The impact of voters' preference diversity on the probability of some electoral outcomes*

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Abstract

Voting rules are known to exhibit various paradoxical or problematical behaviors, typically the failure to meet the Condorcet criterion or the vulnerability to strategic voting. Our basic premise is that the fewer the number of coalitions of voters that exist with similar preference rankings is, the less is the propensity of voting rules to yield undesired results. Surprisingly enough, the results reported by Felsenthal et al. (1990) in an early study do not corroborate this intuition. We reconsider and extend in the present paper the Felsenthal et al. analysis by using a modified Impartial Anonymous Culture (IAC) model. It turns out that the results obtained with this probabilistic assumption are much more consistent with our intuitive premise.

* The authors are indebted to Dan Felsenthal for bringing this area of research to their attention and for very helpful comments on an earlier draft of this paper. However, all conclusions that are drawn in this study should of course only be attributed to the authors.
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1. Introduction

1.1. Preliminary

A significant area of research has evolved around the basic idea of assessing the probability that various outcomes might be observed in elections with three candidates $\{A, B, C\}$ for n voters, where $A \succ B$ denotes that an individual voter prefers Candidate *A* to Candidate *B*. Assumptions are always made regarding the types of preferences that individual voters might have on candidates. It is usually assumed that individual voter indifference between candidates is not allowed, so that either $A \succ B$ or $B \succ A$ for all *A* and *B*. It is almost universally assumed that intransitive voter preferences, such as $\succ B$, $B \succ C$ and $C \succ A$, are prohibited as a requirement of individual rationality. These same assumptions are made in the current study, so there are therefore only six remaining possible complete preference rankings that each voter might have on the candidates, as shown in Fig.1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig.1. Possible individual voter preference rankings on three candidates.

Here, n_i denotes the number of voters who have the complete preferences on the candidates that are in agreement with the associated i^{th} preference ranking. There are for example n_3 voters with a preference ranking that has *B* being most preferred, *C* being least preferred and *A* being ranked in the middle between *B* and *C*. Let a voting situation be a six-dimensional vector of such n_i terms for which $\sum_{i=1}^{6} n_i = n$.

Studies that examine the probability that various voting outcomes might be observed typically include analyses of outcomes that are based on Pairwise Majority Rule (PMR). Candidate *A* will beat Candidate *B* by PMR, which we denote as ***AMB***, when more voters have $A \succ B$ than $B \succ A$. That is, ***AMB*** whenever $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$. Candidate *A* will be the Strict PMR Winner (Strict PMRW), or Condorcet Winner, if both ***AMB*** and ***AMC*** with no PMR ties. If some candidate is the Strict PMRW in a given voting situation, then that candidate would certainly be deemed to be a good candidate to represent the overall most preferred candidate according to the preference rankings of the electorate.

A PMRW does not always exist, so that it is possible that a voting situation can result in a cyclical PMR relationship like *AMB*, *BMC* and *CMA*. Such an outcome is defined as an occurrence of Condorcet's Paradox, such that the PMR comparisons result in an intransitive relationship on pairs of candidates while each of the individual voters is prohibited from having such intransitive preferences. Given that such cyclic PMR relationships can occur, the Condorcet Criterion states that the PMRW should always be selected as the winner of an election whenever such a winner exists. Most commonly used voting rules cannot always meet the Condorcet Criterion; so in keeping with the intent of this criterion, the Condorcet Efficiency of a voting rule is the conditional probability that it will select the PMRW, given that such a winner exists.

Felsenthal et al. (1990) conducted a study to determine the likelihood that some electoral outcomes might be observed in three-candidate elections. Among other things, they considered the probability that a PMRW will exist and the Condorcet Efficiency of Plurality Rule, which is the commonly used voting rule in which each voter casts a vote for their most preferred candidate. The probability that any election outcome might be observed will clearly be driven by the likelihood that various voting situations will be observed, and they use an assumption that is unique to studies of this type. In particular, they developed definitions of the various categories of voting situations that might exist in terms of the general size relationships between the n_i terms when $k = \#K$, with $K = \{i: n_i > 0, for\ 1 \leq i \leq 6\}$. That is, k is the number of non-zero n_i terms that correspond to the number of coalitions of voters that exist with similar preference rankings.

If $k \leq 2$, then some candidate is never ranked first in the preferences of any voter, some candidate is never ranked last by any voter, and some candidate is never middle-ranked by any voter. It is widely known that a Strict PMRW must exist under any of these conditions for three-candidate elections when *n* is odd (see Black, 1958, Vickery, 1960 and Ward, 1965). It follows that Condorcet's Paradox can only exist if at least one Latin Square triple is contained in K for $k \geq 3$, with $\{1,4,5\} \subseteq K$ or $\{2,3,6\} \subseteq K$. There are 20 triples of possible rankings with $k = 3$, and only the two Latin Square triples will ever allow the possible existence of the Paradox at all, while the remaining 18/20 will never allow it. The Paradox probability will be nonzero for

$k = 3$, but it will be very small unless there is some very unusual argument to explain why the 18/20 possible combinations of three rankings cannot be observed. When we go to $k = 5$, one of the Latin Square triples of rankings must be included in the allowable set of rankings, to suggest an increased likelihood of observing the paradox. When we go to $k = 6$, both of the Latin Square triples are allowable, to give the most possible options for introducing cycles.

Our basic premise generalizes this observation, by suggesting that allowing the entry of additional ranking types into voting situations should tend to increase the number of 'dimensions' or 'degrees of freedom' to make it easier to create a PMR cycle, or any other paradoxical result, from voters' preference rankings. Felsenthal et al. (1990) perform a very insightful analysis that can be used to test this intuitive premise with a unique assumption that is described in the following sub-section.

1.2. Felsenthal-Maoz-Rapoport Model

Suppose that k = 3, with $K = \{x, y, z\}$ and assume without loss of generality that $n_x \geq n_y \geq n_z$. There are three possible configurations of the relative size values for these three non-zero n_i terms for which $n_x > n_y > n_z$, with:

Configuration 1	Configuration 2	Configuration 3
$n_x > n_y > n_z$	$n_x > n_y > n_z$	$n_x > n_y > n_z$
$n_x > n_y + n_z$	$n_x = n_y + n_z$	$n_x < n_y + n_z$

For each of these three configurations there are $\binom{6}{3} = 20$ combinations of three of the six possible n_i terms that could be non-zero for k = 3, and there are 3! = 6 possible $n_x > n_y > n_z$ rankings for each of these combinations. The total number of possible voting situation scenarios for these three configurations is therefore given by 3*20*6 = 360.

When k = 3, there are four possible configurations of n_i terms with $n_x \geq n_y \geq n_z$ that have exactly one equality term:

Configuration 4	Configuration 5
$n_x > n_y = n_z$	$n_x > n_y = n_z$
$n_x > n_y + n_z$	$n_x = n_y + n_z$

Configuration 6

$n_x > n_y = n_z$
$n_x < n_y + n_z$

Configuration 7

$n_x = n_y > n_z$
$n_x < n_y + n_z$

For each of these four configurations there are 20 combinations of three of the six possible n_i terms that could be non-zero, as above. But, the presence of the equality now requires that there are only three possible rankings for each of these combinations. The total number of possible voting situation scenarios for these four configurations is therefore given by 4*20*3 = 240.

When k = 3, there is only one possible configuration of n_i terms with $n_x \geq n_y \geq n_z$ that has two equality terms.

