Majority Efficient Representation of the Citizens in a Federal Union*

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Abstract

All the federal unions, like the United States of America or the European Union, face the issue of finding a "good" indirect voting mechanism. In particular, a crucial question is to know how many mandates should be given to each country or state in a two-tier voting system, given that the majority rule is used at each level. We here propose a new normative criterion to evaluate these voting rules: An apportionment of the seats among the states is majority efficient if the probability of electing the candidate who receives less than 50% of the votes in a two candidate competition over the whole union is minimized. Using computer simulations, we suggest that either the proportional or the square root rule can emerge as an optimal apportionment method depending on the probability model we use to describe the electoral process.

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1 Introduction

In federal unions, a decision (or an election) involves often two steps, either because it is impossible to call the electors (decisions in the European Union, for example, where a minister

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†Marc Feix passed away on July 4^{th} 2005 at the age of 78. Though he never saw the last version of this paper, he greatly contributed to this piece of work.

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represents his country and has a certain number of mandates) or for historical reasons (the US presidential election case, for example, as the states did not want to get rid of their sovereignty in the early years of the union). In both cases, a crucial question is the choice of the "best" two tiers voting system. More specifically, how many mandates should be given to countries or states in a two tiers voting system? And which quota should be reached for a decision to be passed? Very different answers to these questions have been adopted by the different federal structures.

The simplest and most natural way is to allocate seats proportionally to the population, in order to give "equal rights" to each citizen. In the United States, the number of representatives attributed to each state is directly proportional to its population, and is derived after each census[1]. The American Electoral College process follows roughly the same principle: A state i of population n_i gets $S + R$ electors, where S comes from the number of senators (2 per state) and R comes from the number of representatives of the state in the house, which is proportional to its population. The candidate who obtains a majority of votes in states i gets all its electors [2] and the candidate who gets a majority of vote in the Electoral College becomes the new president.

In the EU, the number of mandates attributed to each country by the treaty of Nice is very roughly proportional to a $n_i^{1/2}$ law, but with huge fluctuations and a staircase type curve. Each enlargement gave rise to a negotiation among the member states, and no specific rule has never prevailed either on the number of mandates per state or about the threshold needed to pass a decision. The project for an European constitution was an attempt to define a more rigorous method, and proposed to conciliate the "one man-one vote" and the "one state-one vote" justifications by the 'double key vote' : to be approved, a proposal should have been supported by 55% of the countries, their population gathering at least 65% of the total EU population.

Thus, when we look at the American and European cases, we find systems that try to navigate in between the pure federal system of "one state-one vote" (the first threshold in the European constitution, the +2 premium per states in the Electoral College) and the more democratic representation of the states proportionally to their population. Clearly, for all these schemes, the outcome was the result of a political bargain between the small states and the big states. There is also barely no reference to any specific normative criterion that could be used in order to precise what should be the good federal decision process. This apparent confusing situation is partly due to the fact that the democratic equal treatment for each citizen principle is of course easily implemented in a direct elections (for example, imagine a direct popular vote for the US presidential election), but may have different solutions in a two-tier system.

In this paper, we propose a new normative criterion to evaluate the different two-tier voting methods and study its consequences : *a method is said to be majority efficient if it minimizes the probability that a decision is taken with a majority of mandates at the federal level though it is supported by a minority of voters over the whole union.* It is equivalent to the concept of *Condorcet efficiency* that has been developed in Social Choice Theory in order to discriminate

[1] However, there has been of lot of debates in the United States since two centuries in order to find the right mechanism to round off the number of representatives per state proportionally to the population. This problem is very well documented, see for example the book by Balinski and Young (1982). In this paper, by proportionality, we mean that the weight attached to a state is exactly its population, and we will not discuss the rounding off issue.

[2] The "winner takes all" principle is used in all the states but Nebraska and New Hampshire.

among the voting rules on their capacity to pick out the Condorcet winner whenever it exists[3], the main difference being that the majority winner is always well defined in a two candidate election. More specifically we will seek for the best two-tier voting systems, using this normative criterion, when all the decisions are made through majority voting among two candidates. In other words, we wish to minimize the likelihood of the so called *referendum paradox* (see Nurmi (1998)): A referendum paradox occurs whenever a decision taken by representatives elected in local jurisdictions conflicts with the decision that would have been adopted if the voters had directly given their opinion through a referendum. The question of the best threshold to use to pass decisions is not further investigated in this paper.

