Chapter X: The Impact of Group Coherence on the Condorcet Ranking Efficiency of Voting Rules

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Abstract: We consider in this study the effectiveness of some voting rules at matching the complete Pairwise Majority Rule (PMR) ranking on candidates. We define the *Condorcet Ranking Efficiency* of a voting rule as the conditional probability that candidate rankings are identical for both PMR and that voting rule, given that a strict PMR ranking exists. We compute and compare in this study the Condorcet Ranking Efficiency of five common voting rules by considering various scenarios, corresponding to various ways of introducing some degree of group coherence in voters' preferences. It turns out that the Borda Rule and the Negative Plurality Elimination Rule (or Coombs' Rule) exhibit a good performance in most of the scenarios we have investigated. A "maxmin" argument leads us to suggest that the Borda Rule constitutes the best compromise on the basis of Condorcet Ranking Efficiency.

X.1 Introduction

Recent developments in voting theory, based on both probabilistic and empirical considerations, have led to the conclusion that Condorcet's Paradox should be a rare event in actual election settings with a small number of candidates, as soon as voters display any significant level of group mutual coherence, *i.e.* as soon as voters tend, in one way or another, to have similar preferences (see e.g. Gehrlein, 2011, and for an empirical point of view, Regenwetter et al., 2006). In the light of this conclusion, the Condorcet criterion, which requires that the Pairwise Majority Rule Winner (PMRW) – or Condorcet winner – should be elected when such a candidate exists, appears as being very relevant. It is therefore of particular interest to investigate the propensity of common voting rules to be in agreement with Pairwise Majority Rule (PMR) when group coherence is taken into consideration.

Five voting rules are considered in this study, where attention is restricted to three-candidate elections: the plurality rule (PR), the negative plurality rule (NPR), the Borda rule (BR), the plurality elimination rule (PER) and the negative plurality elimination rule (NPER). PR, BR and NPR are the most often used examples of (single stage) scoring rules: A scoring rule is defined in terms of weights $(1,\lambda,0)$, with $1 \geq \lambda \geq 0$ for a three-candidate election; each voter assigns a score of one to their most preferred candidate, a score of zero to their least preferred candidate and a score of λ to their middle-ranked candidate, with $\lambda = 0$ for PR, $\lambda = ½$ for BR, $\lambda = 1$ for NPR, and the scoring rule winner is the candidate that receives the greatest score from all voters[1]. PER and NPER are two-stage scoring rules (associated respectively with PR and NPR) in which the candidate with the lowest score is eliminated at the first stage and the winner is selected at the second stage by using PMR. Notice that, as we only consider three-candidate elections, PER coincides here with the very often used “Two-round Majority” and also with “Alternative Vote” (if we assume sincere voting); on the other hand, NPER corresponds –under the same assumption of sincere voting– to “Coombs Rule”.

A number of studies have been conducted to compute the *Condorcet Efficiency* of these voting rules, with the notion of Condorcet Efficiency of a voting rule VR being defined as the conditional probability that the PMRW be elected under VR, given that a PMRW exists. In these studies, it is implicitly assumed that the objective of the collective choice process is to determine only the winning candidate. We consider in this paper a somewhat different context in which the objective is to obtain a complete ranking of the candidates and we define the *Condorcet Ranking Efficiency* of VR as the conditional probability that candidate rankings are identical for both PMR and VR, given that a (strict) PMR ranking exists. The notion of Condorcet Ranking Efficiency is clearly more demanding than the notion of Condorcet Efficiency: in three-candidate elections, it is required that the voting rule under consideration not only ranks the PMRW in first position but also ranks the Pairwise Majority Rule Loser (PMRL or Condorcet loser) in third (and last) position.

Let n be the number of voters. In an election with three candidates *A*, *B*, and *C*, there are six possible complete preference orders that voters may have, as shown in Figure X.1.

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[1] Equivalently, NPR can be implemented by asking each voter to cast a negative vote against his or her least preferred candidate and the candidate who receives the fewest number of votes is the election winner.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. X.1 The six possible linear preference rankings on three candidates.

Here, n_i denotes the number of voters that have the associated complete preference ranking on the candidates. Any given combination of $n_i's$ such that $\sum_{i=1}^{6} n_i = n$ is referred to as a voting situation and reflects the voters' opinion. A well known assumption in voting theory is the so-called Impartial Anonymous Culture (IAC) assumption, which supposes that every possible voting situation is equally likely to be observed. Let $CRE(VR, n)$ be the Condorcet Ranking Efficiency of VR in three-candidate elections with n voters when IAC is assumed. Cervone et al. (2005) give a general representation for the limiting Condorcet Ranking Efficiency of scoring rules as $n \to \infty$ with IAC, from which it follows that:

$$CRE(PR,\infty) = CRE(NPR,\infty) = 303/540 = .5611, \tag{X.1}$$

$$CRE(BR,\infty) = 111/135 = .8222. \tag{X.2}$$

Using recent developments for obtaining IAC probability representations (see Gehrlein, 2002, Lepelley et al., 2008), it is easy to show that:

$$CRE(PER,\infty) = 17/27 = .6296, \tag{X.3}$$

$$CRE(NPER,\infty) = 119/135 = .8815\,. \tag{X.4}$$

The aim of the present paper is to analyze the impact that various degrees of group coherence in voters' preferences might have on the Condorcet Ranking Efficiency of these five common voting rules[2]. We begin by introducing various measures of group coherence.