Configuration 8

$n_x = n_y = n_z$
$n_x < n_y + n_z$

For this configuration, there are 20 combinations of three of the six possible n_i terms that could be non-zero, and the presence of the two equalities now requires that there is only one possible ranking for each of these combinations. The total number of possible voting situation scenarios for this configuration is therefore given by 1*20*1 = 20. As shown in Table 1 for *k* = 3, there are therefore a total of eight possible configurations with 620 possible voting situation scenarios that are associated with them.

k	Configurations	Scenarios	Scenarios with a Strict PMRW	$P^S_{PMRW}(3, k, ELS)$
3	8	620	504	.8129
4	88	22,785	19,002	.8340
5	4,672	2,898,666	2,498,034	.8618

Table 1. Number of voting configurations and scenarios for *k* = 3,4,5.

Felsenthal et al. (1990) continue the same type of analysis to obtain the number of possible configurations and scenarios for *k* = 4 and *k* = 5, and the results are listed in Table 1. Results could not be obtained for the case with *k* = 6 due to the complexity of the problem. One specific combination of numerical values of n_i is then found to meet the criteria of each scenario, and computer enumeration was then used to determine the number of these possible scenarios for which a Strict PMRW exists for *k* = 3,4,5. The results are summarized in Table 1.

Let ELS denote the condition for which each of the specified scenarios is equally likely to be observed for a specified k. Then, $P^S_{PMRW}(3, k, ELS)$ defines the probability that a Strict PMRW exists for three candidates with the ELS condition for a specified k. Then, $P^S_{PMRW}(3,3, ELS) = \frac{504}{620} = .8129$, along with the associated results for k = 4,5 in Table 1. Given the earlier observation that it must be true that $P^S_{PMRW}(3,1, ELS) = P^S_{PMRW}(3,2, ELS) = 1$ for odd n, consistent behavior of $P^S_{PMRW}(3, k, ELS)$ as k increases would require that $P^S_{PMRW}(3, k, ELS) \geq P^S_{PMRW}(3, k+1, ELS)$ over the interval $1 \leq k \leq 4$, and this clearly is not the case. Some additional observations are also made regarding the relative values of the Condorcet Efficiency of some voting rules and regarding the impact of strategic voting.

It is noted in Felsenthal et al. (1990) that their reported results might be sensitive to the ELS Assumption, since there can be many more possible combinations of specific n_i values that meet the restrictions of some scenarios than others, while their study only attributed one specific assignment of possible n_i values to each possible scenario. The objective of this current study is to reconsider and extend the analysis that was performed by Felsenthal et al. (1990) to somewhat account for this factor, leading to results that are much more consistent with our intuitive premise.

We propose in Section 2 an alternative probabilistic model based on the often used notion of Impartial Anonymous Culture and we compute with the help of this model the probability that a Strict PMRW exists for k = 1, 2,…, 6. Section 3 is devoted to the Condorcet Efficiency of three well known voting rules (including Plurality) and we consider in Section 4 the vulnerability of these voting rules to strategic manipulation by coalitions of voters. Our main conclusions are given in Section 5.

2. A Modified Impartial Anonymous Culture Model

The Impartial Anonymous Culture (IAC) Model is more general than the Felsenthal-Maoz-Rapoport Model and it does more to account for the different number of possible voting situations that can be associated with each of the scenarios that were defined above.

The total number of unique voting situations that can exist with $n_i \geq 0$ for n voters with a specified number m of possible voter preference rankings on three candidates is $\binom{n+m-1}{m-1}$. It is assumed with IAC that every one of these possible voting situations that might exist with $m = 6$, as in Fig. 1, is equally likely to be observed. This assumption has been used in numerous studies to obtain representations for the probability that voting outcomes might be observed.

For example, Gehrlein and Fishburn (1976) use the IAC assumption to develop a representation for the probability $P^S_{PMRW}(3,n,IAC)$ that a Strict PMRW exists for three candidates with odd n:

$$P^S_{PMRW}(3,n,IAC) = \frac{15(n+3)^2}{16(n+2)(n+4)}, \text{ for odd } n. \tag{1}$$

Lepelley (1989) developed a similar representation for even n, with

$$P^S_{PMRW}(3,n,IAC) = \frac{15n(n+2)(n+4)}{16(n+1)(n+3)(n+5)}, \text{ for even } n. \tag{2}$$

Computed values of $P^S_{PMRW}(3,n,IAC)$ are listed in Table 2 for odd $n = 1(2)29$, along with $n = 39, 49, \infty$, and in Table 3 for even $n = 2(2)50$. It is observed that $P^S_{PMRW}(3,n,IAC)$ decreases as odd n increases, while it increases as even n increases. So, different patterns are present for odd and even n, and the rate of convergence to the limiting value as $n \to \infty$ is very slow for even n while it is quite rapid for odd n.

A number of algorithms have been developed to obtain IAC-based probability representations of this type. See for example: Huang and Chua (2000), Gehrlein (2002), Wilson and Pritchard (2007), Lepelley et al (2008), Mbih et al (2009). The IAC assumption is refined in this current study to consider voting situations with a specified value of parameter k, as defined above.

Let $N_{VS}(n,k)$ denote the number of possible voting situations with a specified value of k for n voters on three candidates. Obviously, we require that $n \geq k$. There are $\binom{6}{k}$ possible combinations of the six rankings that could have $\#K = k$. If one voter is arbitrarily assigned to each of the k rankings to require that $n_i \geq 1$ for these rankings, there are then $\binom{n-1}{k-1}$ combinations of ways of assigning the remaining $n-k$ voters to these k different rankings. It follows directly that

$$N_{VS}(n,k) = \binom{6}{k} \times \binom{n-1}{k-1}. \quad (3)$$

The IAC(k) Assumption specifies that each of the $N_{VS}(n,k)$ voting situations is equally likely to be observed. It is possible to develop representations for $P_{PMRW}^{S}(3,n,IAC(k))$ quite easily for small values of k. If $k = 1$, every voter must have the same preference ranking, so it is obvious that $P_{PMRW}^{S}(3,n,IAC(1)) = 1$. For larger values of k, we start by calculating the number of voting situations $N_{PMRW}^{A}(n,k)$ for which Candidate A is the strict PMRW when $\#K = k$.

For the case with $k = 2$, let voting situation type T denote the doublet $\{x,y\}$ of integers with $1 \leq x < y \leq 6$ for which $n_x, n_y \geq 1$, with $n_x + n_y = n$. The preference rankings in Fig. 1 indicate that A will be the most preferred candidate for every voter when $T = \{1,2\}$, and there are $n-1$ different ways in which $n_1 + n_2 = n$ for either odd or even n since n_1 takes values from 1 to $n-1$. Voting situations of type $T = \{1,x\}$ with $3 \leq x \leq 6$ will have A as the strict PMRW for the $\frac{n-2}{2}$ voting situations in which $\frac{n+2}{2} \leq n_1 \leq n-1$ for even n, and for the $\frac{n-1}{2}$ voting situations in which $\frac{n+1}{2} \leq n_1 \leq n-1$ for odd n. The same arguments apply for voting situations of type $T = \{2,x\}$ with $3 \leq x \leq 6$. The results in Fig. 1 indicate that A is never the most preferred candidate in voting situation of type $T = \{x,y\}$ when $3 \leq x < y \leq 6$, so it can never be the PMRW in these cases. All of these results are summarized in Fig. 2 to lead to a representation for $N_{PMRW}^{A}(n,2)$.

Types (T)	$N_{PMRW}^{A}(n,2)$	
	Even n	Odd n
{1,2}	$n-1$	$n-1$
{1,3},{1,4},{1,5} {1,6},{2,3} {2,4},{2,5},{2,6}	$\frac{n-2}{2}$	$\frac{n-1}{2}$
{3,4},{3,5},{3,6} {4,5},{4,6},{5,6}	0	0

Fig. 2. Possible voting situation types with k = 2 for computing $N_{PMRW}^{A}(n,2)$.

