$ implies $q = \frac{w}{2} + 1$ which is not possible by hypothesis. QED

Proof of Lemma 3 Consider the following game $[q; w-m-n+1, m+1, 1, ..., 1]$ We have to show that $q > w_1 \geq w_2 \geq ... \geq w_n$ (1) and since $w_n = 1$ we have to show that there is no coalition $S \subseteq N/\{n\}$ such that $\sum_{i \in S} w_i = q-1$ (2).

(1) we have just to show that $w_1 \geq w_2$ and $w_2 \geq 1$. We have $w_1 \geq w_2$ if $w-m-n+1 \geq m+1$ or $m \leq \frac{w-n}{2}$. We know that $m < 2n-5$, thus $w_1 \geq w_2$ if $2n-5 < \frac{w-n}{2}$ or $w \geq 5n-10$. This is true by hypothesis.

(2) remark first that $w_1 + w_3 + w_4 ... + w_{n-1} = q-2$. Thus the players 1 and 2 must belong to the coalition $S \subseteq N/\{n\}$ if we want $\sum_{i \in S} w_i = q-1$. But $w_1 + w_2 = w - n + 2$ or $w_1 + w_2 = w - (n-2)$. By hypothesis, $m \geq n-2$, thus we can write $w_1 + w_2 \geq w - m$ or $w_1 + w_2 = q$. It is not possible to obtain a total weight equal to $q-1$. QED

Proof of Lemma 4 Consider the following game $[q; w-(n-m-1)(m+1)-m, m+1, ...m+1, 1..., 1]$ knowing that the number of players with a weight equal to $m+1$ is equal to $n-m-1$ and the number of players with a weight equal to 1 is equal to m. We have to show that $q > w_1 \geq w_2 \geq ... \geq w_n$ (1) and since $w_n = 1$ we have to show that there is no coalition $S \subseteq N/\{n\}$ such that $\sum_{i \in S} w_i = q-1$ (2).

(1) we show first that $w_1 < q$. Actually, this is the case if $w-(n-m-1)(m+1)-m < w-m$ or $(n-m-1)(m+1) > 0$. This inequality is true if $n-1 > m$ which is our hypothesis ($m \leq n-2$). We have to show now that $w_1 \geq w_2$. This is the case if $w-(n-m-1)(m+1)-m \geq m+1$ or $w > (m+1)n - m^2$ which is our hypothesis.

(2) Remark that $\sum_{i \in S} w_i = q$ with $S = \{1, 2, ..., n-m\}$. This sum becomes equal to $q-(m+1)$ if one individual (different from the player 1) leaves the coalition S. Since there are $m-1$ players different from the player n with a weight equal to 1, it is not possible to obtain a total weight equal to $q-1$ if the player 1 belongs to S. Remark now that $\sum_{i \in S} w_i = (n-m-1)(m+1)+m-1$ with $S = \{2, ..., n-1\}$. To obtain $\sum_{i \in S} w_i = q-1$, we must have $(n-m-1)(m+1)+m-1 = q-1$ or $(n-m-1)(m+1)+m-1 = w-m-1$. We must have $w = (n-m-1)(m+1)+2m$. A short manipulation gives us $w = n(m+1)-m^2-2$ which is not possible by hypothesis. QED

Proof of Lemma 5 Observe first that 3 is the only player who can be a dummy: if 2 **and** 3 are dummies, then the player 1 is a dictator, in contradiction with our assumptions. Consider the possibly winning coalitions to which the player 3 is susceptible to belong:{1,3}, {1,2} and {1,2,3}. The player 3 is a dummy if either these coalitions are losing, or they are winning and remain winning when the player 3 is removed, *i.e.* if and only if: ($w_1 + w_3 < q$ or ($w_1 + w_3 \geq q$ and $w_1 \geq q$)) and ($w_2 + w_3 < q$ or ($w_2 + w_3 \geq q$ and $w_2 \geq q$)) and ($w_1 + w_2 + w_3 < q$ or ($w_1 + w_2 + w_3 \geq q$ and $w_1 + w_2 \geq q$)). Given that the grand coalition is always winning and a coalition with only one player cannot be winning (no dictator), these inequalities reduce to: $w_1 + w_3 < q$ and $w_2 + w_3 < q$ and $w_1 + w_2 \geq q$. We obtain the desired result by noticing that $w_1 + w_3 < q$ implies $w_2 + w_3 < q$ (recall that $w_2 \leq w_1$). QED

Proof of Lemma 6 Notice that the only possible dummies are the players 3 and 4 and if 3 is a dummy, so is 4. Consider first the player 4. The possibly winning coalitions to which the player 4 can belong are {1,4}, {2,4}, {3,4}, {1,2,4}, {1,3,4}, {2,3,4} and {1,2,3,4}. Proceeding as in the proof of Lemma 5, and after eliminating the redundant inequalities, we obtain that the player 4 is a dummy if and only if: $w_1 + w_4 < q$ and ($w_1 + w_2 + w_4 < q$ or $w_1 + w_2 \geq q$) and ($w_1 + w_3 + w_4 < q$ or $w_1 + w_3 \geq q$) and ($w_2 + w_3 + w_4 < q$ or $w_2 + w_3 \geq q$) and $w_1 + w_2 + w_3 \geq q$. This set of inequalities can be rewrite in the following way:

($w_1 + w_4 < q$ and $w_1 + w_2 + w_4 < q$ and $w_1 + w_3 + w_4 < q$ and $w_2 + w_3 + w_4 < q$ and $w_1 + w_2 + w_3 \geq q$), (a)

or

($w_1 + w_4 < q$ and $w_1 + w_2 + w_4 < q$ and $w_1 + w_3 + w_4 < q$ and $w_2 + w_3 \geq q$ and $w_1 + w_2 + w_3 \geq q$), (b)

or

($w_1 + w_4 < q$ and $w_1 + w_2 + w_4 < q$ and $w_1 + w_3 \geq q$ and $w_2 + w_3 + w_4 < q$ and $w_1 + w_2 + w_3 \geq q$), (c)

or

($w_1 + w_4 < q$ and $w_1 + w_2 + w_4 < q$ and $w_1 + w_3 \geq q$ and $w_2 + w_3 \geq q$ and $w_1 + w_2 + w_3 \geq q$), (d)

or

($w_1 + w_4 < q$ and $w_1 + w_2 \geq q$ and $w_1 + w_3 + w_4 < q$ and $w_2 + w_3 + w_4 < q$ and $w_1 + w_2 + w_3 \geq q$), (e)

or

($w_1 + w_4 < q$ and $w_1 + w_2 \geq q$ and $w_1 + w_3 + w_4 < q$ and $w_2 + w_3 \geq q$ and $w_1 + w_2 + w_3 \geq q$), (f)

or

($w_1 + w_4 < q$ and $w_1 + w_2 \geq q$ and $w_1 + w_3 \geq q$ and $w_2 + w_3 + w_4 < q$ and $w_1 + w_2 + w_3 \geq q$), (g)

or

($w_1 + w_4 < q$ and $w_1 + w_2 \geq q$ and $w_1 + w_3 \geq q$ and $w_2 + w_3 \geq q$ and $w_1 + w_2 + w_3 \geq q$). (h)

It is easy to check that (b), (c), (d) and (f) contain contradictory inequalities and can be ruled out. Moreover, (a), (e), (g) and (h) can be reduced to:

$w_1 + w_2 + w_4 < q$ and $w_1 + w_2 + w_3 \geq q$, (case 1)

or $w_1 + w_3 + w_4 < q$ and $w_1 + w_2 \geq q$, (case 2)

or $w_2 + w_3 + w_4 < q$ and $w_1 + w_3 \geq q$ and $w_1 + w_4 < q$, (case 3)

or $w_2 + w_3 \geq q$, (case 4)

in accordance with Lemma 6. Consider now the player 3. 3 is not a dummy in case

1: coalition {1,2,3} is winning and coalition {1,2} is losing ($w_1 + w_2 + w_4 < q$ implies $w_1 + w_2 < q$). We obtain the same conclusion for cases 3 and 4. By contrast, the player 3 has no power in case 2 and is a dummy: in this case, the player 3 is not pivotal in the winning coalition to which he (she) belongs ({1,2,3} and {1,2,3,4}) because $w_1 + w_2 \geq q$. QED