X.2 Measuring the Level of Group Coherence in Voters' Preferences

Given a voting situation, how can we evaluate the level of group coherence in voters' preferences associated with this voting situation? Two sets of measures are used in our study.

X.2.1 Weak Measures of Group Coherence.

The first measure of group coherence we consider is directly inspired from Black's single-peakedness. We introduce a *Parameter b*, that measures the mini-

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[2] This paper completes two recent studies by the same authors and H. Smaoui (Gehrlein et al., 2008, 2010) that consider the impact of group coherence on the Condorcet Efficiency of voting rules.

mum number of times that some candidate is bottom ranked, or is least preferred, in the preferences of the *n* voters in a voting situation, to serve as a simple measure of the proximity of a voting situation to representing perfectly single-peaked preferences in a three-candidate election, where

$$b = Min\{n_1 + n_3, n_2 + n_4, n_5 + n_6\}. \tag{X.5}$$

If *b* is equal to zero for a voting situation with three candidates, it means that the *n* voters agree that some candidate is never ranked as least preferred, so the voting situation represents the condition in which voters have perfectly single-peaked preferences. This would happen, for example if $n_1 + n_3 = 0$, where the definitions from Fig. X.1 indicate that this requires that Candidate *C* is never the least preferred candidate for any voter in the associated voting situation. When *b* is maximized at $n/3$, a voting situation reflects very disperse preferences of voters over candidates to reflect a situation that is very far removed from perfect single-peakedness.

As Parameter *b* increases in voting situations, the preferences of voters in a voting situation become more removed from the condition of perfect single-peakedness. Another perspective on this issue is that a voting situation with a small Parameter *b* reflects a situation in which there is some candidate that very few voters think is the worst of the three candidates. The electorate would be somewhat united by their *weak* support of, or lack of complete opposition to, the election of such a candidate. In that sense, this candidate can be viewed as a *Weak Positively Unifying Candidate* that voters would not generally think of as reflecting the worst possible outcome if that candidate were to be elected.

Following the development of Parameter *b* above, *Parameter t* measures the proximity of a voting situation to meeting the condition of perfectly single-troughed preferences (see Vickery, 1960), with

$$t = Min\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{X.6}$$

The definition of $n_i's$ in Fig. X.1 are used to define Parameter *t* as the minimum number of times that some candidate is top-ranked as the most preferred candidate in the voters' preference rankings, so that a voting situation is perfectly single-troughed if $t = 0$, and the value of *t* then reflects the relative proximity of a voting situation to the condition of perfect single-troughedness. Any candidate that very few voters rank as the most preferred candidate in a voting situation can be viewed as a *Weak Negatively Unifying Candidate* since none of the voters would generally think of the election of this candidate as reflecting the best possible outcome. The electorate would be weakly unified by their opposition to, or lack in complete support of, the election of such a candidate.

According to Ward (1965), a candidate is said to be *perfectly polarizing* if this candidate is never middle ranked, or ranked at the center, of any voter's preference ranking. That is, every voter will either consider this candidate to be either the most preferred or the least preferred. The definition of $n_i's$ in Fig. X.1 are used to define *Parameter c* to reflect the proximity of a voting situation to the condition of perfect polarization, with

$$c = Min\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \quad (X.7)$$

If $c = 0$, some candidate is perfectly polarizing, since all voters will rank that candidate as either least preferred or most preferred, and the value of *c* measures the proximity of a voting situation to the condition of perfect polarization. Any candidate that very few voters rank in the middle of their preference ranking can generally be viewed as a *Weak Polarizing Candidate*.

X.2.2 Strong Measures of Group Coherence

Stronger measures of group coherence are developed in Gehrlein (2011), and each of these measures is a more restrictive variation of Parameters *b*, *t*, and *c*. A Weak Positively Unifying Candidate was defined as some candidate that is ranked as least preferred by a small proportion of voters in a voting situation, and the proximity of a voting situation to having a perfect Weak Positively Unifying Candidate is measured by Parameter *b*. A candidate would more strongly reflect the notion of being a positively unifying candidate by being ranked as most preferred by a large proportion of the voters in a voting situation. *Parameter t** is defined accordingly from the definition of the $n_i's$ in Fig. X.1, with

$$t^* = Max\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \quad (X.8)$$

If $t^* = n$, the same candidate is ranked as most preferred by all voters, making it a perfect *Strong Positively Unifying Candidate*, and Parameter *t** is used as a measure of the proximity of a voting situation to this condition.

The same basic logic can be used to strengthen the definition the proximity of a voting situation to having perfect Weak Negatively Unifying Candidate, as measured by Parameter *t*. *Parameter b** is defined accordingly by

$$b^* = Max\{n_5 + n_6, n_2 + n_4, n_1 + n_3\}. \quad (X.9)$$

If $b^* = n$, the same candidate is ranked as least preferred by all voters, making it a perfect *Strong Negatively Unifying Candidate*, and Parameter *b** is used as a measure of the proximity of a voting situation to this condition.

Parameter *c* measured the proximity of a voting situation to the condition of perfect weak polarization. The strong measure that is associated with this parameter is *Parameter c**, with

$$c^* = Max\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \quad (X.10)$$

If $c^* = n$, the same candidate is middle-ranked in the preferences of all voters, so that this candidate is neither extremely liked nor extremely disliked by any voter, making it a perfect *Strong Centrist Candidate*, and Parameter c^* is used as a measure of the proximity of a voting situation to this condition.

We have finally six different measures of group mutual coherence at our disposal, with an increasing level of mutual coherence when b, t and c move from $n/3$ to 0 and when b^*, t^* and c^* move from $n/3$ to n.

X.3 Obtaining Condorcet Ranking Efficiency Representations

To determine the impact that our measures of group coherence have on the probability that candidate rankings are identical for both PMR and a given voting rule, we develop representations for the conditional probability of that event, given that voting situations have specified values of these measures.

These probability representations are based on a direct extension of the IAC assumption: for any particular X in $\{b, t, c, b^*, t^*, c^*\}$, we assume that only voting situations for which Parameter X has a specified value can be observed, and that each of these possible voting situations is equally likely to be observed.

Using some techniques recently developed in the literature (see Gehrlein, 2005 and Lepelley et al. 2008), it is possible to derive some representations for the desired probabilities as functions of n and X. Unhappily, these representations are often of a very complicated nature, which makes them of limited value. For this reason, we only focus here on the limiting probabilities as n tends to infinity. Let $\alpha_X = X/n$; thus, for example, α_b denotes the minimal proportion of voters that rank a candidate in last position. The limiting representations we present in this paper express the Condorcet Ranking Efficiency of the voting rules as a function of the proportion α_X as $n \to \infty$, with α_X in [0,1/3] for X = b, t, c and α_X in [1/3,1] for X = b*, t*, c*.[3]

Let $CRE(VR, n/X)$ denote the Condorcet Ranking Efficiency of *VR* with n voters, given a specified value of Parameter X. As we consider five voting rules and six distinct group coherence parameters, thirty representations are *a priori* needed. However, some representations do not need to be determined as a result of the following observations.

Consider first the weak measures of group coherence. We have:

Lemma 1 $CRE(PR, n/t) = CRE(NPR, n/b)$,
$CRE(NPR, n/t) = CRE(PR, n/b)$ and
$CRE(BR, n/t) = CRE(BR, n/b)$.

Lemma 2 $CRE(PR, n/c) = CRE(NPR, n/c)$.

[3] For more details on this approach, see Gehrlein et al. (2010).

In order to illustrate the argument on which these results are based, consider a voting situation such that 1) *A* is the PMRW and *C* the PMRL; 2) *A* is the PR Winner and *C* the PR Loser; 3) $b=k$, where *k* is an integer between 0 and $n/3$. To this voting situation, it is possible to associate an equally likely dual voting situation obtained by inverting the preference of each of the voters, with the following 1-1 mapping: $n_1 \leftrightarrow n_6$, $n_2 \leftrightarrow n_5$ and $n_3 \leftrightarrow n_4$. It is easy to check that, in this dual voting situation 1) *C* is the PMRW and *A* the PMRL; 2) *C* is the NPR Winner and *A* the NPR Loser; 3) $t=k$. Consequently, the Condorcet ranking Efficiency of PR given a specified value of *b* will be equal to the Condorcet ranking Efficiency of NPR given the same specified value of *t*.

Furthermore, it can be noticed that the Condorcet Ranking Efficiency of the two-stage scoring rules PER and NPER can be obtained from the Condorcet Efficieny of NPR and PR (respectively). Let $CE(VR, n/X)$ denote the Condorcet Efficiency of VR, given a specified value of *X*.

Lemma 3 $CRE(PER, n/b) = CE(NPR, n/t)$ and
$CRE(NPER, n/b) = CE(PR, n/t)$.

Lemma 4 $CRE(PER, n/t) = CE(NPR, n/b)$ and
$CRE(NPER, n/t) = CE(PR, n/b)$.

Lemma 5 $CRE(PER, n/c) = CE(NPR, n/c)$ and
$CRE(NPER, n/c) = CE(PR, n/c)$.

The proof techniques that are used to obtain these results are very similar to the ones used to prove Lemma 1 and Lemma 2. Lemmas 3 to 5 allow us to use the Condorcet Efficiency results given in Gehrlein et al. (2008) to obtain the Condorcet Ranking Efficiency of PER and NPER with weak measures of group coherence.

Similar observations can be made for the strong measures of coherence.

Lemma 6 $CRE(PR, n/t^*) = CRE(NPR, n/b^*)$,
$CRE(NPR, n/t^*) = CRE(PR, n/b^*)$ and
$CRE(BR, n/t^*) = CRE(BR, n/b^*)$.

Lemma 7 $CRE(PR, n/c^*) = CRE(NPR, n/c^*)$.

Lemma 8 $CRE(PER, n/b^*) = CE(NPR, n/t^*)$ and
$CRE(NPER, n/b^*) = CE(PR, n/t^*)$.

Lemma 9 $CRE(PER, n/t^*) = CE(NPR, n/b^*)$ and
$CRE(NPER, n/t^*) = CE(PR, n/b^*)$.