It is simple to obtain a representation for $N^A_{PMRW}(n,2)$ from the entries of Fig. 2 as

$$N^A_{PMRW}(n,2) = 1(n-1) + 8\left(\frac{n-2}{2}\right) + 6(0) = 5n-9, \text{for even } n \geq 2. \quad (4)$$

$$N^A_{PMRW}(n,2) = 1(n-1) + 8\left(\frac{n-1}{2}\right) + 6(0) = 5(n-1), \text{for odd } n. \quad (5)$$

By the symmetry of the IAC(k) Assumption with respect to candidates, it is obvious that $N^A_{PMRW}(n,k)$ = $N^B_{PMRW}(n,k)$ = $N^C_{PMRW}(n,k)$ for all k, and it follows that

$$P^S_{PMRW}\big(3,n,IAC(k)\big) = \frac{3N^A_{PMRW}(n,k)}{N_{VS}(n,k)}. \quad (6)$$

Eqs. (6) and (3) can now be used with Eqs. (4) and (5) respectively to obtain

$$P^S_{PMRW}\big(3,n,IAC(2)\big) = \frac{5n-9}{5(n-1)}, \text{for even } n \geq 2. \quad (7)$$

$$P^S_{PMRW}\big(3,n,IAC(2)\big) = 1, \text{for odd } n \geq 2. \quad (8)$$

The same general logic can then be extended to the case with $k = 3$, with the results being summarized in Fig. 3.

Types (T)	$N^A_{PMRW}(n,3)$ Even n	$N^A_{PMRW}(n,3)$ Odd n
{1,2,3}, {1,2,4}	$\frac{(3n-4)(n-2)}{8}$	$\frac{(3n-5)(n-1)}{8}$
{1,3,4},{1,5,6},{1,3,6} {1,4,6},{2,4,6}, {1,4,5}, {2,4,5} {2,3,4},{2,5,6},{2,3,6}	$\frac{(n-4)(n-2)}{8}$	$\frac{(n-3)(n-1)}{8}$
{1,3,5},{2,3,5}	$\frac{(n-2)^2}{4}$	$\frac{(n-1)^2}{4}$
{4,5,6},{3,5,6} {3,4,6},{3,4,5}	0	0

Fig. 3. Possible voting situation types with k = 3 for computing $N^A_{PMRW}(n,3)$.

Following the analysis that led to Eqs. (7) and (8) from Fig. 2, the results in Fig. 3 give

$$P^S_{PMRW}\big(3,n,IAC(3)\big) = \frac{39n-96}{40(n-1)}, \text{for even } n \geq 4 \quad (9)$$

$P^S_{PMRW}(3, n, IAC(3)) = \frac{39n-81}{40(n-2)}$, for odd $n \geq 3$. (10)

The probability representations for $P^S_{PMRW}(3, n, IAC(k))$ for $k > 3$ get more complicated and have been obtained directly from the use of the computer algorithms we have mentioned above, with

$P^S_{PMRW}(3, n, IAC(4)) = \frac{19n^2-115n+168}{20(n-1)(n-3)}$, for even $n \geq 4$ (11)

$P^S_{PMRW}(3, n, IAC(4)) = \frac{19n-41}{20(n-2)}$, for odd $n \geq 5$. (12)

$P^S_{PMRW}(3, n, IAC(5)) = \frac{15n^2-98n+144}{16(n-1)(n-3)}$, for even $n \geq 6$ (13)

$P^S_{PMRW}(3, n, IAC(5)) = \frac{(3n-7)(5n-19)}{16(n-2)(n-4)}$, for odd $n \geq 5$ (14)

$P^S_{PMRW}(3, n, IAC(6)) = \frac{15(n-2)(n-4)(n-6)}{16(n-1)(n-3)(n-5)}$, for even $n \geq 6$ (15)

$P^S_{PMRW}(3, n, IAC(6)) = \frac{15(n-3)^2}{16(n-2)(n-4)}$, for odd $n \geq 7$ (16)

Based on these representations, it is easily shown that

$P^S_{PMRW}(3, n, IAC(k)) > P^S_{PMRW}(3, n, IAC(k+1))$ for $1 \leq k \leq 5$, for finite even $n \geq 6$

$P^S_{PMRW}(3, n, IAC(k)) > P^S_{PMRW}(3, n, IAC(k+1))$ for $1 \leq k \leq 4$,
and $P^S_{PMRW}(3, n, IAC(6)) > P^S_{PMRW}(3, n, IAC(5))$, for finite odd $n \geq 11$.

However, in the limit $n \to \infty$, $P^S_{PMRW}(3, \infty, IAC(5)) = P^S_{PMRW}(3, \infty, IAC(6))$.

These results indicate that $P^S_{PMRW}(3, n, IAC(k))$ does decrease as k increases for even $n \geq 6$, following intuition. It is also observed that $P^S_{PMRW}(3, n, IAC(k))$ decreases as k increases for $k \leq 5$ for odd $n \geq 11$, but this is not true in the transition from $P^S_{PMRW}(3, \infty, IAC(5))$ to $P^S_{PMRW}(3, \infty, IAC(6))$, except in the limit $n \to \infty$ where

$P^S_{PMRW}(3, \infty, IAC(5)) = P^S_{PMRW}(3, \infty, IAC(6))$.

The intuitive result is therefore only valid in the limiting case of $n \to \infty$ for odd n.

It is clearly of interest to observe the probability values that are obtained from these representations. Computed values of $P^S_{PMRW}(3, n, IAC(k))$ are listed in Table 2 for odd $n = 1(2)29$, along with $n = 39, 49, \infty$.

n	IAC	k					
		6	5	4	3	2	1
1	1.000						1.000
3	0.964				0.900	1.000	1.000
5	0.952		1.000	0.900	0.950	1.000	1.000
7	0.947	1.000	0.933	0.920	0.960	1.000	1.000
9	0.944	0.964	0.929	0.929	0.964	1.000	1.000
11	0.942	0.952	0.929	0.933	0.967	1.000	1.000
13	0.941	0.947	0.929	0.936	0.968	1.000	1.000
15	0.940	0.944	0.930	0.938	0.969	1.000	1.000
17	0.940	0.942	0.931	0.940	0.970	1.000	1.000
19	0.939	0.941	0.931	0.941	0.971	1.000	1.000
21	0.939	0.940	0.932	0.942	0.971	1.000	1.000
23	0.939	0.940	0.932	0.943	0.971	1.000	1.000
25	0.939	0.939	0.933	0.943	0.972	1.000	1.000
27	0.939	0.939	0.933	0.944	0.972	1.000	1.000
29	0.938	0.939	0.933	0.944	0.972	1.000	1.000
39	0.938	0.938	0.934	0.946	0.973	1.000	1.000
49	0.938	0.938	0.935	0.947	0.973	1.000	1.000
∞	15/16	15/16	15/16	19/20	39/40	1.000	1.000

Table 2. Computed values of $P_{PMRW}^{S}\big(3, n, IAC(k)\big)$ for odd *n*.

The results of Table 2 indicate that $P_{PMRW}^{S}\big(3, n, IAC(k)\big)$ converges to the limiting value as $n \to \infty$ very quickly as *n* increases.

Computed values of $P_{PMRW}^{S}\big(3, n, IAC(k)\big)$ are listed in Table 3 for even $n = 2(2)50$. The computed values in Table 3 indicate that the rate of convergence to the limiting values of $P_{PMRW}^{S}\big(3, \infty, IAC(k)\big)$ is extremely slow for even *n*.

By extending this analysis with the $IAC(k)$ assumption, much more support is observed for the notion that $P_{PMRW}^{S}\big(3, \infty, IAC(k)\big)$ should tend to decrease as *k* increases than that indicated in the earlier work of Felsenthal et al. (1990). This is particularly true for the limiting case as $n \to \infty$.

At this point, we turn our attention to a closer examination of the differences between the ELS and Modified IAC Assumptions. It was stated above that IAC does more to account for the different number of possible voting situations that can be associated with each of the scenarios that were defined for ELS. We illustrate this for the special case with $k = 3$.

		k					
n	*IAC*	6	5	4	3	2	1
2	0.429					0.200	1.000
4	0.571			0.200	0.500	0.733	1.000
6	0.649	0.000	0.400	0.540	0.690	0.840	1.000
8	0.699	0.429	0.571	0.663	0.771	0.886	1.000
10	0.734	0.571	0.659	0.729	0.817	0.911	1.000
12	0.760	0.649	0.712	0.770	0.845	0.927	1.000
14	0.780	0.699	0.748	0.798	0.865	0.938	1.000
16	0.796	0.734	0.774	0.818	0.880	0.947	1.000
18	0.809	0.760	0.794	0.834	0.891	0.953	1.000
20	0.820	0.780	0.810	0.846	0.900	0.958	1.000
22	0.829	0.796	0.822	0.856	0.907	0.962	1.000
24	0.837	0.809	0.832	0.865	0.913	0.965	1.000
26	0.844	0.820	0.841	0.871	0.918	0.968	1.000
28	0.849	0.829	0.848	0.877	0.922	0.970	1.000
30	0.855	0.837	0.854	0.882	0.926	0.972	1.000
32	0.859	0.844	0.860	0.887	0.929	0.974	1.000
34	0.863	0.849	0.865	0.891	0.932	0.976	1.000
36	0.867	0.855	0.869	0.894	0.934	0.977	1.000
38	0.870	0.859	0.873	0.897	0.936	0.978	1.000
40	0.874	0.863	0.876	0.900	0.938	0.979	1.000
42	0.876	0.867	0.879	0.902	0.940	0.980	1.000
44	0.879	0.870	0.882	0.904	0.942	0.981	1.000
46	0.881	0.874	0.884	0.907	0.943	0.982	1.000
48	0.883	0.876	0.887	0.908	0.945	0.983	1.000
50	0.885	0.879	0.889	0.910	0.946	0.984	1.000
∞	15/16	15/16	15/16	19/20	39/40	1.000	1.000

Table 3. Computed values of $P^{S}_{PMRW}\big(3, n, IAC(k)\big)$ for even *n*.

The IAC results are dependent upon *n*, so results were obtained by computer enumeration to get the number of voting situations that result in each of the eight possible ELS configurations for each *n* = 15, 50, 120. The results are summarized in Table 4, along with the number of ELS scenarios for each of the configurations from Table 2.

The first observation is that there are very significant differences between the number of scenarios in each category with ELS and the number of voting situations in each category with IAC. For *n* = 15 (odd *n* in general) that it is not possible ever to observe any voting situations that meet the restrictions of Configurations 2 and 5. For *n* = 50 (even *n* not a multiple of 4 or 12) it is not possible to observe any voting situations that meet the restrictions of Configurations 5 and 8.

Configuration	ELS Scenarios	IAC(n=15) VS	IAC(n=50) VS	IAC(n=120) VS
1	120	1080	15840	100920
2	120	0	1440	3480
3	120	360	4800	32520
4	60	180	720	1740
5	60	0	0	60
6	60	60	240	540
7	60	120	480	1140
8	20	20	0	20
Total	620	1820	23520	140420

Table 4. Computer enumeration comparison of ELS and IAC with $k = 3$.

To equate IAC outcome probabilities with 50 voters to ELS, with an equal number of scenarios for Configurations 1, 2 and 3, it would be necessary for example to come up with a plausible explanation as to why 15,840 – 1,440 =14,400 voting situations in the Configuration 1 category cannot be observed, and 4,800 – 1,400 = 3,400 voting situations in the Configuration 3 category cannot be observed, while every one of the 1,440 voting situations in Configuration 2 are feasible.

Another concern with ELS arises when elections are considered with large electorates as $n \to \infty$, since the probability of observing any ELS Scenarios in Configurations 2, 4, 5, 6, 7 or 8 would all be of measure zero under any reasonable set of assumptions about voters' preferences since they all require tied outcomes for some n_i terms. This outcome is reflected in the IAC values above with n = 120, but it will never be reflected in the ELS Scenario count. If we want to make any comparisons about what happens for small electorates versus large electorates, or even odd versus even numbers of voters, something like IAC must be called on for a comparison. We now continue by extending this analysis to consider other aspects of election procedures.

3. Condorcet Efficiency of Voting Rules

It was noted above that the Condorcet Criterion suggests that the PMRW is a good candidate to represent the most preferred candidate for an electorate, so that it should be selected as the winner whenever such a candidate exists. However, the voting rules that are in common use will not always select the PMRW. The Condorcet Efficiency of a voting rule VR is therefore defined as the conditional probability that VR will select the Strict PMRW, given that such a Strict PMRW exists. When there are three candidates and n voters with the assumption of $IAC(k)$, the Condorcet Efficiency of VR is denoted by $CE^{VR}(3, n, IAC(k))$.

Plurality Rule (PR) is the most commonly used voting rule in which each voter casts a vote for their most preferred candidate and the winner is selected as the candidate receiving the most votes. We use the same algorithms as above to obtain representations for $CE^{PR}(3,n,IAC(k))$, where it is required that the candidate is both the Strict PMRW and the strict winner by PR, so there are no tied votes allowed in the determination of the PR winner. Unfortunately, this representation becomes quite complex since it is found to have a periodicity of six as n increases.

The relevant representations for $CE^{PR}(3,n,IAC(k))$ for each $2 \leq k \leq 6$ with $n \geq k$ are shown here for the sequence with $n = 3,9,15,21$ ..., which we denote as $n = 3(6)$ ....

$$CE^{PR}(3,n,IAC(6)) = \frac{119n^4-1508n^3+6926n^2-12252n+6075}{135(n-1)(n-5)(n-3)^2}, \text{for } n = 9(6) \ldots$$

$$CE^{PR}(3,n,IAC(5)) = \frac{119n^3-955n^2+2169n-1269}{9(n-1)(5n-19)(3n-7)}, \text{for } n = 9(6) \ldots$$

$$CE^{PR}(3,n,IAC(4)) = \frac{4(13n^2-48n+33)}{3(n-1)(19n-41)}, \text{for } n = 9(6) \ldots$$

$$CE^{PR}(3,n,IAC(3)) = \frac{37n-87}{3(13n-27)}, \text{for } n = 3(6) \ldots$$

$$CE^{PR}(3,n,IAC(2)) = 1, \text{for } n = 3(6) \ldots$$

The representations for each of the other sequences of n with periodicity six are listed in the Appendix. All of these representations were used to obtain the computed values of $CE^{PR}(3,n,IAC(k))$ that are listed in Table 5. These representations can also be used to formally verify the observation from Table 5:

$$CE^{PR}(3,n,IAC(k)) > CE^{PR}(3,n,IAC(k+1)) \text{ for } 2 \leq k \leq 4,$$
$$\text{and } CE^{PR}(3,n,IAC(6)) > CE^{PR}(3,n,IAC(5)), \text{ for finite } n \geq 7.$$

Just as in the limiting case of the probability that a PMRW exists as $n \to \infty$, we also find that $CE^{PR}(3,\infty,IAC(5)) = CE^{PR}(3,\infty,IAC(6))$.

As k increases with $n \to \infty$, there is strong evidence that $CE^{PR}(3,n,IAC(k))$ tends to decrease. Differences in patterns of change for $CE^{PR}(3,n,IAC(k))$ as n changes continue to exist with odd and even n, as they did with the IAC assumption. But, as $n \to \infty$, it can be concluded that increasing the number of possible preference ranking types that voters might have in voting situations creates scenarios that increase the expected propensity of PR to become confounded and fail to elect the PMRW when such a winner exists. However, the Condorcet Efficiency of PR always remains above .88 in the limiting case with $n \to \infty$.

	k				
n	2	3	4	5	6
2	1.000	----	----	----	----
3	1.000	0.667	----	----	----
4	1.000	1.000	1.000	----	----
5	1.000	0.842	0.667	0.000	----
6	1.000	0.957	0.963	1.000	----
7	1.000	0.896	0.804	0.643	1.000
8	1.000	0.944	0.922	0.900	1.000
9	1.000	0.911	0.838	0.738	0.889
10	1.000	0.949	0.925	0.904	1.000
11	1.000	0.917	0.850	0.764	0.825
20	1.000	0.947	0.914	0.885	0.914
21	1.000	0.935	0.886	0.837	0.856
30	1.000	0.948	0.914	0.885	0.903
31	1.000	0.940	0.895	0.853	0.863
40	1.000	0.948	0.913	0.884	0.896
41	1.000	0.942	0.899	0.860	0.866
50	1.000	0.948	0.913	0.883	0.892
51	1.000	0.943	0.902	0.865	0.870
60	1.000	0.948	0.913	0.883	0.890
61	1.000	0.944	0.904	0.868	0.872
70	1.000	0.948	0.913	0.882	0.889
71	1.000	0.945	0.905	0.870	0.873
80	1.000	0.948	0.913	0.882	0.888
81	1.000	0.945	0.906	0.871	0.874
90	1.000	0.948	0.913	0.882	0.887
91	1.000	0.946	0.907	0.872	0.875
100	1.000	0.948	0.913	0.882	0.887
101	1.000	0.946	0.907	0.873	0.876
∞	1.000	0.949	0.912	0.881	0.881

Table 5. Computed values of $CE^{PR}\big(3, n, IAC(k)\big)$.

We continue our analysis by considering the Condorcet Efficiency of other voting rules that are based on the notion of a Weighted Scoring Rule (WSR). A WSR is defined as a set of three weights $(1, \lambda, 0)$ in a three-candidate election. A candidate receives a score of one point for each time it is ranked as most preferred by a voter and λ points for each time it is middle-ranked by a voter. No points are assigned to candidates for being least preferred in a voter's ranking of candidates. The winner is the candidate that receives the greatest total number of points from voters. Our earlier analysis of PR was consistent with using a WSR with $\lambda = 0$.

Borda Rule (BR), which uses $\lambda = 1/2$, is another WSR that has received significant attention in the literature. Representations were obtained for $CE^{BR}\big(3, n, IAC(k)\big)$, with periodicity six, for each $2 \leq k \leq 6$. The results are listed in the Appendix, and we restrict our attention to the limiting probabilities as $n \to \infty$. The corresponding values of $CE^{BR}(3, \infty, IAC(k))$ are listed in Table 6 along with values of $CE^{PR}(3, \infty, IAC(k))$.

	VR		
k	*PR*	*BR*	*NPR*
2	1	14/15 = .9333	2/5 = .4000
3	37/39 = .9487	107/117 = .9145	22/39 = .5641
4	52/57 = .9123	52/57 = .9123	137/228 = .6009
5	119/135 = .8815	41/45 = .9111	17/27 = .6296
6	119/135 = .8815	41/45 = .9111	17/27 = .6296

Table 6. Limiting values of $CE^{VR}(3, \infty, IAC(k))$ for $VR \in \{PR, BR, NPR\}$.

Just as observed with PR, $CE^{BR}(3, \infty, IAC(k)) > CE^{BR}(3, \infty, IAC(k+1))$ for $2 \leq k \leq 4$, and $CE^{BR}(3, \infty, IAC(5)) = CE^{BR}(3, \infty, IAC(6))$. So, increasing the number of possible preference ranking types that voters might have in voting situations increases the expected likelihood for BR to fail to elect the PMRW when such a winner exists. The Condorcet Efficiency of BR does however always remain above .91 in the limiting case with $n \to \infty$.

It is very interesting to note that $CE^{PR}(3, \infty, IAC(4)) = CE^{BR}(3, \infty, IAC(4))$. Voting situations with $k = 2{,}3$ reflect scenarios in which voters' preferences are highly restricted, in the sense that a majority of possible preference rankings are not feasible to represent the preferences that any voter in the electorate might have. In these scenarios, PR performs better than BR, and no advantage is obtained by giving some weight to voters' middle-ranked candidates when attempting to elect the PMRW. Voting situations with $k = 4{,}5$ reflect converse scenarios in which voters' preferences are much more diverse, in the sense that a majority of all possible preference rankings ***are*** feasible options to represent the preferences that any voter in the electorate might have. In these diverse scenarios, BR performs better than PR on the basis of Condorcet Efficiency, so an advantage is obtained by giving some weight to the voters' middle-ranked candidates when seeking the PMRW.

This analysis is continued by considering another WSR with $\lambda = 1$ that is known as Negative Plurality Rule (NPR). Representations were obtained for $CE^{NPR}(3, n, IAC(k))$, with periodicity twelve, for each $2 \leq k \leq 6$. The results are listed in the Appendix, and the limiting values of $CE^{NPR}(3, \infty, IAC(k))$ are listed in Table 6. Results that are completely contrary to intuition are observed with NPR, with $CE^{NPR}(3, \infty, IAC(k)) < CE^{NPR}(3, \infty, IAC(k+1))$ for $2 \leq k \leq 4$, and $CE^{NPR}(3, \infty, IAC(5)) = CE^{NPR}(3, \infty, IAC(6))$. Thus, increasing k does not reduce the Condorcet Efficiency of all voting rules. The Condorcet Efficiency of NPR is also completely dominated by both *PR* and *BR* for all $2 \leq k \leq 6$, so the application of too great a weight to middle-ranked candidates can significantly confound a WSR in the pursuit of electing the

PMRW. The potential use of NPR is therefore strongly discouraged on the criterion of Condorcet Efficiency.

4. Strategic Manipulation

A voting rule is subject to strategic manipulation if it is possible that voting situations might exist for which some subset of voters could misrepresent their actual preferences in an election to thereby obtain an outcome that they prefer to the outcome that would have resulted if they had voted sincerely. It is widely known from the work of Gibbard (1973) and Satterthwaite (1975) that effectively all voting rules can be strategically manipulated, so we are left with comparing voting rule based on the relative probability that they might be manipulated by a coalition of voters. Felsenthal et al. (1990) perform some analysis of this type with their assumption of ELS. We extend that work with the assumption of $IAC(k)$ to determine if increasing the number of possible preference ranking types that voters might have on candidates, by increasing *k*, increases the expected probability that PR, BR and NPR are subject to strategic manipulation by a coalition of voters.

Let $MP^{VR}(3, n, IAC(k))$ denote the probability that voting rule VR can be manipulated by a coalition of voters in a three-candidate election with n voters under the $IAC(k)$ assumption. Representations for $MP^{VR}(3, n, IAC(k))$ were developed for PR, with periodicity six, and for NPR, with periodicity three. These results are presented in the Appendix[1]. The representation for $MP^{BR}(3, n, IAC(k))$ was found to be extremely complex, with periodicity 126. As a result, we have restricted our attention to the limiting representation for $MP^{BR}(3, \infty, IAC(k))$, and the associated values are presented in Table 7 for each $2 \leq k \leq 6$. The limiting manipulation probability values for PR and NPR are also listed in Table 7.

	VR		
k	*PR*	*BR*	*NPR*
2	0	1/5 = .2000	4/15 = .2667
3	3/40 = .0750	229/840 = .2726	13/30 = .4333
4	26/180 = .1444	2302/6615 = .3480	62/135 = .4593
5	31/144 = .2153	133751/317520 = .4212	13/27 = .4815
6	21/72 = .2917	132953/264600 = .5025	14/27 = .5185

Table 7. Limiting values of $MP^{VR}(3, \infty, IAC(k))$ for $VR \in \{PR, BR, NPR\}$.

[1] All the representations in this section are based upon preliminary characterizations of those voting situations where manipulation is possible under PR, NPR and BR. These characterizations can be found in Lepelley and Mbih (1987) for PR, in Lepelley and Mbih (1994) for NPR and in Favardin et al (2002) or Wilson and Pritchard (2007) for BR. It is also worth noticing that we assume that the coalitions of voters who manipulate may use ranking types that do not belong to *K*, the set of ranking types under consideration.

The results of Table 7 clearly show that $MP^{VR}(3, \infty, IAC(k))$ increases as k increases for each of PR, BR and NPR, exactly as intuition suggests. Moreover, PR dominates BR on the basis of manipulation probability since $MP^{PR}(3, \infty, IAC(k)) < MP^{BR}(3, \infty, IAC(k))$ for each $2 \leq k \leq 6$, and BR in turn dominates NPR.

5. Conclusions

By extending the seminal ELS Assumption based analysis of Felsenthal et al. (1990) with the alternative assumption of $IAC(k)$, we somewhat account for the fact that there can be many more possible combinations of specific n_i values that meet the restrictions of some voting scenarios than others, while the ELS Assumption based analysis consistently attributes only one specific assignment of possible n_i values to each possible voting scenario.

When attention is focused on limiting results with large electorates with three candidates, very strong evidence is provided to support the intuitive notion that increasing the number of possible preference ranking types that voters might have on candidates will decrease the probability that a PMRW might exist and increase the probability that strategic manipulation by a coalition of voters can occur in voting rules. This provides some limited evidence to support the concept that voting situations that are more likely to have a PMRW should be less likely to be subject to strategic manipulation.

Results from our analysis of the Condorcet Efficiency of PR, BR and NPR are mixed. PR and BR have Condorcet Efficiency values that decrease as the number of possible preference rankings that voters might have on candidates increases, as expected. However, the reverse situation surprisingly occurs with NPR.

Voting situations in which voters' preferences are highly restricted, with a majority of possible preference rankings not being feasible to represent any voter's preferences, have PR outperforming BR on the basis of Condorcet Efficiency. The reverse is true for voting situations that reflect scenarios in which voters' preferences are much more diverse, with a majority of all possible preference rankings being feasible to represent the preferences of any voter. These diverse scenarios therefore provide some Condorcet Efficiency advantage to voting rules that give some weight to the voters' middle-ranked candidates. However, the NPR results indicate that providing too much weight to voters' middle-ranked candidates can confound a WSR in its attempt to determine the PMRW. NPR is dominated by both PR and BR on the basis of both Condorcet Efficiency and the probability that it can be strategically manipulated, to indicate that it is a very poor voting rule for consideration of implementation.

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Appendix

$$CE^{PR}\big(3,n,IAC(6)\big)=\frac{119n^4-1746n^3+9704n^2-23646n+20465}{135(n-5)(n-3)^3}, \text{for } n=7(6)\,...$$

$$=\frac{119n^4-1842n^3+10376n^2-25728n+23360}{135(n-6)(n-2)(n-4)^2}, \text{for } n=8(6)\,...$$

$$=\frac{119n^4-1508n^3+6926n^2-12252n+6075}{135(n-1)(n-5)(n-3)^2}, \text{for } n=9(6)\,...$$

$$=\frac{119n^3-1366n^2+5072n-7200}{135(n-6)(n-2)(n-4)}, \text{for } n=10(6)\,...$$

$$=\frac{119n^4-1270n^3+4940n^2-7530n+3421}{135(n-1)(n-3)^3}, \text{for } n=11(6)\,...$$

$$=\frac{119n^4-1366n^3+6024n^2-12096n+8640}{135(n-2)^2(n-4)^2}, \text{for } n=12(6)\,...$$

$$CE^{PR}\big(3,n,IAC(5)\big)=\frac{(n-5)\big(119n^3-717n^2+1353n-691\big)}{9(n-1)(n-3)(5n-19)(3n-7)}, \text{for } n=5(6)\,...$$

$$=\frac{119n^3-1008n^2+2772n-2592}{9(n-2)(15n^2-98n+144)}, \text{for } n=6(6)\,...$$

$$=\frac{119n^3-1193n^2+3841n-4063}{9(n-3)(5n-19)(3n-7)}, \text{for } n=7(6)\,...$$

$$=\frac{119n^2-770n+1136}{9(15n^2-98n+144)}, \text{for } n=8(6)\,...$$

$$=\frac{119n^3-955n^2+2169n-1269}{9(n-1)(5n-19)(3n-7)}, \text{for } n=9(6)\,...$$

$$=\frac{119n^3-1008n^2+2772n-2720}{9(n-2)(15n^2-98n+144)}, \text{for } n=10(6)\,...$$

$$CE^{PR}\big(3,n,IAC(4)\big)=\frac{52n^3-417n^2+1110n-1024}{3(n-2)(19n^2-115n+168)}, \text{for } n=4(6)\,...$$

$$=\frac{4\big(13n^3-87n^2+171n-89\big)}{3(n-1)(n-3)(19n-41)}, \text{for } n=5(6)\,...$$

$= \frac{52n^3-417n^2+1110n-1008}{3(n-2)(19n^2-115n+168)}$, for $n = 6(6) \dots$

$= \frac{4(13n^2-74n+103)}{3(n-3)(19n-41)}$, for $n = 7(6) \dots$

$= \frac{52n^2-313n+460}{3(19n^2-115n+168)}$, for $n = 8(6) \dots$

$= \frac{4(13n^2-48n+33)}{3(n-1)(19n-41)}$, for $n = 9(6) \dots$

$CE^{PR}(3, n, IAC(3)) = \frac{37n-87}{3(13n-27)}$, for $n = 3(6) \dots$ and $n = 7(6) \dots$

$= \frac{37n^2-166n+192}{3(n-2)(13n-32)}$, for $n = 4(6) \dots$ and $n = 6(6) \dots$

$= \frac{37n^2-124n+79}{3(n-1)(13n-27)}$, for $n = 5(6) \dots$

$= \frac{37n-92}{3(13n-32)}$, for $n = 8(6) \dots$

$CE^{PR}(3, n, IAC(2)) = 1$ for $n \geq 2$

$CE^{BR}(3, n, IAC(6)) = \frac{123n^4-1782n^3+9628n^2-22682n+20185}{135(n-5)(n-3)^3}$, for $n = 7(6) \dots$

$= \frac{123n^4-1884n^3+10252n^2-23776n+20800}{135(n-2)(n-6)(n-4)^2}$, for $n = 8(6) \dots$

$= \frac{123n^4-1536n^3+6802n^2-11904n+7155}{135(n-1)(n-5)(n-3)^2}$, for $n = 9(6) \dots$

$= \frac{123n^4-1638n^3+7468n^2-14408n+12640}{135(n-4)(n-6)(n-2)^2}$, for $n = 10(6) \dots$

$= \frac{123n^4-1290n^3+4690n^2-7510n+3397}{135(n-1)(n-3)^3}$, for $n = 11(6) \dots$

$= \frac{123n^4-1392n^3+5668n^2-10272n+8640}{135(n-2)^2(n-4)^2}$, for $n = 12(6) \dots$

$CE^{BR}(3, n, IAC(5)) = \frac{41n^3-309n^2+711n-379}{3(n-1)(3n-7)(5n-19)}$, for $n = 5(6) \dots$

$= \frac{41n^4-492n^3+2124n^2-4080n+3456}{3(n-2)(n-4)(15n^2-98n+144)}$, for $n = 6(6) \dots$