The majority efficient criterion seems very natural, as it is not only a theoretical object; such strange political situations often happen, a well known case being the election of George W. Bush against Al. Gore in 2000 (for other examples in US, United Kingdom and France, see Feix *et al.* (2004)). Moreover, it is quite obvious that the existence of a federal union may be put in danger if it is plagued too often by these situations: A majority of the citizens would lose confidence in the institutions, leading to a political crisis. We also believe that, as the referendum paradox has been popularized by the media recently, the criterion of majority efficiency could be more easily accepted by the public opinion than other normative criteria that have been presented in the literature; We will review this literature and present some related definitions and concepts in section 2.

Though the criterion of majority efficiency is simple to identify and popularize, the search of the apportionment rule which minimizes the occurrence of the referendum paradox is not that simple: We have not been able to resolve analytically the problem for more than three states, and we cannot prove mathematically that it points toward a clear optimal apportionment method. The situation is similar to the one we encounter in Social Choice Theory for the evaluation of the Condorcet efficiency of different voting rules: Unless the number of parameters (voters, alternatives, and here states) is very small, we have to rely on computer simulations. As we cannot test all the possible apportionment methods, we will focus our analysis on a particular class of rules, the $\delta - rules$. More precisely, through simulations, we will try to identify which parameter δ minimizes the probability of the paradox if we allocate the a_i seats for state i according to the law $a_i = n_i^{\delta}$. Although this formula does not take into account all the possibilities, it covers the pure federal case ($\delta = 0$), the square-root rule ($\delta = 1/2$,), the proportional case ($\delta = 1$), and even the dictatorship of the biggest state ($\delta \to \infty$).

In order to compute the probability of the referendum paradox, we need to set some *a priori* assumptions on the behavior of the voters. The probabilistic models we will use throughout the paper convey some notions of impartiality: We assume that each party is equally likely to win, and we simulate the votes without any reference to a precise political context under different probability models. Though it may be possible to specify some parameters for the distribution of the votes with applied works in some specified contexts, we do not consider this possibility here. Thus we will focus our analysis on two models previously introduced in the Social Choice literature, the Impartial Culture (IC) and the Impartial Anonymous Culture (IAC) assumptions. The model, the probability assumptions and the methodology concerning simulations will be presented in detail in Section 3.

Section 4 presents the only case we could completely resolve analytically, that is the three state case.

Section 5 is devoted to a more general study of the δ-rules. In particular, we test some

[3]For more on this literature, see Gehrlein (2006).

conjectures on what should be the best voting rule according to the probabilistic assumptions, based upon the results of the numerous simulations we have carried on. Conclusions from these extensive studies will be drawn in section 6.

2 Normative Criteria for Two-Tier Voting Rules

In this section, we present the different normative criteria that have been proposed to evaluate two-tier voting systems. The contribution can be gathered in two categories. Historically, the first contributions were linked to game theory (more specifically to the power indices literature), and their objective was to give to each voter the same influence on the decision process. It is only recently that new criteria, all based on some kinds of utility principle, have been proposed.

2.1 Equalizing power and influence

Perhaps, the most widely-publicized normative political claim about two-tier systems comes from game theorists and the voting power literature. Since the works of Penrose (1946) and Banzhaf (1965), many scholars defend the so called *Penrose square root law*, on the basis that, under a very simple model, the voting power or influence of an individual from state i is proportional to the inverse of the square root of the population of his home state ($1/\sqrt{n_i}$). As the power of a state in the Union is also roughly proportional to its number of mandates, equal treatment in term of power is achieved when each state gets a number of mandates proportional to the square root of its population. The recent book by Felsenthal and Machover (1998) and their papers on the European Union (2001,2004) are perfect examples of this tradition.

Let us now describe more precisely the tools and concepts from the power index literature[4]. The model begins with the assumption that n voters have to choose between two proposals A and B. These two options are exclusive, abstention is not allowed, and there is no bias in favor of one alternative (such as a statu quo alternative). We next assume that each vote is determined by flipping independently a fair coin randomly; In game theory, this hypothesis has been called the Independence assumption by Straffin (1977), and it is equivalent to the Impartial Culture model used in social choice literature for the computation of voting paradox probabilities (see the book by Gehrlein (2006)). A vote configuration is the list of the ballots chosen by the voters. Under the IC random voting model, all the 2^n vote configurations are equally likely, and the power of voter j is simply the proportion of the configurations of the other $n-1$ votes for which voter j is decisive. By decisive, we mean that voter j, by changing his vote, can affect the final outcome[5]. The power of voter j is then the number of situations where he is decisive divided by the total number of vote configurations, 2^n. In fact, we have just described a well known measure of power the (non normalized) *Banzhaf Power index*:

$$\text{Banzhaf power of voter j} = \frac{\text{Number of configurations for which voter j is decisive}}{\text{Total number of voting configurations}} \tag{1}$$

In a federal union, a voter casts his vote in his home state for party A or B. The winner in state i is the party which obtains a majority of votes on his side (abstention is not allowed)

[4]The reader can also find a very nice introduction to these concepts in the recent papers by Gelman *et al.* (2002) and Gelman *et al.* (2004).

[5]If the number of mandates is even, ties may occur. A way to avoid such situation is to assume that the number of mandates is odd, or to flip a fair coin to take a decision in case of a draw, or to ask for a new election until a clear decision is obtained, etc.

among the n_i citizens. Each state i is represented at the federal level by a_i mandates, and the winner in state i catches all these mandates. Then, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level. Thus, the probability for a voter to be decisive is the product of the probability that he is decisive in his home state times the probability his state is decisive at the federal level.

$$\begin{aligned}\text{Probability that } j \text{ is decisive in the union} &= \text{Probability that state } i \text{ is decisive} \\ &\quad \times \text{Probability that voter } j \text{ is decisive in state } i\end{aligned} \tag{2}$$

It is well know that if certain conditions hold - if the number of states is large enough, if no single state or handful of states controls almost all the mandates, and there are no discrete features in the mandates - that the Banzhaf power of state i with a_i mandates is approximatively proportional to its number of mandates[6].

$$\frac{\text{Banzhaf power of state } i \text{ with } a_i \text{ mandates in the union}}{\text{Banzhaf power of state } i' \text{ with } a_{i'} \text{ mandates in the union}} \approx \frac{a_i}{a_{i'}} \tag{3}$$

But in two-tier voting, we also have to estimate the power of voter j in state i. If n_i is odd, voter j is decisive for

$$\binom{n_i - 1}{(n_i - 1)/2} \tag{4}$$

configurations among the $2^{n_i - 1}$ possible vote configurations of the other citizens of state i. For n_i large, this can be approximated as $\sqrt{\frac{2}{\pi n_i}}$. When the behavior of the voters is governed by the IC assumption, we immediately deduce that equation (2) gives:

$$\frac{\text{Probability that voter } j \text{ from state } i \text{ is decisive in the union}}{\text{Probability that voter } j' \text{ from state } i' \text{ is decisive in the union}} \approx \frac{a_i}{a_{i'}} \frac{\sqrt{n_{i'}}}{\sqrt{n_i}} \tag{5}$$

and that equal treatment in term of power is achieved if a_i is proportional to $\sqrt{n_i}$.

A problem with this approach is that, in real life, voters seldom flip coins independently before casting their vote. By analyzing electoral data from the least fifty years, Gelman *et al.* (2004) have recently showed that Straffin's Independence assumption had to be rejected for the elections of the senators, the representatives and president in the United States; similar conclusions are drawn from the electoral data collected over Europe. A way out of this problem is to recognize that the probability of being decisive depends on the probability of the configurations for which a voter is decisive. Then other probability assumptions, also modelling a particular instance of the veil of ignorance hypothesis, can be used. In particular, Straffin (1977) and Berg (1999) clearly show that if each repartition of the votes between A and B is equally likely

[6] Penrose's limit theorem (Penrose (1952)) says that, in simple weighted majority game, if the number of voters increases indefinitely and the relative quota is pegged, then, under certain conditions, the ratio between the voting power of any two voters converge to the ratio between their weights. However, this result in only implicit in Penrose's works, and the result is only valid "most of the time", as counter examples exist. Feix *et al.* (2007) have recently shown that proportionality of the power is met for in the enlarged European Union if we attribute to each state a number of mandate proportional to the square root of its population. Lindner and Machover (2004) have proposed a formal version of Penrose's statement. Using simulations, Chang *at al.* (2006) have tested the validity of the approximation for numerous repartition of the population among states and have shown that it is valid with a probability close to one for the non normalized Banzhaf index and a quota of 50%. Independently, Feix *et al.* (2007) and Slomiczyǹski and Zyczkowski (2008) have pointed out that the proportionality is even better for some super majority rules.

(that is, the probability that A gets 0% of the vote is equal to the probability that he gets 15%, 51% or 89% of the vote), the probability attached to a configuration with t votes for A and $(n-t)$ votes for B is no longer $1/2^n$, but is now given by:

$$\frac{1}{(n+1)\begin{pmatrix} n \\ t \end{pmatrix}} \tag{6}$$

This is Straffin Homogeneity assumption; one should also notice that the Impartial Anonymous Culture used in Social Choice theory to compute the likelihood of the Condorcet paradox gives the same probability when used in binary elections (see Gehrlein, (2006)). Then, the right measure of power in state i, that is the probability of being decisive in state i, is now given by the *Shapley-Shubick index* (Shapley and Shubik (1954)). There are $\binom{n_i-1}{(n_i-1)/2}$ configurations of the other voters which split equally between A and B in state i, for which voter j is pivotal whether he votes for A or B. Thus:

$$\begin{aligned} \text{Shapley-Shubick power for player } j \text{ in state } i &= \frac{2\begin{pmatrix} n_i - 1 \\ (n_i-1)/2 \end{pmatrix}}{(n_i+1)\begin{pmatrix} n_i \\ (n_i-1)/2 \end{pmatrix}} \\ &= \frac{1}{n_i} \end{aligned} \tag{7}$$

and when the IAC assumption is used to model the behavior of the voters:

$$\frac{\text{Probability that } j \text{ from state } i \text{ is decisive in the union}}{\text{Probability that } j' \text{ from state } i' \text{ is decisive in the union}} \approx \frac{a_i}{n_i}\frac{n_{i'}}{a_{i'}} \tag{8}$$

Notice that each state still vote for A or B with probability 1/2, ensuring that approximation (3) still holds. Equal treatment is now obtained when the number of mandates is proportional to the population.

Thus, the objective of equalizing the probability of being decisive in two-tier systems has no clear answer. To illustrate this fact, notice that Napel and Maaser (2007) have recently recovered the square root rule law in a model with a continuum of options, while Owen *et al.* (2006) found a completely different picture by adapting the Shapley Shubik index to cope with the US electoral data. The choice of the right apportionment rule is completly driven by the characteristic of the underlying probability model governing the behavior of the voters, which in turn defines a particular measure of the power.

2.2 Utility Based Arguments

The argument that the citizens of the different states should be given equal power, that is equal probability to be decisive, was for a long time the only normative argument proposed by mathematical economists to evaluate the merits of a federal constitution. Nevertheless, the literature based upon the notion of pivotal voters have been often criticized: a classical argument against the power indices approach is that the influence of a single voter is in any case extremely low in federal bodies like the USA or the EU, of magnitude a_i/n_i or $a_i/\sqrt{n_i}$! Thus, it is unlikely that citizens would fight for a fairer representation based upon the notion of power.

Concepts such as equal satisfaction or equal opportunity of success are surely more bitting. Rae (1969) was the first author who proposed a clear definition of success: Using the independence assumption, he defines his index as the probability of being in the winning side. However, it is well known that the Rae index can be linked to the Banzhaf index and should then give equivalent normative recommendation (see for example Felsenthal and Machover (1998), Laruelle and Valenciano (2005)). Thus, again, the same argument applies: Laruelle and Valenciano (2008) argue that for many voting rules, the differences in term of success are so thin among the citizens that this concept does not bite either.

In fact, the development of new criteria is due to a shift to the aggregated level, with measures based upon the total utilities of the members of the society.

Ten years ago, Felsenthal and Machover (1999) suggested that, for a federal union, the mean difference between the size of the majority camp among all citizens and the number of citizens who agree with the decision taken by the majority of the delegates of the states should be minimized. They showed that under the Independence assumption, the *mean majority deficit* could be linked to the Banzhaf index, and concluded that the Penrose's square root rule holds again.

Beisbart *et al.* (2005) have more recently compared seven different possible decision schemes for the European Union on their capacity to select the motions that will have a positive total utility for its citizens, while rejecting the bad ones (which give a negative total utility). They distinguish between the benchmark rules (Simple majority with weights proportional to population, simple majority with equal weights, the combination of simple majorities of states and populations, Penrose weights with 62% threshold) and the political rules (Nice treaty, original draft constitution, present draft constitution). For each country, the utility of a policy for the representative citizen in state i is drawn from a normal law of mean μ and standard deviation σ. Using computer simulations, they compute the average utility brought by each decision rule. They also check whether some countries (the small ones, the big ones) are systematically harmed. Again, they discover that no decision rule performs best independently of all model parameters. For bad motions (μ/σ values significatively smaller than 0) the political rules tends to outperform other rules in terms of expected utility, because they are less permissive and thus effectively block motions with too many negative utilities. On the other hand, when $\mu/\sigma > 0$ benchmark rules tend to outperform political rules.

The normative criterion proposed by Barberà and Jackson (2006) uses similar ideas, but is more general with respect to several points. They assume that in a two-candidate election, the partisans of candidate A get a utility of $u_j = 1$ if he is elected (and 0 otherwise), while the partisans of B get a utility of $u_j = v, v \in [0, +\infty[$ if their preferred candidate is elected (and 0 otherwise). Thus, one camp may enjoy a higher utility when winning. They also assume, that, at the federal level, a motion is passed if it is supported by q% of the mandates, with q possibly different from 1/2. Then, the optimal voting rule for two-tier elections should be the one that maximizes the total expected utility of the voters. We call such a criteria the *utility efficient* principle. Their first results are very general in the sense that they do not depend upon a particular probability model; The *utility efficient* two-tier voting rule should be such as:

- The quota of the mandates in order to select A against B is $\frac{v}{v+1}$. This result justifies the existence of a quota superior to one-half when one camp (here, the partisan of B if $v > 1$) is particularly harmed when defeated. However, interpersonal comparisons of utility govern the level of the quota.

- The optimal weight of state i is proportional to the total expected utility of the voters

i knowing that A is selected:

$$a_i = E\left[\sum_{j=1}^{n_i} u_i \,|\, \text{A is elected}\right].$$

Then depending on the assumption that are made to model *a priori* the behavior of the voters, different efficient weights can emerge. They retrieve that the independence assumption points towards the square root rule, while another probability assumption, which assumes that the variance of the results in a state is independent from the size of the state, leads to the proportional rule.

Clearly, the majority efficient criterion that we defend in this paper belongs to the same class as the ones proposed by Felsenthal and Machover (1999), Beisbart *et al.* (2005), and Barberà and Jackson (2006). The main difference with the utilitarian efficiency criterion discussed above is that we do not here take into account the magnitude of the paradox: We try to evaluate the number of the cases for which a majority of voters is frustrated, but we do not weight our results by the magnitude of the paradox, either by counting the number of dissatisfied voter (as in Felsenthal and Machover (1999)) or adding up the utilities (as in Barberà and Jackson (2006) and Beisbart *et al.* (2005).

Perhaps the most important conclusion that we can draw from this review of the literature is that, whatever the criteria (equalizing power or success, maximizing the utility), the choice of the optimal apportionment rule seems to be driven by the underlying probability assumptions. The independence assumption seems to always point toward the square root rule, while models which assume that the variance of the results are constant favor mandates proportional to the size of the population. Thus, it is clear that one of the question at stake in this paper is to know whether the square root rule (resp. the proportional rule) will be the optimal apportionment rule is terms of majority efficiency when the IC (resp. IAC) assumption is used.

3 Methodology

3.1 The model

Consider a finite set $I = \{1, \ldots, i, \ldots, N\}$ of states (or regions, districts, etc.) which have to take decisions altogether in a political union. We assume that n_i voters live in state i, and $\sum_{i=1}^{N} n_i = n$. The vector $\tilde{n} = (n_1, \ldots, n_i, \ldots, n_N)$ describes the repartition of the population among the N states. Without loss of generality, we will assume throughout the paper that $n_1 \geq n_2 \ldots n_N > 0$. Two parties, A and B, compete in all the states; the winner in state i is the party who obtains a majority of voters on his side (abstention is not allowed). Each state is represented by a_i mandates in the union, and the winner in state i gets all the mandates. For the sake of simplicity, we set that $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$, with at least a_1 strictly positive. Thus, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level. Notice that we always use throughout the paper the quota of 50% for all the decisions (votes in the states, vote of the delegates and popular vote nationwide).

In our search of the apportionment rules that minimize the probability of the referendum paradox, we have decided to focus our study on the family of δ-rules. That is, we assume that the vector of mandates, $\tilde{a}$, is entirely characterized by the parameter δ, $\delta \in [0, \infty[$ as $a_i = n_i^{\delta}$ $\forall i = 1, \ldots, N$. Though restrictive, this assumption enable us to encompass the pure federalist case ($\delta = 0$ and each state has one mandate), the square root rule case ($\delta = 1/2$), the pure proportionality case ($\delta = 1$) and even the dictatorship of the biggest state ($\delta \to \infty$). But clearly, our main objective is to check whether the recommendations we should adopt when we wish to minimize the likelihood of the referendum paradox are compatible with solutions that have been put forward when one wishes to equalize the power of the citizens ($\delta = 1$ for IAC, $\delta = 0.5$ for IC).

3.2 On probability assumptions and their implications

As seen in section 2, there are several ways to model theoretically the behavior of the citizens; we present here these assumptions with more details and discuss their relevance. We model the people's vote inside each state and we assume that their behavior is described by the same probability distribution in every state. Thus, the probabilistic behavior of a given state at the federal level is totally driven by the behavior of its voters.

In the IC model, each citizen votes independently of the others and selects among the two issues with equal probability. Due to the law of large numbers, the scale of the mean difference of ballots between the two issues will vary in $n_i^{1/2}$. The other classical model in social choice theory is the IAC model. It assumes that every repartition of the votes between A and B is equally likely. Many interpretations of this model have been given (going from Polya urns to quantum Bose-Einstein statistics), a fine one being the one proposed by Berg (1999). The idea is the following: In state i, for a given election, a "public opinion" emerges, *i.e.* an individual probability p_i for selecting one of the issue is drawn from the uniform distribution on $[0, 1]$. Thus, the probability of picking A, may be 0.1, 0.5, 0.7 or whatever you want in [0,1], with equal probability. Of course, p_i varies from one election to the other, but in average, there no bias in favor of one alternative.

It is possible to consider IC and IAC models as specific cases of a Generalized Impartial Anonymous Culture model or GIAC. To develop Berg's reasoning, suppose that the choice of a probability p is itself of probabilistic nature through the introduction of a probability distribution function $f(p)$. The choice of $f(p)$ is a first step for a better description of the electorate behavior. In particular, it could be determined after the study of real data. The distribution $f(p)$ is defined on $0 \leq p \leq 1$, with $f(p) \geq 0$ and $\int_0^1 f(p)dp = 1$. The probability of a given configuration of n identified voters with t votes for A and $(n-t)$ votes for B is $p^t(1-p)^{n-t}$, and for a large number of elections it reads

$$\int_0^1 f(p)p^t(1-p)^{n-t}dp. \tag{9}$$

If $f(p) = \delta(p - 1/2)$, where δ is the Dirac distribution function, that is if p is equal to $1/2$ for all elections, the IC model is recovered with a probability of $1/2^n$ for all configurations. For $f(p) = 1$, we get the usual IAC and we obtain

$$\frac{1}{n+1\begin{pmatrix} n \\ t \end{pmatrix}} \tag{10}$$

for the probability of a configuration with t votes for A and $n-t$ votes for B.

A third case of GIAC is of special interest, it is Rescaled IAC assumption (RIAC) for which

Voter 1	A	A	A	A	B	B	B	B
Voter 2	A	A	B	B	A	A	B	B
Voter 3	A	B	A	B	A	B	A	B
IC	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
IAC	1/4	1/12	1/12	1/12	1/12	1/12	1/12	1/4
$RIAC$	0	1/6	1/6	1/6	1/6	1/6	1/6	0
UC	1/2	0	0	0	0	0	0	1/2

Table 1: The 8 voting configurations for 3 voters and the associated probabilities according to IC, IAC, $RIAC$ and UC.

the p_i are independent and are drawn from the distribution

$$f_i(p_i) = \begin{cases} \frac{1}{2\Delta_i} & \text{if} \quad \frac{1}{2} - \Delta_i < p_i < \frac{1}{2} + \Delta_i \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

for $i = 1, \ldots, M$. