On the Likelihood of Dummy Players in Weighted Majority Games

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Abstract

When the number of players is small in a weighted majority voting game, it can occur that one of the players has no influence on the result of the vote, in spite of a strictly positive weight. Such a player is called a "dummy" player in game theory. The purpose of this paper is to investigate the conditions that give rise to such a phenomenon and to compute its likelihood. It is shown that the probability of having a dummy player is surprisingly high and some paradoxical results are observed.

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1 Introduction

The main teaching of the literature on power indices is that, in a collective choice process, voting power or influence need not to be proportional to the relative number of votes (*weight*) an individual or a group (*player*) is entitled to. An extreme and striking consequence of this non proportionality is that a player can have a positive weight but never be a member of a minimal winning coalition (a coalition that wins and the removal of a single player does not allow the coalition to win any longer). Such players have no voting power and are known as *dummies*.

The most famous example of this somewhat paradoxical phenomenon is offered by Luxembourg in the Council of Ministers of the EU between 1958 and 1973. Luxembourg held one vote, whereas the quota for a proposition to be approved was 12 out of 17. Since other member states held an even number of votes (4 for Germany, France and Italy, 2 for Belgium and The Netherlands), Luxembourg formally was never able to make any difference in the voting process and was a dummy.

Another well known case of dummies involves Nassau County, New York (Banzhaf, 1965). Nassau County's government took the form of a Board of Supervisors, one representative for each of various municipalities, who cast a block of votes. Here are the weighted voting systems used at various times by Nassau County. The passing quota shown reflects the number of votes needed to pass "ordinary legislation".

	1958	1964
Hempstead 1	9	31
Hempstead 2	9	31
North Hempstead	7	21
Oyster Bay	3	28
Long Beach	1	2
Glen Cove	1	2
Total votes	30	115
Quota	16	58

The numerical weights were chosen to try to take into account the populations of the different municipalities, which were quite disparate. It is easy to see that in 1958, Oyster Bay, Long Beach and Glen Cove were dummies. It can also be checked that, in 1964, there were three dummies (North Hempstead, Glen Cove and Long Beach). After 1964, the quota was raised to guarantee that no municipality was a dummy.

A third example of dummy has recently been discovered by one of the authors (see Blancard and Lepelley, 2010) in a community of municipalities in La Réunion Island (France). This community, called CIVIS (**C**ommunauté **I**ntercommunale des **VI**lles **S**olidaires), gathered between 1997 and 2008 five municipalities: Saint-Pierre (15 representatives in the community council), Saint-Louis (10 representatives), L'Etang-Salé (5), Petite-Ile (5) and Cilaos (4), the number of representatives being roughly proportional to the municipality population. In the community

Council, 20 votes were necessary for a proposition to be accepted. If we suppose that the representatives of a municipality were voting as a block, it can be seen that Cilaos was a dummy: all the winning coalitions containing Cilaos remain winning when this municipality is removed. It is worth noticing that in 2008 a sixth municipality has entered the community and Cilaos is no more a dummy.

The possibility of dummy players is clearly problematic from a democratic point of view and the diversity of the examples given above suggests that the occurrence of dummies in voting games is of practical concern and could be less rare that expected in first analysis. What is the likelihood of such an undesired phenomenon ? How the distribution of weights should be arranged in order to avoid the occurrence of dummies in voting games?

We propose in this paper a theoretical investigation of these issues in the context of weighted *majority* games, where the quota is equal to the half of the total number of votes, plus one. Our framework and our main assumptions are introduced in Section 2. We propose some analytical results in Section 3 for weighted voting games with 4, 5 and 6 players: in each case, we characterize the distributions of weights giving rise to the occurrence of the "dummy paradox" and deduce from these characterizations some representations for the likelihood of the paradox as a function of the total number of votes. Section 4 proposes both exact and estimated numerical results for the likelihood of dummy players for more than 6 players. Our results are discussed in Section 5, where we study the impact of a reduction of the weight scattering on the probability of having some dummies. Section 6 concludes the paper.

2 Framework and assumptions

We will adopt the following notation:

Let m be the number of players, with $m \geq 3$. The players are denoted by $J1$, $J2$, ..., Jm.

Let n_i be the *weight* of player i and $n = \sum_i n_i$. Hence, n_i can be interpreted as the number of votes assigned to a member Ji of a voting body. Notice however that, when the players are parties in a political assembly, the n_i's correspond to the number of representatives of each party and n is the total number of votes in the assembly.

As mentioned above, we only consider in the present study Weighted *Majority* Games (WMG): a proposition is adopted if and only if the total weight of the players in favor of this proposition is greater or equal to $n/2 + 1$ if n is even and to $(n + 1)/2$ if n is odd. In what follows, this majority quota will be denoted by $Q = [n/2]^+$, where $[x]^+$ is the smallest integer strictly higher than x. So, a coalition S is *winning* if and only if $\sum_{i \in S} n_i \geq Q$; otherwise, the coalition is said to be *loosing*. A player Ji is a *dummy* if and only if, for each winning coalition S including Ji, $S - \{Ji\}$ is still winning.

Our main assumptions are the following:

(1) the n_i's are integer,

(2) $n/2 \geq n_1 \geq n_2 \geq ... \geq n_m \geq 1$,

(3) m and n being given, all the distributions of the n_i's verifying (1), (2) and $n = \sum_i n_i$ are equally likely to occur.

The two first assumptions are rather innocuous. The third one is reminiscent of the Impartial Anonymous Culture condition often used in voting theory to compute the probability of various voting events: all the admissible "voting situations" are considered as equiprobable. This assumption generally allows to obtain close form representations for the probabilities we are interested in (see *e.g.* Gehrlein, 2002 ; Wilson and Pritchard, 2007).
Notice that this framework fits well with the (recent) French local entities called EPCI (Etablissement Public de Cooperation Intercommunale) where each municipality belonging to the EPCI is given a number of delegates approximately proportionate to its number of inhabitants[1]. In this context, n_1 is the number of delegates of the biggest municipality in the EPCI council, n_m the number of delegates of the smallest, and n is the total number of delegates in the EPCI council (we suppose that, in this council, the delegates of a given municipality vote as a block). Of course, the biggest municipality should not be a dictator ($n_1 \leq n/2$) and the smallest one should obtain at least one delegate. In the EPCI council, the current decisions are taken with a quota $Q = [n/2]^+$.

3 Some analytical results

We begin our analysis with a preliminary result, which is true whatever the number of players.

Proposition 1 *In a m-player WMG, (i) the maximum number of possible dummies is equal to $m-3$ and (ii) the number of dummies is exactly $m-3$ if and only if $n_2 + n_3 \geq Q$.*

Proof. In order to prove (i), we have to show that $J3$ cannot be a dummy in a m-player majority game, $m \geq 3$. Suppose the contrary: $J3$ is a dummy. A first consequence is that $J4, J5, ..., Jm$ are also dummies. Furthermore, the coalition $\{J1, J3\}$ is loosing (if this coalition was winning, the fact that $J3$ is a dummy would imply that $n_1 > Q$, contradicting our assumptions). Now, if $\{J1, J3\}$ is loosing, then $\{J1, J3, J4\}$ is also loosing since $J4$ is a dummy. Similarly, as $J5$ is a dummy, the coalition $\{J1, J3, J4, J5\}$ is loosing and we can set in the same way that $\{J1, J3, J4, ..., Jm\}$ is loosing, which implies $n_1 + n_3 + n_4 + n_5 + ... + n_m < Q$. As $\sum_i n_i = n$, we would have $n_2 \geq Q - 1$ and thus $n_2 + n_3 \geq Q$ which is not possible since $J3$ is a dummy.
Consider now assertion (ii) and suppose that $n_2 + n_3 \geq Q$. Let's show this implies that $J4$ is a dummy. Consider the winning coalitions including $J4$. As $n_2 + n_3 \geq Q$ implies $n_1 + n_4 + n_5 + ... + n_m < Q$, it can be observed that both the coalition $\{J4, J5, ..., Jm\}$ and the coalition $\{J1, J4\}$ are loosing. It follows from this observation that the only winning coalitions with $J4$ must include two players among $\{J1, J2, J3\}$. As $n_2 + n_3 \geq Q$ and $n_1 \geq n_2 \geq n_3$, we have $n_1 + n_3 \geq Q$ and $n_1 + n_2 \geq Q$. Consequently, the defection of $J4$ in these coalitions lets them winning and $J4$ is a dummy.

[1]The CIVIS we have mentioned in the Introduction is an example of EPCI.

Finally, suppose that $J4$ is a dummy. This implies that the coalition $\{J1, J4\}$ is loosing (if not, $J4$ dummy would imply $n_1 \geq Q$, a contradiction) and thus $n_1 < Q$ or $n_1 \leq Q - 1$. As $J5, J6, ..., Jm$ are (also) dummies and $n_i > 0$, it follows that $n_1 + n_4 + n_5 + ... + n_m < Q - (m-3)$ (with $m \geq 4$). This implies that $n_2 + n_3 \geq Q$. □

Two straightforward corollaries are the following.

Corollary 1 *There is no dummy player in a 3-player WMG.*

Corollary 2 *In a 4-player WMG, J4 is a dummy player if and only if $n_2 + n_3 \geq Q$.*

Notice that Corollary 1 is only true when WMG's are under consideration; if the quota differs from $Q = [n/2]^+$, then it is easy to check that a dummy can exist in a 3-player weighted voting game (see *e.g.* Leech, 2002).

Corollary 2 gives a complete (and simple) characterization of the admissible distributions of weights that give rise to a dummy in 4-player WMG's. For instance, for $n = 13$ (and $Q = 7$), $J4$ is a dummy with the weight distribution given by the vector $(n_1, n_2, n_3, n_4) = (4, 4, 3, 2)$, but not with the distribution $(6, 4, 2, 1)$.

For 5-player and 6-player WMG's, the characterizations are more involved, as shown in the following Proposition.

Proposition 2 *(i) In a 5-player WMG, J5 is a dummy player if and only if one of the following cases holds:*

- case 1: $n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_4 \geq Q$;

- case 2: $n_2 + n_3 \geq Q$.

In case 2 (and only in this case), both J4 and J5 are dummy players.

(ii) In a 6-player WMG, J6 is a dummy player if and only if one of the following cases holds:

- case 1: $n_2 + n_3 + n_4 + n_5 \geq Q$ and $n_1 + n_5 \geq Q$;

- case 2: $n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_4 \geq Q$;

- case 3: $n_2 + n_3 + n_5 \geq Q$ and $n_1 + n_4 + n_5 \geq Q$ and $n_1 + n_3 \geq Q$;

- case 4: $n_2 + n_3 \geq Q$;

- case 5: $n_2 + n_4 + n_5 \geq Q$ and $n_1 + n_2 \geq Q$;

- case 6: $n_3 + n_4 + n_5 \geq Q$.

In case 2, J5 and J6 are dummy players; in case 4, J4, J5 and J6 are dummy players.

The proof of this proposition is given in Appendix.

Corollary 2 and Proposition 2 allow us to enumerate the distributions of the weights that give rise to dummy players and to compute the probability of their occurrence in m-player WMG's, with $m \in \{4, 5, 6\}$. Moreover, it is possible to derive from Corollary 2 and Proposition 2 some representations for this probability as a function of n, the total number of votes. This probability is denoted by $P(m, n)$ in what follows.

Proposition 3 *For $n \equiv 9$ modulo 12, the probability of having a dummy player in a 4-player WMG is given as:*

$$P(4, n) = \frac{n^2 - 33}{2(n^2 + 3n - 12)}.$$

As a consequence, $lim_{n\to\infty} P(4,n) = \frac{1}{2}$.

Proof. Given our assumption (3) and for a given value of n, we have to divide the number of those distributions of the n_i's that give rise to the occurrence of a dummy player (denoted by $D(4,n)$) by the total number of possible distributions with 4 players (denoted by $T(4,n)$). We begin by evaluating $T(4,n)$. A vector of integers (n_1, n_2, n_3, n_4) is a possible distribution of the weights if and only if

$$n_1 \geq n_2,\ n_2 \geq n_3,\ n_3 \geq n_4,\ n_4 \geq 1,\ n_1 \leq n/2 \text{ and } n_1 + n_2 + n_3 + n_4 = n.$$

We know from Ehrhart's theory and its recent developments (the reader is referred to Lepelley *et al.* (2008) for a presentation of this theory) that the number of integer solutions of such a set of (in)equalities is a *quasi polynomial* in n with periodic coefficients (or Ehrhart's polynomial). A periodic coefficient takes various values according to n and to a given period. For example, $c(n) = [\frac{1}{2}, \frac{3}{4}, 1]_n$ is a periodic coefficient with period 3, $c(n) = \frac{1}{2}$ if $n \equiv 0$ *modulo* 3, $c(n) = \frac{3}{4}$ if $n \equiv 1$ *modulo* 3 and $c(n) = 1$ if $n \equiv 2$ *modulo* 3. Numerous algorithms exist to derive the expression of such a quasi polynomial (see, once again, Lepelley *et al.* (2008)). Using one of these algorithms, we obtain

$$T(4,n) = \tfrac{1}{288}\, n^3 + [\tfrac{1}{32}, \tfrac{1}{48}]_n\, n^2 + [\tfrac{1}{24}, -\tfrac{1}{96}]_n\, n + [0, -\tfrac{1}{72}, -\tfrac{17}{72}, -\tfrac{1}{4}, \tfrac{1}{9}, \tfrac{7}{72}, -\tfrac{1}{8}, -\tfrac{5}{36}, -\tfrac{1}{9}, -\tfrac{1}{8}, -\tfrac{1}{72}, -\tfrac{1}{36}]_n.$$

The period of such a quasi polynomial is the least common multiple of the periods of its coefficients, here 12. Consequently, the expression of $T(4,n)$ corresponds to 12 distinct polynomials; for instance, we obtain[2] for $n \equiv 9$ *modulo* 12:

$$T(4,n) = \frac{1}{288}n^3 + \frac{1}{48}n^2 - \frac{1}{96}n - \frac{1}{8} = \frac{(n+3)(n^2+3n-12)}{288}.$$

Now, according to Corollary 2, a dummy player exists if and only if

$$n_1 \geq n_2,\ n_2 \geq n_3,\ n_3 \geq n_4,\ n_4 \geq 1,\ n_1 \leq n/2,\ n_2 + n_3 > n/2 \text{ and } n_1 + n_2 + n_3 + n_4 = n.$$

The number of associated distributions of the n_i's is given as

$$D(4,n) = \tfrac{1}{576}\, n^3 + [-\tfrac{1}{96}, \tfrac{1}{192}]_n\, n^2 + [-\tfrac{1}{24}, -\tfrac{11}{192}, \tfrac{1}{48}, \tfrac{1}{192}]_n\, n + \\ [0, \tfrac{29}{576}, -\tfrac{7}{72}, -\tfrac{7}{64}, \tfrac{2}{9}, -\tfrac{35}{576}, -\tfrac{1}{8}, \tfrac{65}{576}, \tfrac{1}{9}, -\tfrac{11}{64}, \tfrac{7}{72}, \tfrac{1}{576}]_n,$$

which implies, for $n \equiv 9$ *modulo* 12:

$$D(4,n) = \frac{1}{576}n^3 + \frac{1}{192}n^2 - \frac{11}{192}n - \frac{11}{64} = \frac{(n+3)(n^2-33)}{576}.$$

The expression of $P(4,n) = D(4,n)/T(4,n)$ for $n \equiv 9$ *modulo* 12 directly follows, as well as the limiting value[3] $P(4,\infty) = \frac{\frac{1}{576}}{\frac{1}{288}} = \frac{1}{2}$. □

[2]Of course, the 11 other polynomials can be derived in the same way and are available from the authors upon request.

[3]It is important to note that the coefficient of the leading term of the quasi polynomials is not periodic. This peculiarity allows to easily obtain the desired probabilities for n large by considering only this coefficient in the quasi polynomials.

The two following Propositions are obtained along the same lines as Proposition 3 and their proofs are omitted.

Proposition 4 *For $n \equiv 15$ modulo 120, the probability of having dummy player(s) in a 5-player WMG is:*

$$P(5,n) = \frac{5(n+9)(7n^3-51n^2+165n-801)}{6(11n^4+120n^3+350n^2+960n+4815)}.$$

Consequently, the probability for P5 to be a dummy when n is large is: $lim_{n\to\infty}P(5,n) = \frac{35}{66}$. And the limiting probability of having two dummy players (J4 and J5) when n is large is given as $\frac{5}{22}$.

Proposition 5 *The limiting probability of having at least one dummy player in a 6-player WMG is given by: $lim_{n\to\infty}P(6,n) = \frac{155}{312}$. The limiting probability of having two dummies (J5 and J6) is $\frac{5}{39}$ and the limiting probability of having three (J4, J5 and J6) is $\frac{5}{52}$.*

Numerical values derived from our analytical results are shown in Tables 1 and 2. Table 1 gives the probability of having at least one dummy in a WMG with 4, 5 or 6 players as a function of the total number of votes n. These probabilities are surprisingly high and tend to increase with n: for large values of n, a dummy exists in more than 40% of the weight distributions! It is worth noticing that the probabilities are consistently higher with n odd than with n even. In addition, the results in Table 2 show that even the probability of having more than one dummy cannot be considered as negligible: in a 6-player WMG, three among the six players are dummies in almost 10% of the possible weight distributions.
The next section deals with the cases with more than six players.

Table 1

Probability $P(m, n)$ of having a dummy player as a function of n (the total number of votes) for $m = 4,\ 5,\ 6$.

n	4-player WMG	5-player WMG	6-player WMG
15	0.4375	0.2609	0.1818
18	0.2258	0.1778	0.0196
21	0.4146	0.2973	0.1978
24	0.2537	0.2302	0.0529
27	0.4578	0.3696	0.2731
30	0.3089	0.2827	0.1002
33	0.4490	0.3818	0.2905
36	0.3235	0.3087	0.1259
39	0.4684	0.4145	0.3310
42	0.3535	0.3398	0.1601
45	0.4637	0.4213	0.3407
48	0.3624	0.3553	0.1809
51	0.4747	0.4402	0.3641
54	0.3813	0.3757	0.2072
57	0.4718	0.4443	0.3711
60	0.3873	0.3859	0.2232
63	0.4789	0.4564	0.3869
66	0.4002	0.4002	0.2431
69	0.4770	0.4593	0.3918
72	0.4045	0.4075	0.2558
75	0.4820	0.4678	0.4028
78	0.4139	0.4181	0.2716
81	0.4806	0.4699	0.4066
84	0.4172	0.4235	0.2818
87	0.4842	0.4761	0.4149
90	0.4243	0.4316	0.2943
93	0.4832	0.4777	0.4177
96	0.4269	0.4359	0.3027
99	0.4860	0.4825	0.4241
.	.	.	
199	0.4928	0.5061	0.4594
202	0.4645	0.4624	0.3935
.	.	.	
limit	1/2	$\frac{35}{66} = 0.530$	$\frac{155}{312} = 0.497$

Table 2

Probability $P(m, \infty)$ of having one, two or three dummies for $m = 4,\ 5,\ 6$.

m	1 dummy	2 dummies	3 dummies	Total
4	0.5	0	0	0.5
5	0.3030	0.2273	0	0.5303
6	0.2724	0.1282	0.0962	0.4968

4 Results for more than six players

Intuition suggests that the probability of having a dummy should decrease when the number of players increases[4]. This intuition is based upon the so-called Penrose's limit theorem (Penrose, 1952). Penrose's limit theorem says that, in Weighted Majority Games, if the number of players increases, then under certain regularity conditions, the ratio between the voting powers of any two players converges to the ratio between their weights. As we have assumed that every voter has a strictly positive weight, this theorem suggests that dummies should not exist when the number of players is large enough.

The aim of this Section is to check and precise this conclusion. Unhappily, we cannot use the same approach as above for more than 6 players. First, a complete characterization of the weight distributions giving rise to dummies becomes too complex as soon as $m \geq 7$; second, the computing time necessary to implement the algorithms for obtaining Erhart polynomials (see Proof of Proposition 3) is exponentially increasing when the number of variables involved in the problem increases[5] and it becomes practically impossible to obtain a result when the number of variables is too large.

In order to obtain the desired probabilities for more than 6 players, we make use of two alternative approaches. The first one gives exact results whereas the second one is based on simulations and provides estimated probabilities.

Exact computations are done with an exhaustive list of all the possible vectors of weights for a given number n of votes. For all these vectors $(n_1, \ldots, n_m)$, we check whether or not the last player is pivotal (decisive) (remember that $n_1 \geq n_2 \geq \ldots \geq n_m$). This is done using the Banzhaf power index[6]: indeed, for most of power indices, a player is a dummy if and only if his (her) power index value is null. This equivalence is true for the Banzhaf power index, which is the easiest to calculate. We compute this index using a generating function approach which leads to exact values (this point is fundamental as we are looking for a null value; this peculiarity prevents from using approximation methods). Finally, the exact probability of having at least one dummy player is the ratio between the number of times the last player J_m is never pivotal (decisive) and the number of vectors $(n_1, \ldots, n_m)$ considered as admissible (with a uniform

[4]Notice however that $P(3, \infty) < P(4, \infty)$, as shown in Section 3.

[5]Here, the variables are the n_i's and their number is equal to m.

[6]For a clear and simple presentation, see Straffin (1994).

distribution of weights vectors, as assumed in the previous section).

Our simulations are based on random vectors of weights. The estimated probability of having at least one dummy player is then obtained by dividing the number of times the last player J_m is never pivotal (decisive) by the number of vectors $(n_1, \ldots, n_m)$ randomly generated. In order to estimate the probability of a dummy player, two steps have to be considered. First, we have to simulate a vector of weights for a given n and a given number of players m. This can be done by using for instance the `Rancom` algorithm proposed by Nijenhuis and Wilf (1978). Second, we have to check if there is at least one dummy player in the WMG associated to these weights. This is done as mentioned above by using the Banzhaf power index. Then repeating these two steps k times, we obtain the estimated probability as the proportion of weight distributions leading to a dummy player:

$$\hat{P}(m, n, q) = \frac{1}{k} \sum_{j=1}^{k} 1\!\!1_{\beta_m^{(j)}\left([q; n_1^{(j)}, \ldots, n_i^{(j)}, \ldots, n_m^{(j)}]\right) = 0}$$

where $\beta_m^{(j)}\left([q; n_1^{(j)}, \ldots, n_i^{(j)}, \ldots, n_m^{(j)}]\right)$ is the Banzhaf power index value for the m^{th} player in the j^{th} simulation, and $\forall j = 1, \ldots, k, \quad n^{(j)} = (n_1^{(j)}, \ldots, n_i^{j}, \ldots, n_m^{(j)})$, corresponds to the j^{th} simulated vector of weights.

Table 3

Probability $P(m, n)$ of having a dummy player as a function of m for $n = 45$, $n = 50$, $n = 95$ and $n = 100$.

m	$n = 45$	$n = 50$	$n = 95$	$n = 100$
4	0.4637	0.3735	0.4855	0.4297
5	0.4213	0.3020	0.4806	0.4003
6	0.3407	0.1931	0.4215	0.3091
7	0.2135	0.0858	0.3173	0.1869
8	0.1050	0.0299	0.2017	0.0862
9	0.0434	0.0091	0.0963	0.0304
10	0.0185	0.0030	0.0447	0.0108
11	0.0086	0.0012	0.0194*	0.0044*
12	0.0044	0.0005	0.0098*	0.0017*
13	0.0021	0.0002	0.0060*	0.0008*
14	0.0007	0.0000	0.0038*	0.0005*
15	0.0004	0.0000	0.0025*	0.0003*

*Simulated probabilities

Table 4

Simulated[7] probability $P(m, \infty)$ of having one, two ... or x dummies

	Number of dummy players					
m	≥ 1	1	2	3	4	5
4	0.4930	0.4930	0	0	0	0
5	0.5249	0.2983	0.2266	0	0	0
6	0.4931	0.2714	0.1275	0.0924	0	0
7	0.4353	0.2453	0.1046	0.0457	0.0398	0
8	0.3447	0.2103	0.0797	0.0293	0.0130	0.0124
9	0.2575	0.1725	0.0493	0.0196	0.0079	0.0037
10	0.1775	0.13160	0.0271	0.0100	0.00340	0.0023
11	0.1184	0.0963	0.0144	0.0041	0.0014	0.0007
12	0.0714	0.0604	0.0076	0.0018	0.0006	0.0002
13	0.0434	0.0394	0.0029	0.0006	0.0002	0^+
14	0.0228	0.0213	0.0011	0.0002	0^+	0^+
15	0.0123	0.0117	0.0005	0^+	0^+	0^+

Table 3 gathers both exact results and estimates for $P(m, n)$, with $4 \leq m \leq 15$ and for some specific values of n. As expected, $P(m, n)$ monotonically decreases as m increases and this decrease accelerates with m. However, for 10 players, the probability cannot be considered as completely negligible (around 2% for $n = 45$ and around 5% for $n = 95$). Once again, the discrepancy between odd and even values of n is worth noticing: for $m = 13$, the probability is divided by 10.5 when n moves from 45 to 50, and by 7.5 for n moving from 95 to 100. This observation strongly suggests to adopt an even value for n in order to minimize the probability of having dummies.

The figures given in Table 4 have been obtained *via* simulations by taking $n = 99,999$ and provide estimates for $P(m, \infty)$. Notice that these results give some information about the probability of having dummies when the weight vector is constituted by real numbers between 0 and 1/2 and summing up to 1 (all the possible weight vectors being considered as equally likely). It turns out that, in this case, the probability of having at least one dummy is still higher than 1% for $m = 15$.

5 Discussion and further results

The theoretical risk of having a dummy appears to be very high in WMG's with a small number of players. It can be suggested, however, that our calculations possibly overestimate this risk in the case of the French EPCI's, which very often try to reduce the spread of the numbers of representatives in each city. How can we introduce more realism in our analysis ?

[7]10,000 simulations have been conducted for $m < 11$ and 50,000 when $m \geq 11$. The 0 notation corresponds to an exact null probability value (see Proposition 1), while the 0^+ notation corresponds to a null estimated probability using 50,000 replications (in this case, the true probability could be positive).

One approach is the following. Let k be the maximal fraction of the total weight given to the "biggest" player: $n_1/n \leq k$. We wish to study the impact of parameter k on the probability of having a dummy player. Under our other assumptions, k belongs to $[\frac{1}{m}, \frac{1}{2}]$. When $k = \frac{1}{2}$, we recover the situation we have studied in the preceding sections. With $k = \frac{1}{m}$, each player obtains the same weight (assuming that kn is an integer), hence the same power and there is no dummy. Let $P(m, n, k)$ be the probability of having a dummy when $J1$ gets no more than $100k\%$ of the total weight. It seems natural to conjecture that $P(m, n, k)$ decreases when k moves from $\frac{1}{2}$ to $\frac{1}{m}$. The following results show that this conjecture does not hold for small values of m and n large. We will only give the proof of the first Proposition.[8]

Proposition 6 *In a 4-player WMG with n large, the probability of having a dummy player as a function of k is given by the following representation:*

$$P(4, \infty, k) = \frac{3}{4} \quad \text{for } \frac{1}{4} \leq k < \frac{1}{3}$$
$$= \frac{240k^3 - 288k^2 + 108k - 13}{4(44k^3 - 60k^2 + 24k - 3)} \quad \text{for } \frac{1}{3} \leq k \leq \frac{1}{2}.$$

Proof. Let K be the maximal weight of $J1$, with $k = K/n$. In order to compute the desired probability, we begin by evaluate the total number $T(4, n, K)$ of distributions on the n_i's when n_1 is constrained to be lower or equal to K. $T(4, n, K)$ is the number of integer solutions of the following inequalities:

$$n_1 \geq n_2, \; n_2 \geq n_3, \; n_3 \geq n_4, \; n_4 \geq 1, \; n_1 \leq K \quad n_1 + n_2 + n_3 + n_4 = n \text{ and } K \leq n/2.$$

We have now two parameters, n and K, and the number of integer solutions is given by bivariate quasi polynomials (see Lepelley *et al.*(2008)). Using an algorithm recently developed by Barvinok (see *e.g.* Barvinok, 2008), we obtain for n even two distinct quasi polynomials associated with two validity domains:

For $\frac{n}{4} \leq K < \frac{n}{3}$:

$$T(4, n, K) = -\tfrac{1}{144}n^3 + (\tfrac{1}{12}K + \tfrac{5}{48})n^2 + (-\tfrac{1}{3}K^2 - \tfrac{5}{6}K - \tfrac{1}{2})n + \tfrac{4}{9}K^3 + \tfrac{5}{3}K^2 + 2K + c_1;$$

For $\frac{n}{3} \leq K \leq \frac{n}{2}$:

$$T(4, n, K) = \tfrac{1}{48}n^3 + (-\tfrac{1}{6}K - \tfrac{3}{16})n^2 + (\tfrac{5}{12}K^2 + -\tfrac{11}{12}K + \tfrac{5}{12})n - \tfrac{11}{36}K^3 - \tfrac{23}{24}K^2 - 2K + c_2,$$

where c_1 and c_2 are periodic constants the value of which depends on both n and K.
Consider the first domain. As $K = kn$, it follows that, for $\frac{n}{4} \leq K < \frac{n}{3}$, *i.e.* for $\frac{1}{4} \leq k < \frac{1}{3}$:

$$T(4, n, k) = -\tfrac{1}{144}n^3 + (\tfrac{1}{12}kn + \tfrac{5}{48})n^2 + (-\tfrac{1}{3}k^2n^2 - \tfrac{5}{6}kn - \tfrac{1}{2})n + \tfrac{4}{9}k^3n^3 + \tfrac{5}{3}k^2n^2 + 2kn + c_1$$
$$= (-\tfrac{1}{144} + \tfrac{1}{12}k - \tfrac{1}{3}k^2 + \tfrac{4}{9}k^3)n^3 + (\tfrac{5}{48} - \tfrac{5}{6}k + \tfrac{5}{3}k^2)n + (-\tfrac{1}{2} + 2k)n + c_1.$$

Observe that, in order to compute the limiting probability $P(4, \infty, k)$, only the coefficient of the leading term in n^3 matters. For this reason, we will only give the coefficient of n^3 of the quasi polynomials we exhibit in the remaining of this proof.

[8] Although more cumbersome, the proofs of Proposition 7 and 8 are quite similar.

Proceeding as above, we obtain for the second domain, $\frac{1}{3} \leq k \leq \frac{1}{2}$:

$$T(4,n,k) = (\tfrac{1}{48} - \tfrac{1}{6}k + \tfrac{5}{12}k^2 - \tfrac{11}{36}k^3)n^3 + ...$$

Consider now the number $D(4,n,K)$ of distributions with a dummy player with $n_1 \leq K$. All we have to do is to add to the above set of inequalities $n_2 + n_3 > n/2$. Replacing K by kn in the quasi polynomials associated with this new set on inequalities gives:
For $\frac{1}{4} \leq k < \frac{1}{3}$: $D(4,n,k) = (-\frac{1}{192} + \frac{1}{16}k - \frac{1}{4}k^2 + \frac{1}{3}k^3)n^3 + ...$
For $\frac{1}{3} \leq k \leq \frac{1}{2}$: $D(4,n,k) = (\frac{13}{576} - \frac{3}{16}k + \frac{1}{2}k^2 - \frac{5}{12}k^3)n^3 + ...$

We finally obtain:
For $\frac{1}{4} \leq k < \frac{1}{3}$:

$$P(4,\infty,k) = \frac{D(4,\infty,k)}{T(4,\infty,k)} = \frac{-\frac{1}{192} + \frac{1}{16}k - \frac{1}{4}k^2 + \frac{1}{3}k^3}{-\frac{1}{144} + \frac{1}{12}k - \frac{1}{3}k^2 + \frac{4}{9}k^3} = \frac{\frac{(4k-1)^3}{192}}{\frac{(4k-1)^3}{144}} = \frac{3}{4},$$

and for $\frac{1}{4} \leq k < \frac{1}{3}$:

$$P(4,\infty,k) = \frac{D(4,\infty,k)}{T(4,\infty,k)} = \frac{\frac{13}{576} - \frac{3}{16}k + \frac{1}{2}k^2 - \frac{5}{12}k^3}{\frac{1}{48} - \frac{1}{6}k + \frac{5}{12}k^2 - \frac{11}{36}k^3} = \frac{240k^3 - 288k^2 + 108k - 13}{4(44k^3 - 60k^2 + 24k - 3)}. \quad \square$$

Proposition 7 *In a 5-player WMG with n large, the probability of having at least one dummy player depends on k as shown in the following representation:*

$$\begin{aligned} P(5,\infty,k) &= 0 \quad \textit{for } \frac{1}{5} < k < \frac{1}{4} \\ &= \frac{5(4k-1)^3(44k-23)}{32(655k^4 - 780k^3 + 330k^2 - 60k + 43)} \quad \textit{for } \frac{1}{4} < k < \frac{1}{3} \\ &= \frac{-5(3264k^4 - 3840k^3 + 1440k^2 - 192k + 5)}{96(155k^4 - 300k^3 + 210k^2 - 60k + 6)} \quad \textit{for } \frac{1}{3} < k < \frac{1}{2}. \end{aligned}$$

Proposition 8 *In a 6-player WMG with n large, the probability of having at least one dummy player depends on k as shown in the following representation:*

$$\begin{aligned} P(6,\infty,k) &= \frac{5}{12} \quad \textit{for } \frac{1}{6} < k < \frac{1}{5} \\ &= \frac{186120k^5 - 192600k^4 + 79200k^3 - 16200k^2 + 1650k - 67}{12(10974k^5 - 12270k^4 + 5340k^3 - 1140k^2 + 120k - 5)} \quad \textit{for } \frac{1}{5} < k < \frac{1}{4} \\ &= -\frac{5034240k^5 - 7027200k^4 + 3916800k^3 - 1094400k^2 + 153600k - 8669}{768(2193k^5 - 3465k^4 + 2130k^3 - 630k^2 + 90k - 5)} \quad \textit{for } \frac{1}{4} < k < \frac{3}{10} \\ &= \frac{5(1153152k^5 - 1834560k^4 + 1160640k^3 - 364320k^2 + 56760k - 3515)}{768(2193k^5 - 3465k^4 + 2130k^3 - 630k^2 + 90k - 5)} \quad \textit{for } \frac{3}{10} < k < \frac{1}{3} \\ &= -\frac{5(282240k^5 - 486720k^4 + 331200k^3 - 110880k^2 + 18360k - 1211)}{768(237k^5 - 585k^4 + 570k^3 - 270k^2 + 60k - 5)} \quad \textit{for } \frac{1}{3} < k < \frac{3}{8} \\ &= \frac{5(307584k^5 - 619200k^4 + 498240k^3 - 200160k^2 + 39960k - 3163)}{768(237k^5 - 585k^4 + 570k^3 - 270k^2 + 60k - 5)} \quad \textit{for } \frac{3}{8} < k < \frac{1}{2}. \end{aligned}$$

Numerical values for the probability of having at least one dummy player are displayed in Table 5 for $m = 4,\ 5$ and 6. The results for four players show that this probability increases monotonically when the spread of the weights is reduced and this conclusion is exactly the converse of what could be expected. This somewhat paradoxical result can be explained as follows. With four players, only two categories of weight distributions need to be considered when n is large: either $n_2 + n_3 > n/2$ and there is a dummy, or $n_1 + n_4 > n/2$ and there is no dummy. For $k = 50\%$, the two categories have exactly the same likelihood (under our probabilistic assumption) but reducing the value of k makes the distributions with $n_2+n_3 > n/2$ more frequent than the distributions with $n_1 + n_4 > n/2$. With five players, the results are much more in accordance with our intuition: when k increases, the probability of observing at least one dummy tends to increase. Notice however that this probability decreases when k moves from 47% to 50%. For six players, the results we observe are once again surprising: when k increases, the probability first decreases, then increases when k is about 27% and decreases again when k becomes close to 50%. For small values of k (between 1/6 and 1/5), the probability of having a dummy remains quite significant (more than 40%).

What is the impact of parameter k when considering more than six players? In order to answer this question, we have conducted a simulation study allowing to estimate the desired probabilities for $m = 7,\ 8,\ 9$ and 10. The results are shown in Table 6[9] and demonstrate that our conjecture is true for seven players or more: the probability of having at least one dummy monotonically decreases when the fraction of the total weight given to $J1$ decreases. Observe however that this decrease is rather slow and that $P(m, \infty, k)$ remains high for $k = 20\%$.

6 Concluding remark

We have shown in this paper that the probability of having at least one dummy player in Weighted Majority Games with a small number of player is very high. This probability can reach about 50% for 4, 5 or 6 players ; for more than 6 players, the probability decreases but we have to consider more than 15 players for obtaining results lower than 1%. Of course, it can be suspected that our probabilistic assumption (all admissible weight distributions are supposed to be equally likely to occur) could tend to exaggerate the probability of having a dummy. We have proved however that, for a very small number of players, the introduction of some degree of homogeneity in the distribution of the weights has a weak impact on this probability.

Finally, it is worth to emphasize that our results are limited to *majority* games, in which the quota for a proposition to be approved is equal to 50% of the total weight. It should be of interest to consider the impact of the quota value on the probability of having a dummy player. We plan to study this question in another paper.

[9] It is difficult to obtain a precise estimate when parameter k is close to its minimum value because the number of associated distributions is too small. It is the reason why we only consider in this Table some values of k higher or equal to 20%.

Table 5

Exact probability $P(m, \infty, k)$ of having a dummy player as a function of k for large n and $m = 4,\ 5,\ 6$.

k	4-player WMG	5-player WMG	6-player WMG
17%	-	-	0.4167
19%	-	-	0.4167
21%	-	0	0.4157
23%	-	0	0.3811
25%	0.7500	0	0.2821
27%	0.7500	0.0601	0.2694
29%	0.7500	0.1737	0.3106
31%	0.7500	0.2699	0.3539
33%	0.7500	0.3444	0.3908
35%	0.7480	0.4033	0.4184
37%	0.7374	0.4519	0.4393
39%	0.7185	0.4924	0.4552
41%	0.6930	0.5251	0.4687
43%	0.6615	0.5494	0.4821
45%	0.6241	0.5631	0.4953
47%	0.5802	0.5634	0.5053
49%	0.5288	0.5466	0.5047
50%	0.5000	0.5303	0.4968

Table 6

Simulated[10] probability limit $P(m, \infty, k)$ of having a dummy player as a function of k for large n and $m = 6, 7, 8, 9, 10$.

k	6-player WMG	7-player WMG	8-player WMG	9-player WMG	10-player WMG
20%	0.4187	0.0973	0.1199	0.1035	0.0802
21%	0.4172	0.1360	0.1419	0.1183	0.0902
23%	0.3790	0.2002	0.1750	0.1398	0.1042
25%	0.2825	0.2350	0.1990	0.1556	0.1127
27%	0.2692	0.2591	0.2200	0.1719	0.1233
29%	0.3106	0.2898	0.2421	0.1865	0.1308
31%	0.3530	0.3189	0.2605	0.1981	0.1378
33%	0.3910	0.3435	0.2766	0.2100	0.1433
35%	0.4175	0.3592	0.2890	0.2183	0.1486
37%	0.4378	0.3749	0.3005	0.2248	0.1528
39%	0.4533	0.3885	0.3082	0.2308	0.1560
41%	0.4650	0.3986	0.3158	0.2365	0.1600
43%	0.4797	0.4088	0.3220	0.2409	0.1625
45%	0.4929	0.4162	0.3267	0.2454	0.1655
47%	0.5035	0.4250	0.3326	0.2495	0.1690
49%	0.5033	0.4309	0.3378	0.2535	0.1728
50%**	0.4955	0.4353	0.3447	0.2575	0.1775

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[10] Probabilies are estimated with $n = 9,999$ and using 50,000 replications. Moreover the probabilities corresponding to $k = 50\%$ are those of Table 4, first column. Finally, Table 6 gives estimated probabilities for the 6-player WMG. Comparison with the exact values given in Table 5 illustrates the relative quality of the estimation method.

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8 Appendix: Proof of Proposition 2

(i) In order to characterize the distributions of the n_i's for which $J5$ is a dummy player, we consider the set of coalitions to which $J5$ is susceptible to belong: $\{J5\}$, $\{J1, J5\}$, $\{J2, J5\}$, $\{J3, J5\}$, $\{J4, J5\}$, $\{J1, J2, J5\}$, $\{J1, J3, J5\}$, $\{J1, J4, J5\}$, $\{J2, J3, J5\}$, $\{J2, J4, J5\}$, $\{J3, J4, J5\}$, $\{J1, J2, J3, J5\}$, $\{J1, J2, J4, J5\}$, $\{J1, J3, J4, J5\}$, $\{J2, J3, J4, J5\}$ and $\{J1, J2, J3, J4, J5\}$. Given that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq 1$, by assumption (2), it can be checked that $\{J5\}$, $\{J2, J5\}$, $\{J3, J5\}$ and $\{J4, J5\}$ are loosing, whereas $\{J1, J2, J3, J5\}$, $\{J1, J2, J4, J5\}$, $\{J1, J3, J4, J5\}$, and $\{J1, J2, J3, J4, J5\}$ are winning and remain winning when $J5$ is removed. Consequently, we have just to examine the coalitions which are left. Consider first the two-player coalition $\{J1, J5\}$; $J5$ is a dummy player if and only if this coalition is loosing (if not, a zero power for $J5$ would imply that a coalition with only one player is winning, contradicting our assumptions) and this two-player coalition will be loosing if and only if $n_1 + n_5 < Q$. Consider now the coalitions $\{J1, J2, J5\}$ and $\{J1, J3, J5\}$; these coalitions are necessarily winning. $J5$ is a dummy if and only if we have $n_1 + n_3 \geq Q$ (which implies $n_1 + n_2 \geq Q$). The next three-player coalitions $\{J1, J4, J5\}$, $\{J2, J3, J5\}$, $\{J2, J4, J5\}$ and $\{J3, J4, J5\}$ can be *a priori* winning or loosing. Observe however that, as we have just seen that $n_1 + n_3 \geq Q$, the coalitions $\{J2, J4, J5\}$ and $\{J3, J4, J5\}$ are necessarily loosing and we have only to consider the two first coalitions. In these coalitions, $J5$ is a dummy if and only if either the coalition is loosing either it remains winning when $J5$ is removed, which is equivalent to: ($n_1 + n_4 + n_5 < Q$ or $n_1 + n_4 \geq Q$) and ($n_2 + n_3 + n_5 < Q$ or $n_2 + n_3 \geq Q$). Finally, consider the coalition with four players $\{J2, J3, J4, J5\}$. $J5$ is a dummy player in this coalition[11] if and only if $n_2 + n_3 + n_4 \geq Q$.
To summing up, $J5$ is a dummy player if and only if we have:
$n_1 + n_5 < Q$ and $n_1 + n_3 \geq Q$ and ($n_1 + n_4 + n_5 < Q$ or $n_1 + n_4 \geq Q$) and ($n_2 + n_3 + n_5 < Q$ or $n_2 + n_3 \geq Q$) and $n_2 + n_3 + n_4 \geq Q$.
Recalling that $\sum_i n_i = n$ and eliminating redundant inequalities, we obtain:
$n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_3 \geq Q$ and ($n_2 + n_3 \geq Q$ or $n_1 + n_4 \geq Q$),
which can be reduced to:
($n_2 + n_3 + n_4 \geq Q$ and $n_2 + n_3 \geq Q$) or ($n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_4 \geq Q$).
As $n_2 + n_3 \geq Q$ implies $n_2 + n_3 + n_4 \geq Q$, we finally obtain:

[11] Notice that this coalition could be loosing when n is even, by taking $n_1 = n/2$. However this event cannot occur as we have $n_1 + n_5 < Q$.

$$n_2 + n_3 \geq Q \text{ or } (n_2 + n_3 + n_4 \geq Q \text{ and } n_1 + n_4 \geq Q),$$

in accordance with Proposition 2 (i). Furthermore, it follows from Proposition 1 that $J4$ and $J5$ are both dummy players if and only if $n_2 + n_3 \geq Q$.

(ii) $J6$ belongs to 32 coalitions: $\{J6\}$, $\{J1, J6\}$, $\{J2, J6\}$, $\{J3, J6\}$, $\{J4, J6\}$, $\{J5, J6\}$, $\{J1, J2, J6\}$, $\{J1, J3, J6\}$, $\{J1, J4, J6\}$, $\{J1, J5, J6\}$, $\{J2, J3, J6\}$, $\{J2, J4, J6\}$, $\{J2, J5, J6\}$, $\{J3, J4, J6\}$, $\{J3, J5, J6\}$, $\{J4, J5, J6\}$, $\{J1, J2, J3, J6\}$, $\{J1, J2, J4, J6\}$, $\{J1, J2, J5, J6\}$, $\{J1, J3, J4, J6\}$, $\{J1, J3, J5, J6\}$, $\{J1, J4, J5, J6\}$, $\{J2, J3, J4, J6\}$, $\{J2, J3, J5, J6\}$, $\{J2, J4, J5, J6\}$, $\{J3, J4, J5, J6\}$, $\{J1, J2, J3, J4, J6\}$, $\{J1, J2, J3, J5, J6\}$, $\{J1, J2, J4, J5, J6\}$, $\{J1, J3, J4, J5, J6\}$, $\{J2, J3, J4, J5, J6\}$ and $\{J1, J2, J3, J4, J5, J6\}$. Assumption (2) in our model implies that $\{J6\}$, $\{J2, J6\}$, $\{J3, J6\}$, $\{J4, J6\}$, $\{J5, J6\}$, $\{J2, J4, J6\}$, $\{J2, J5, J6\}$, $\{J3, J4, J6\}$, $\{J3, J5, J6\}$ and $\{J4, J5, J6\}$ are loosing whereas $\{J1, J2, J3, J6\}$, $\{J1, J2, J4, J6\}$, $\{J1, J2, J5, J6\}$, $\{J1, J3, J4, J6\}$, $\{J1, J3, J5, J6\}$, $\{J1, J2, J3, J4, J6\}$, $\{J1, J2, J3, J5, J6\}$, $\{J1, J2, J4, J5, J6\}$, $\{J1, J3, J4, J5, J6\}$, and $\{J1, J2, J3, J4, J5, J6\}$ are winning and remain winning when $J6$ is removed. We have to studied the other 12 coalitions (among which $\{J1, J2, J6\}$ and $\{J1, J3, J6\}$ are winning).
Proceeding as above, it is easily checked that $J6$ is a dummy player if and only if
a) $n_1 + n_6 < Q$ and
b) $n_1 + n_2 \geq Q$ and $n_1 + n_3 \geq Q$ and $(n_1 + n_4 + n_6 < Q$ or $n_1 + n_4 \geq Q)$ and $(n_1 + n_5 + n_6 < Q$ or $n_1 + n_5 \geq Q)$ and $(n_2 + n_3 + n_6 < Q$ or $n_2 + n_3 \geq Q)$ and
c) $(n_1 + n_4 + n_5 + n_6 < Q$ or $n_1 + n_4 + n_5 \geq Q)$ and $(n_2 + n_3 + n_4 + n_6 < Q$ or $n_2 + n_3 + n_4 \geq Q)$ and $(n_2 + n_3 + n_5 + n_6 < Q$ or $n_2 + n_3 + n_5 \geq Q)$ and $(n_2 + n_4 + n_5 + n_6 < Q$ or $n_2 + n_4 + n_5 \geq Q)$ and $(n_3 + n_4 + n_5 + n_6 < Q$ or $n_3 + n_4 + n_5 \geq Q)$ and
d) $n_2 + n_3 + n_4 + n_5 \geq Q$.
A tedious process of reduction of this set of inequalities leads to the six cases given in Proposition 2 (ii): $(n_2 + n_3 + n_4 + n_5 \geq Q$ and $n_1 + n_5 \geq Q)$ or $(n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_4 \geq Q)$ or $(n_2 + n_3 + n_5 \geq Q$ and $n_1 + n_4 + n_5 \geq Q$ and $n_1 + n_3 \geq Q)$ or $n_2 + n_3 \geq Q$ or $(n_2 + n_4 + n_5 \geq Q$ and $n_1 + n_2 \geq Q)$ or $n_3 + n_4 + n_5 \geq Q$.
To complete the proof, it remains to observe that, in case 4, $J4$, $J5$ and $J6$ are dummy players (by Proposition 1); and if $J6$ is a dummy and $J4$ is not $(n_2+n_3 < Q)$, then it results from part (i) of Proposition 2 that $J5$ is also a dummy player if and only if $n_1+n_4 \geq Q$ and $n_2+n_3+n_4 \geq Q$.□