$= \frac{41n^2-268n+443}{3(3n-7)(5n-19)}$, for $n = 7(6) \dots$

$= \frac{41n^3-410n^2+1304n-1344}{3(n-4)(15n^2-98n+144)}$, for $n = 8(6) \ldots$

$= \frac{41n^3-309n^2+711n-507}{3(n-1)(3n-7)(5n-19)}$, for $n = 9(6) \ldots$

$= \frac{41n^3-328n^2+812n-768}{3(n-2)(15n^2-98n+144)}$, for $n = 10(6) \ldots$

$CE^{BR}\big(3, n, IAC(4)\big) = \frac{52n^3-399n^2+990n-832}{3(n-2)(19n^2-115n+168)}$, for $n = 4(6) \ldots$

$= \frac{4\left(13n^3-84n^2+171n-92\right)}{3(n-1)(n-3)(19n-41)}$, for $n = 5(6) \ldots$

$= \frac{52n^3-399n^2+1014n-1008}{3(n-2)(19n^2-115n+168)}$, for $n = 6(6) \ldots$

$= \frac{4\left(13n^2-71n+100\right)}{3(n-3)(19n-41)}$, for $n = 7(6) \ldots$

$= \frac{52n^2-295n+400}{3(19n^2-115n+168)}$, for $n = 8(6) \ldots$

$= \frac{4\left(13n^2-45n+42\right)}{3(n-1)(19n-41)}$, for $n = 9(6) \ldots$

$CE^{BR}\big(3, n, IAC(3)\big) = \frac{107n^2-360n+333}{9(n-1)(13n-27)}$, for $n = 3(6) \ldots$

$= \frac{107n^2-454n+464}{9(n-2)(13n-32)}$, for $n = 4(6) \ldots$

$= \frac{107n^2-356n+257}{9(n-1)(13n-27)}$, for $n = 5(6) \ldots$

$= \frac{107n^2-462n+576}{9(n-2)(13n-32)}$, for $n = 6(6) \ldots$

$= \frac{107n-245}{9(13n-27)}$, for $n = 7(6) \ldots$

$= \frac{107n-244}{9(13n-32)}$, for $n = 8(6) \ldots$

$CE^{BR}\big(3, n, IAC(2)\big) = \frac{14n-25}{3(5n-9)}$, for $n = 2(6) \ldots$

$= \frac{2(7n-9)}{15(n-1)}$, for $n = 3(6) \ldots$

$= \frac{14n-23}{3(5n-9)}$, for $n = 4(6) \ldots$

$= \frac{2(7n-8)}{15(n-1)}$, for $n = 5(6)$ ...

$= \frac{14n-27}{3(5n-9)}$, for $n = 6(6)$ ...

$= \frac{14}{15}$, for $n = 7(6)$ ...

$CE^{NPR}\big(3, n, IAC(6)\big) = \frac{(n-7)(68n^4-655n^3+2403n^2-4013n+2845)}{108(n-1)(n-5)(n-3)^3}$, for $n = 7(12)$ ...

$= \frac{68n^5-1279n^4+9184n^3-32368n^2+58496n-46592}{108(n-6)(n-2)^2(n-4)^2}$, for $n = 8(12)$ ...

$= \frac{68n^5-1131n^4+7116n^3-20826n^2+27432n-10611}{108(n-1)(n-5)(n-3)^3}$, for $n = 9(12)$ ...

$= \frac{68n^5-1279n^4+9056n^3-30056n^2+46144n-24176}{108(n-6)(n-2)^2(n-4)^2}$, for $n = 10(12)$ ...

$= \frac{68n^3-723n^2+2166n-2227}{108(n-1)(n-3)(n-5)}$, for $n = 11(12)$ ...

$= \frac{68n^5-1279n^4+9184n^3-32112n^2+56448n-41472}{108(n-6)(n-2)^2(n-4)^2}$, for $n = 12(12)$ ...

$= \frac{68n^4-1063n^3+5925n^2-14261n+12787}{108(n-5)(n-3)^3}$, for $n = 13(12)$ ...

$= \frac{68n^4-1143n^3+6898n^2-17924n+17464}{108(n-2)(n-6)(n-4)^2}$, for $n = 14(12)$ ...

$= \frac{68n^3-723n^2+2166n-1971}{108(n-1)(n-3)(n-5)}$, for $n = 15(12)$ ...

$= \frac{68n^3-735n^2+2088n-2240}{108(n-6)(n-2)^2}$, for $n = 16(12)$ ...

$= \frac{68n^4-791n^3+3161n^2-5277n+2583}{108(n-1)(n-3)^3}$, for $n = 17(12)$ ...

$= \frac{68n^4-871n^3+3958n^2-7716n+4968}{108(n-2)^2(n-4)^2}$, for $n = 18(12)$ ...

$CE^{NPR}\big(3, n, IAC(5)\big) = \frac{85n^4-933n^3+3861n^2-6671n+3786}{9(n-1)(n-3)(3n-7)(5n-19)}$, for $n = 5(12)$ ...

$= \frac{(5n-18)\big(17n^3-155n^2+492n-468\big)}{9(n-2)(n-4)(15n^2-98n+144)}$, for $n = 6(12)$ ...

$= \frac{85n^3-678n^2+1785n-1516}{9(n-1)(3n-7)(5n-19)}$, for $n = 7(12)$ …

$= \frac{85n^3-741n^2+2340n-2720}{9(n-2)(15n^2-98n+144)}$, for $n = 8(12)$ …

$= \frac{85n^4-933n^3+3861n^2-6543n+3402}{9(n-1)(n-3)(3n-7)(5n-19)}$, for $n = 9(12)$ …

$= \frac{85n^4-1081n^3+5154n^2-10492n+7144}{9(n-2)(n-4)(15n^2-98n+144)}$, for $n = 10(12)$ …

$= \frac{85n^3-678n^2+1881n-1676}{9(n-1)(3n-7)(5n-19)}$, for $n = 11(12)$ …

$= \frac{85n^3-741n^2+2340n-2592}{9(n-2)(15n^2-98n+144)}$, for $n = 12(12)$ …

$= \frac{85n^3-848n^2+2917n-3306}{9(n-3)(3n-7)(5n-19)}$, for $n = 13(12)$ …

$= \frac{85n^3-911n^2+3428n-4468}{9(n-4)(15n^2-98n+144)}$, for $n = 14(12)$ …

$= \frac{85n^3-678n^2+1881n-1548}{9(n-1)(3n-7)(5n-19)}$, for $n = 15(12)$ …

$= \frac{85n^3-741n^2+2244n-2722}{9(n-2)(15n^2-98n+144)}$, for $n = 16(12)$ …

$CE^{NPR}\left(3, n, IAC(4)\right) = \frac{137n^3-1158n^2+3288n-3104}{12(n-2)(19n^2-115n+168)}$, for $n = 4(12)$ …

$= \frac{137n^3-963n^2+2139n-1441}{12(n-1)(n-3)(19n-41)}$, for $n = 5(12)$ …

$= \frac{137n^3-1158n^2+3276n-2952}{12(n-2)(19n^2-115n+168)}$, for $n = 6(12)$ …

$= \frac{137n^3-963n^2+2151n-1613}{12(n-1)(n-3)(19n-41)}$, for $n = 7(12)$ …

$= \frac{137n^3-1158n^2+3384n-3520}{12(n-2)(19n^2-115n+168)}$, for $n = 8(12)$ …

$= \frac{137n^3-963n^2+2139n-1377}{12(n-1)(n-3)(19n-41)}$, for $n = 9(12)$ …

$= \frac{137n^3-1158n^2+3180n-2600}{12(n-2)(19n^2-115n+168)}$, for $n = 10(12)$ …

$= \frac{137n^3-963n^2+2247n-1837}{12(n-1)(n-3)(19n-41)}$, for $n = 11(12)$ …

$= \frac{137n^3-1158n^2+3384n-3456}{12(n-2)(19n^2-115n+168)}$, for $n = 12(12)$ ...

$= \frac{137n^2-826n+1217}{12(n-3)(19n-41)}$, for $n = 13(12)$ ...

$= \frac{137n^2-884n+1508}{12(19n^2-115n+168)}$, for $n = 14(12)$ ...

$= \frac{137n^2-552n+591}{12(n-1)(19n-41)}$, for $n = 15(12)$ ...

$CE^{NPR}\big(3, n, IAC(3)\big) = \frac{2(11n^2-37n+36)}{3(n-1)(13n-27)}$, for $n = 3(12)$ ... and $n = 11(12)$ ...

$= \frac{2(11n^2-49n+56)}{3(n-2)(13n-32)}, for\ n = 4(12)$ ...

$= \frac{2(n-2)(11n-15)}{3(n-1)(13n-27)}$, for $n = 5(12)$ ... and $n = 9(12)$ ...

$= \frac{2(11n-27)}{3(13n-32)}$, for $n = 6(12)$ ... and $n = 14(12)$ ...

$= \frac{2(11n^2-37n+32)}{3(n-1)(13n-27)}$, for $n = 7(12)$ ...

$= \frac{2(11n^2-49n+60)}{3(n-2)(13n-32)}$, for $n = 8(12)$ ... and $n = 12(12)$

$= \frac{2(11n^2-49n+50)}{3(n-2)(13n-32)}$, for $n = 10(12)$ ...

$= \frac{2(11n-26)}{3(13n-27)}$, for $n = 13(12)$ ...

$= \frac{2(11n-27)}{3(13n-32)}$, for $n = 14(12)$ ...

$CE^{NPR}\big(3, n, IAC(2)\big) = \frac{2n-3}{5n-9}$, for $n = 2(2)$ ...

$= \frac{2}{5}$, for $n = 3(2)$ ...

$MP^{PR}\big(3, n, IAC(6)\big) = \frac{(n-6)(21n^4-299n^3+1926n^2-5184n+4320)}{72(n-1)(n-2)(n-3)(n-4)(n-5)}, for\ n = 6(6)$ ...

$= \frac{(n-7)(21n^2-194n+125)}{72(n-2)(n-4)(n-5)}$, for $n = 7(6)$ ...

$$= \frac{(n-8)(21n^3-215n^2+914n-1280)}{72(n-1)(n-3)(n-4)(n-5)}, \text{for } n = 8(6) \dots$$

$$= \frac{(n-9)(21n^3-173n^2+267n+45)}{72(n-1)(n-2)(n-4)(n-5)}, \text{for } n = 9(6) \dots$$

$$= \frac{21n^4-341n^3+2356n^2-7316n+7440}{72(n-1)(n-2)(n-3)(n-5)}, \text{for } n = 10(6) \dots$$

$$= \frac{(n-11)(n+1)(3n-11)(7n-11)}{72(n-1)(n-2)(n-3)(n-4)}, \text{for } n = 11(6) \dots$$

$$MP^{PR}(3, n, IAC(5)) = \frac{(n-5)(n+1)(31n^2-284n+525)}{144(n-1)(n-2)(n-3)(n-4)}, \text{for } n = 5(6) \dots$$

$$= \frac{(n-6)(31n^3-330n^2+1368n-1728)}{144(n-1)(n-2)(n-3)(n-4)}, \text{for } n = 6(6) \dots$$

$$= \frac{31n^3-377n^2+1513n-1599}{144(n-2)(n-3)(n-4)}, \text{for } n = 7(6) \dots$$

$$= \frac{(n-8)(31n^2-206n+408)}{144(n-1)(n-3)(n-4)}, \text{for } n = 8(6) \dots$$

$$= \frac{31n^3-315n^2+945n-405}{144(n-1)(n-2)(n-4)}, \text{for } n = 9(6) \dots$$

$$= \frac{31n^3-392n^2+1780n-2688}{144(n-1)(n-2)(n-3)}, \text{for } n = 10(6) \dots$$

$$MP^{PR}(3, n, IAC(4)) = \frac{(n-4)(26n^2-241n+566)}{180(n-1)(n-2)(n-3)}, \text{for } n = 4(6)$$

$$= \frac{(n-5)(n+1)(13n-53)}{90(n-1)(n-2)(n-3)}, \text{for } n = 5(6)$$

$$= \frac{(n-6)(26n^2-189n+396)}{180(n-1)(n-2)(n-3)}, \text{for } n = 6(6)$$

$$= \frac{(13n^2-92n+187)}{90(n-2)(n-3)}, \text{for } n = 7(6)$$

$$= \frac{(n-8)(26n-85)}{180(n-1)(n-3)}, \text{for } n = 8(6)$$

$$= \frac{(n-3)(13n-27)}{90(n-1)(n-2)}, \text{for } n = 9(6)$$

$$MP^{PR}(3, n, IAC(3)) = \frac{3(n-3)}{40(n-2)}, \text{for } n = 3(6) \dots \text{and } n = 7(6)..$$

$$= \frac{3(n-4)(n-6)}{40(n-1)(n-2)}, \text{for } n = 4(6) \dots \text{ and } n = 6(6) \dots$$

$= \frac{3(n+1)(n-5)}{40(n-1)(n-2)}$, for $n = 5(6) \dots$

$= \frac{3(n-8)}{40(n-1)}$, for $n = 8(6) \dots$

$MP^{PR}\big(3, n, IAC(2)\big) = 0$, for $n \geq 2$

$MP^{NPR}\big(3, n, IAC(6)\big) = \frac{2(n-6)(n-9)(7n^2-39n+60)}{27(n-1)(n-2)(n-4)(n-5)}$, for $n = 6(3) \dots$

$= \frac{2(n-7)(n-10)(7n-31)}{27(n-2)(n-3)(n-5)}$, for $n = 7(3) \dots$

$= \frac{2(n-8)(7n^2-40n+73)}{27(n-1)(n-3)(n-4)}$, for $n = 8(3) \dots$

$MP^{NPR}\big(3, n, IAC(5)\big) = \frac{(n-5)(13n^2-103n+208)}{27(n-1)(n-3)(n-4)}$, for $n = 5(3) \dots$

$= \frac{(n-6)(13n^2-105n+180)}{27(n-1)(n-2)(n-4)}$, for $n = 6(3) \dots$

$= \frac{(n-7)(13n^2-107n+130)}{27(n-1)(n-2)(n-3)}$, for $n = 7(3) \dots$

$MP^{NPR}\big(3, n, IAC(4)\big) = \frac{2(n-4)(n-7)(31n-67)}{135(n-1)(n-2)(n-3)}$, for $n = 4(3)$

$= \frac{2(n-5)(31n-95)}{135(n-1)(n-3)}$, for $n = 5(3)$

$= \frac{2(n-6)(31n-81)}{135(n-1)(n-2)}$, for $n = 6(3)$

$MP^{NPR}\big(3, n, IAC(3)\big) = \frac{(n-3)(13n-42)}{30(n-1)(n-2)}$, for $n = 3(3)$

$= \frac{(n-4)(13n-43)}{30(n-1)(n-2)}$, for $n = 4(3)$

$= \frac{13n-41}{30(n-1)}$, for $n = 5(3)$

$MP^{NPR}\big(3, n, IAC(2)\big) = \frac{4(n-2)}{15(n-1)}$, for $n = 2(3)$

$= \frac{4(n-3)}{15(n-1)}$, for $n = 3(3)$

$= \frac{4(n-4)}{15(n-1)}$, for $n = 4(3)$