Lemma 10 $CRE(PER, n/c^*) = CE(NPR, n/c^*)$ and

$CRE(NPER, n/c^*) = CE(PR, n/c^*)$.

Thanks to Lemmas 8-10, the Condorcet Ranking Efficiency of PER and NPER for strong measures of coherence can be deduced from the recent work by Gehrlein et al. (2010), where the Condorcet Efficiency of PR and NPR is derived for Parameters b^*, t^* and c^*.

Finally, our preliminary observations allow to reduce the number of needed representations from thirty to ten. These ten (limiting) representations are given in Appendix.

X.4 Results

The representations given in Appendix are used to obtain computed values of $CRE(VR, \infty/\alpha_X)$ for $VR \in \{PR, NPR, BR, PER, NPER\}$ and for various values of α_X. The results are listed in Tables X.1 to X.6. Tables X.1, X.2 and X.3 allow to analyze the relationship between Condorcet Ranking Efficiency of voting rules and weak measures of group mutual coherence (Parameters b, t, c), whereas Table X.4, X.5 and X.6 deal with strong measures of group coherence (Parameters b^*, t^*, c^*).

Table X.1 Computed values of $CRE(VR, \infty/\alpha_b)$ for $VR \in \{PR, NPR, BR, PER, NPER\}$

α_b	PR	NPR	BR	PER	NPER
.00	.5764	.7500	.8333	.6528	1.0000
.02	.5759	.7279	.8334	.6463	.9925
.04	.5730	.7065	.8335	.6385	.9850
.06	.5678	.6855	.8337	.6296	.9772
.08	.5606	.6647	.8338	.6199	.9689
.10	.5518	.6438	.8336	.6098	.9598
.12	.5418	.6225	.8331	.5997	.9494
.14	.5315	.6003	.8318	.5907	.9370
.16	.5222	.5767	.8295	.5841	.9218
.18	.5163	.5509	.8252	.5826	.9020
.20	.5178	.5213	.8177	.5897	.8750
.22	.5327	.4858	.8055	.6112	.8357
.24	.5717	.4394	.7905	.6578	.7736
.26	.6491	.3756	.7862	.7430	.6737
.28	.7269	.3111	.7973	.8213	.5698
.30	.7885	.2515	.8166	.8783	.4731
.32	.8257	.1983	.8303	.9104	.3857
1/3	.8333	.1667	.8333	.9163	.3460

It is worth noticing that, in many cases, the results do not correspond to what intuition suggests: it could be expected that the Condorcet Ranking Efficiency of voting rules would monotonically increase when the group coherence in voters' preferences increases. It turns out that this pattern of behavior is actually rather rare since it is only observed with NPR and NPER for Parameter *b*, with PR and PER for Parameter *t* and with NPER for Parameter *b**. The somewhat paradoxical opposite behavior, where the Condorcet Ranking Efficiency monotonically decreases when the level of group coherence increases, appears to be more frequent: this behavior occurs with NPER for Parameter *t*, with PR, NPR, PER and NPER for Parameter *c*, and with PR, NPR, PER and NPER for Parameter *c**. In all other combinations of voting rules and scenarios that we have considered, the behavior of the Condorcet Ranking Efficiency is not monotonic when the degree of group mutual coherence increases.

It is also of interest to remark that the Condorcet Ranking Efficiencies of PR and NPR can be very different when some measures of group coherence are taken in consideration, despite the fact that, on average, PR and NPR display the same level of Condorcet Ranking Efficiency ($CRE(PR,\infty) = CRE(NPR,\infty)$ = 303/540 = .5611, as seen in Section X.1). To illustrate, Table X.1 shows that NPR continuously decreases from ¾ to 1/6 as α_b increases, to reflect voting situations that are farther removed from having a perfect Positively Unifying Candidate,whereas the results for PR slowly decrease over the range $0 \le \alpha_b \le .21$, and then continues to increase significantly up to 5/6 as α_b continues to increase to 1/3.

Table X.2 Computed values of $CRE(VR,\infty/\alpha_t)$ for $VR \in \{PR,NPR,BR,PER,NPER\}$

α_t	PR	NPR	BR	PER	NPER
.00	.7500	.5764	.8333	.7500	.8611
.02	.7279	.5759	.8334	.7353	.8674
.04	.7065	.5730	.8335	.7210	.8728
.06	.6855	.5678	.8337	.7071	.8774
.08	.6647	.5606	.8338	.6933	.8811
.10	.6438	.5518	.8336	.6796	.8839
.12	.6225	.5418	.8331	.6657	.8857
.14	.6003	.5315	.8318	.6515	.8861
.16	.5767	.5222	.8295	.6367	.8850
.18	.5509	.5163	.8252	.6209	.8818
.20	.5213	.5178	.8177	.6037	.8768
.22	.4858	.5327	.8055	.5840	.8717
.24	.4394	.5717	.7905	.5601	.8710
.26	.3756	.6491	.7862	.5279	.8826
.28	.3111	.7269	.7973	.4839	.8965
.30	.2515	.7885	.8166	.4311	.9080
.32	.1983	.8257	.8303	.3732	.9151
1/3	.1667	.8333	.8333	.3433	.9166

Table X.3 Computed Values of $CRE(VR, \infty / \alpha_c)$ for $VR \in \{PR, NPR, BR, PER, NPER\}$

α_c	PR	NPR	BR	PER	NPER
.00	.4097	.4097	.8333	.4236	.8611
.02	.4147	.4147	.8386	.4351	.8662
.04	.4231	.4231	.8419	.4501	.8702
.06	.4349	.4349	.8433	.4685	.8733
.08	.4502	.4502	.8428	.4906	.8755
.10	.4689	.4689	.8405	.5165	.8769
.12	.4909	.4909	.8364	.5463	.8776
.14	.5162	.5162	.8307	.5800	.8780
.16	.5447	.5447	.8236	.6178	.8783
.18	.5768	.5768	.8156	.6590	.8793
.20	.6128	.6128	.8076	.7022	.8818
.22	.6518	.6518	.8009	.7459	.8864
.24	.6921	.6921	.7970	.7883	.8924
.26	.7302	.7302	.7867	.8275	.8974
.28	.7630	.7630	.7996	.8608	.9002
.30	.7891	.7891	.8044	.8863	.9023
.32	.8059	.8059	.8085	.9016	.9043
1/3	.8095	.8095	.8095	.9046	.9047

Table X.4 Computed values of $CRE(VR, \infty / \alpha_{b*})$ for $VR \in \{PR, NPR, BR, PER, NPER\}$

α_{b*}	PR	NPR	BR	PER	NPER
.33	.8333	.1667	.8333	.9162	.3233
.35	.8220	.2053	.8285	.9074	.3833
.40	.7029	.3323	.7642	.7990	.5366
.45	.5495	.4797	.7023	.6388	.7161
.50	.4074	.6875	.7292	.4676	1.0000
.55	.4647	.6534	.8541	.5259	1.0000
.60	.5300	.6250	.9106	.5918	1.0000
.65	.6034	.6010	.9327	.6636	1.0000
.70	.6792	.5804	.9464	.7321	1.0000
.75	.7500	.5625	.9583	.7917	1.0000
.80	.8125	.5469	.9688	.8438	1.0000
.85	.8676	.5331	.9779	.8897	1.0000
.90	.9167	.5208	.9861	.9306	1.0000
.95	.9605	.5099	.9934	.9671	1.0000
1.00	1.0000	.5000	1.0000	1.0000	1.0000

Table X.5 Computed values of $CRE(VR,\infty/\alpha_{t*})$ for $VR\in\{PR,NPR,BR,PER,NPER\}$

α_{t*}	PR	NPR	BR	PER	NPER
.33	.1667	.8333	.8333	.3208	.9166
.35	.2053	.8220	.8285	.3955	.9143
.40	.3323	.7029	.7642	.5640	.8864
.45	.4797	.5495	.7023	.6770	.8432
.50	.6875	.4074	.7292	.6875	.8148
.55	.6534	.4647	.8541	.6534	.8750
.60	.6250	.5300	.9106	.6250	.9118
.65	.6010	.6034	.9327	.6009	.9326
.70	.5804	.6792	.9464	.5803	.9464
.75	.5625	.7500	.9583	.5625	.9583
.80	.5469	.8125	.9688	.5469	.9688
.85	.5331	.8676	.9779	.5331	.9779
.90	.5208	.9167	.9861	.5208	.9861
.95	.5099	.9605	.9934	.5099	.9934
1.00	.5000	1.0000	1.0000	.5000	1.0000

Table X.6 Computed values of $CRE(VR,\infty/\alpha_{c*})$ for $VR\in\{PR,NPR,BR,PER,NPER\}$

α_{c*}	PR	NPR	BR	PER	NPER
.33	.8095	.8095	.8095	.9047	.9047
.35	.8040	.8040	.8080	.8999	.9040
.40	.7494	.7494	.7969	.8475	.8972
.45	.6850	.6850	.7908	.7738	.8889
.50	.6179	.6179	.7900	.6988	.8691
.55	.5392	.5392	.8075	.6181	.8536
.60	.4738	.4738	.8359	.5339	.8591
.65	.4128	.4128	.8671	.4452	.8802
.70	.3458	.3458	.8950	.3571	.9052
.75	.2713	.2713	.9188	.2766	.9267
.80	.2024	.2024	.9394	.2063	.9453
.85	.1422	.1422	.9574	.1450	.9616
.90	.0892	.0892	.9733	.0909	.9759
.95	.0421	.0421	.9874	.0429	.9886
1.00	.0000	.0000	1.0000	.0000	1.000

We now turn to the overall comparison of the five voting rules. In what follows, we will say that Voting Rule VR1 dominates Voting Rule VR2 for Parameter X if $CRE(VR1,\infty/\alpha_X) \geq CRE(VR2,\infty/\alpha_X)$ for every possible value of α_X and there exists some value of α_X for which $CRE(VR1,\infty/\alpha_X)$ > $CRE(VR2,\infty/\alpha_X)$.

The results from Table X.1 to X.6 show that:

- BR dominates PR and NPR for every X in {b, t, c, b^*, t^*, c^*}, *i.e.* for all weak and strong measures of group mutual coherence we have considered.
- Similarly, the two-stage voting rules (PER and NPER) dominate the corresponding single stage voting rules (PR and NPR) for every X in {b, t, c, b^*, t^*, c^*}.
- NPER dominates each of the four other voting rules for $X \in \{t, c, c^*\}$. Moreover, NPER dominates PR, NPR and PER for $X=t^*$.[4]
- BR dominates PER for $X \in \{t, t^*\}$.

Thus, BR and NPER clearly exhibit the best performances on the basis of Condorcet Ranking Efficiency. But the most striking observation from Tables X.1 to X.6 is that BR remains relatively stable over the entire range of the various parameters we have introduced and it turns out that the Condorcet Ranking Efficieny of BR is never lower than .70. When BR and NPER are compared, it can be seen that $CRE(NPER,\infty/\alpha_X)$ is higher that $CRE(BR,\infty/\alpha_X)$ in many cases but the margin of dominance for NPER over BR remains always rather low. Moreover, the possibility exists in which NPER could behave very poorly with large values of Parameter b or with low values of Parameter b^*, reflecting scenarios in which voting situations are far removed either from having a perfect Weak Positively Unifying Candidate or from having a perfect Strong Negatively Unifying Candidate. In these circumstances, the Condorcet Ranking Efficiency of NPER can be lower than .35.

X.5 Conclusion

The results that are observed for Condorcet Ranking Efficiency in the presence of group coherence are very similar to the results obtained previously, in the same context, for Condorcet Efficiency when the objective is to select a single winner (see Gehrlein et al., 2008, Gehrlein et al., 2010). The Condorcet Ranking Efficiency of BR remains somewhat stable across the complete range of all measures of group mutual coherence. BR dominates both PR and NPR for all weak and

[4] NPER does not dominate BR for $X=t^*$ because $CRE(BR,\infty/\alpha_{t^*})$ > $CRE(NPER,\infty/\alpha_{t^*})$ for $.62 \leq \alpha_{t^*} \leq .65$.

strong measures of group mutual coherence, particularly for Parameter *c* and Parameter *c**. While PER does display superior performance to BR over a small range of some parameters, it very frequently exhibits extremely poor performance on the basis of Condorcet Ranking Efficiency and it is not a viable option for consideration. The efficiency of NPER is very often superior to that of BR, but there are ranges in which NPER performs very poorly for both Parameter *b* and Parameter *b**, while BR does not so. Since we cannot exclude the possibility that voters are obtaining preference rankings with some model that will fall into the ranges in which NPER performs very poorly, the Borda Compromise suggested in Gehrlein et al. (2008) still has a good foundation for Condorcet Ranking Efficiencies.

X.6 Appendix: Condorcet Ranking Efficiency Representations

Limiting Condorcet Ranking Efficiency of PR for Parameter *b*:

$$CRE(PR,\infty/\alpha_b) = \frac{83-243\alpha_b-828\alpha_b^2+2898\alpha_b^3}{144\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 0 \le \alpha_b \le 1/6$$

$$\frac{-121+4800\alpha_b-30240\alpha_b^2+74304\alpha_b^3-66528\alpha_b^4}{3456\alpha_b\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 1/6 \le \alpha_b \le 1/4$$

$$\frac{46-549\alpha_b+2457\alpha_b^2-3267\alpha_b^3}{54\left(1-6\alpha_b+18\alpha_b^2-18\alpha_b^3\right)}, \text{ for } 1/4 \le \alpha_b \le 1/3$$

Limiting Condorcet Ranking Efficiency of NPR for Parameter *b*:

$$CRE(NPR,\infty/\alpha_b) = \frac{3(1-2\alpha_b)\left(2-5\alpha_b-6\alpha_b^2\right)}{8\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 0 \le \alpha_b \le 1/4$$

$$\frac{3(1-2\alpha_b)^2(1-\alpha_b)}{4\left(1-6\alpha_b+18\alpha_b^2-18\alpha_b^3\right)}, \text{ for } 1/4 \le \alpha_b \le 1/3$$

Limiting Condorcet Ranking Efficiency of BR for Parameter *b*:

$$CRE(BR,\infty/\alpha_b) = \frac{40-120\alpha_b-152\alpha_b^2+369\alpha_b^3}{48\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 0 \le \alpha_b \le 1/6$$

$$\frac{1+136\alpha_b-264\alpha_b^2-1472\alpha_b^3+2772\alpha_b^4}{192\alpha_b\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 1/6 \le \alpha_b \le 1/5$$

$$\frac{41-664\alpha_b+5736\alpha_b^2-21472\alpha_b^3+27772\alpha_b^4}{192\alpha_b\left(1-3\alpha_b-4\alpha_b^2+11\alpha_b^3\right)}, \text{ for } 1/5 \le \alpha_b \le 1/4$$

$$\frac{-77+520\alpha_b-840\alpha_b^2-1568\alpha_b^3+4484\alpha_b^4}{96(3\alpha_b-1)\left(1-6\alpha_b+18\alpha_b^2-18\alpha_b^3\right)}, \text{ for } 1/4 \le \alpha_b \le 2/7$$

$$\frac{525-6792\alpha_b+33768\alpha_b^2-75264\alpha_b^3+62744\alpha_b^4}{96(1-3\alpha_b)\left(1-6\alpha_b+18\alpha_b^2-18\alpha_b^3\right)},\ 2/7\le\alpha_b\le 3/10$$

$$\frac{5-43\alpha_b+153\alpha_b^2-177\alpha_b^3}{4\left(1-6\alpha_b+18\alpha_b^2-18\alpha_b^3\right)},\ 3/10\le\alpha_b\le 1/3$$

Limiting Condorcet Ranking Efficiency of BR for Parameter *c*:

$$CRE(BR,\infty/\alpha_c)=\frac{5(16-48\alpha_c-96\alpha_c^2+309\alpha_c^3)}{6\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }0\le\alpha_c\le 1/6$$

$$\frac{-2+448\alpha_c-1632\alpha_c^2-672\alpha_c^3+5133\alpha_c^4}{30\alpha_c\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }1/6\le\alpha_c\le 1/5$$

$$\frac{-27+948\alpha_c-5382\alpha_c^2+11828\alpha_c^3-10492\alpha_c^4}{30\alpha_c\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }1/5\le\alpha_c\le 1/4$$

$$\frac{3+468\alpha_c-2502\alpha_c^2+4148\alpha_c^3-2812\alpha_c^4}{30(3\alpha_c-1)\left(1-29\alpha_c+63\alpha_c^2-39\alpha_c^3\right)},\ \text{for }1/4\le\alpha_c\le 2/7$$

$$\frac{23-151\alpha_c+693\alpha_c^2-933\alpha_c^3}{6\left(-1+29\alpha_c-63\alpha_c^2+39\alpha_c^3\right)},\ 2/7\le\alpha_c\le 1/3$$

Limiting Condorcet Ranking Efficiency of PR for Parameter *c*:

$$CRE(PR,\infty/\alpha_c)=\frac{236-702\alpha_c+1716\alpha_c^2-6421\alpha_c^3)}{36\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }0\le\alpha_c\le 1/8$$

$$\frac{3+140\alpha_c+450\alpha_c^2-4428\alpha_c^3+5867\alpha_c^4}{36\alpha_c\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }1/8\le\alpha_c\le 1/6$$

$$\frac{-1+1488\alpha_c-4860\alpha_c^2+10584\alpha_c^3-30894\alpha_c^4}{216\alpha_c\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }1/6\le\alpha_c\le 1/5$$

$$\frac{89-312\alpha_c+8640\alpha_c^2-34416\alpha_c^3+25356\alpha_c^4}{216\alpha_c\left(16-54\alpha_c-28\alpha_c^2+139\alpha_c^3\right)},\ \text{for }1/5\le\alpha_c\le 1/4$$

$$\frac{-163+2136\alpha_c+3456\alpha_c^2-45936\alpha_c^3+62220\alpha_c^4}{216(3\alpha_c-1)\left(1-29\alpha_c+63\alpha_c^2-39\alpha_c^3\right)},\ \text{for }1/4\le\alpha_c\le 3/10$$

$$\frac{263-2727\alpha_c+12933\alpha_c^2-17685\alpha_c^3}{54\left(-1+29\alpha_c-63\alpha_c^2+39\alpha_c^3\right)},\ 3/10\le\alpha_c\le 1/3$$

Limiting Condorcet Ranking Efficiency of PR for Parameter *b:**

$$CRE\,(PR,\infty/\alpha_{b*})=\frac{-3591\alpha_{b*}^3+4401\alpha_{b*}^2-1737\,\alpha_{b*}+208}{54(18\alpha_{b*}^3-18\alpha_{b*}^2+6\alpha_{b*}-1)},\quad \text{for}$$

$1/3\le\alpha_{b*}\le 3/8$

$$\frac{61776\alpha_{b*}^4 - 102816\alpha_{b*}^3 + 60480\alpha_{b*}^2 - 14448\alpha_{b*} + 1141}{864(1-3\alpha_{b*})(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*} - 1)}, \text{ for } 3/8 \le \alpha_{b*} \le 5/12$$

$$\frac{20304\alpha_{b*}^4 - 33696\alpha_{b*}^3 + 17280\alpha_{b*}^2 - 2448\alpha_{b*} - 109}{24(3\alpha_{b*}-1)(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*} - 1)}, \text{ for } 5/12 \le \alpha_{b*} \le 1/2$$

$$\frac{432\alpha_{b*}^4 - 1728\alpha_{b*}^3 + 2592\alpha_{b*}^2 - 1392\alpha_{b*} + 149}{3456\alpha_{b*}(\alpha_{b*}-1)^3}, \text{ for } 1/2 \le \alpha_{b*} \le 2/3$$

$$\frac{32\alpha_{b*}^4 - 96\alpha_{b*}^2 + 80\alpha_{b*} - 15}{128\alpha_{b*}(1-\alpha_{b*})^3}, \text{ for } 2/3 \le \alpha_{b*} \le 3/4$$

$$\frac{7\alpha_{b*} - 3}{4\alpha_{b*}}, \text{ for } 3/4 \le \alpha_{b*} \le 1.$$

Limiting Condorcet Ranking Efficiency of BR for Parameter *b**:

$$CRE(BR, \infty / \alpha_{b*}) = \frac{15\alpha_{b*}^3 + 9\alpha_{b*}^2 - 11\alpha_{b*} + 1}{4(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*} - 1)}, \text{ for } 1/3 \le \alpha_{b*} \le 3/8$$

$$\frac{3961\alpha_{b*}^4 - 6180\alpha_{b*}^3 + 3582\alpha_{b*}^2 - 906\alpha_{b*} + 84}{12(1-3\alpha_{b*})(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*k} - 1)}, \text{ for } 3/8 \le \alpha_{b*} \le 2/5$$

$$\frac{1453\alpha_{b*}^4 - 2640\alpha_{b*}^3 + 1836\alpha_{b*}^2 - 588\alpha_{b*} + 72}{24(3\alpha_{b*}-1)(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*} - 1)}, \text{ for } 2/5 \le \alpha_{b*} \le 1/2$$

$$\frac{621\alpha_{b*}^4 - 1680\alpha_{b*}^3 + 1692\alpha_{b*}^2 - 744\alpha_{b*} + 118}{48\alpha_{b*}(\alpha_{b*}-1)^3}, \text{ for } 1/2 \le \alpha_{b*} \le 2/3$$

$$\frac{9\alpha_{b*} - 1}{8\alpha_{b*}}, \text{ for } 2/3 \le \alpha_{b*} \le 1.$$

Limiting Condorcet Ranking Efficiency of NPR for Parameter *b**:

$$CRE(NPR, \infty / \alpha_{b*}) = \frac{87\alpha_{b*}^3 - 99\alpha_{b*}^2 + 31\alpha_{b*} - 3}{8(18\alpha_{b*}^3 - 18\alpha_{b*}^2 + 6\alpha_{b*} - 1)}, \text{ for } 1/3 \le \alpha_{b*} \le 1/2$$

$$\frac{5\alpha_{b*} + 3}{16\alpha_{b*}}, \text{ for } 1/2 \le \alpha_{b*} \le 1.$$

Limiting Condorcet Ranking Efficiency of BR for Parameter *c**:

$$CRE(BR, \infty / \alpha_{c*}) = \frac{-525\alpha_{c*}^3 + 765\alpha_{c*}^2 - 335\alpha_{c*} + 31}{6(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 1/3 \le \alpha_{c*} \le 3/8$$

$$\frac{41911\alpha_{c*}^4 - 56004\alpha_{c*}^3 + 28746\alpha_{c*}^2 - 7404\alpha_{c*} + 831}{90(3\alpha_{c*}-1)(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 3/8 \le \alpha_{c*} \le 3/7$$

$$\frac{71\alpha_{c*}^4 - 10684\alpha_{c*}^3 + 11706\alpha_{c*}^2 - 3884\alpha_{c*} + 391}{60(1-3\alpha_{c*})(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 3/7 \le \alpha_{c*} \le 1/2$$

$$\frac{1991\alpha_{c*}^4 - 4604\alpha_{c*}^3 + 4026\alpha_{c*}^2 - 1724\alpha_{c*} + 351}{60(17\alpha_{c*} - 1)(1-\alpha_{c*})^3}, \text{ for } 1/2 \le \alpha_{c*} \le 2/3$$

$$\frac{1249\alpha_{c*} - 289}{60(17\alpha_{c*} - 1)}, \text{ for } 2/3 \le \alpha_{c*} \le 1.$$

Limiting Condorcet Ranking Efficiency of PR for Parameter *c**:

$$CRE(PR, \infty / \alpha_{c*}) = \frac{-17901\alpha_{c*}^3 + 22653\alpha_{c*}^2 - 9135\alpha_{c*} + 1055}{54(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 1/3 \le \alpha_{c*} \le 3/8$$

$$\frac{21390\alpha_{c*}^4 - 42696\alpha_{c*}^3 + 27540\alpha_{c*}^2 - 6456\alpha_{c*} + 409}{108(1-3\alpha_{c*})(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 3/8 \le \alpha_{c*} \le 5/12$$

$$\frac{20082\alpha_{c*}^4 - 26424\alpha_{c*}^3 + 15660\alpha_{c*}^2 - 5544\alpha_{c*} + 841}{108(3\alpha_{c*} - 1)(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 5/12 \le \alpha_{c*} \le 3/7$$

$$\frac{70503\alpha_{c*}^4 - 112860\alpha_{c*}^3 + 71226\alpha_{c*}^2 - 21420\alpha_{c*} + 2542}{108(3\alpha_{c*} - 1)(123\alpha_{c*}^3 - 99\alpha_{c*}^2 + 25\alpha_{c*} - 5)}, \text{ for } 3/7 \le \alpha_{c*} \le 1/2$$

$$\frac{1545\alpha_{c*}^4 - 4644\alpha_{c*}^3 + 5670\alpha_{c*}^2 - 3372\alpha_{c*} + 815}{108(17\alpha_{c*} - 1)(1-\alpha_{c*})^3}, \text{ for } 1/2 \le \alpha_{c*} \le 2/3$$

$$\frac{565\alpha_{c*}^4 - 1620\alpha_{c*}^3 + 1566\alpha_{c*}^2 - 540\alpha_{c*} + 27}{36(17\alpha_{c*} - 1)(\alpha_{c*} - 1)^3}, \text{ for } 2/3 \le \alpha_{c*} \le 3/4$$

$$\frac{51(1-\alpha_{c*})}{4(17\alpha_{c*} - 1)}, \text{ for } 3/4 \le \alpha_{c*} \le 1.$$

X.7 References

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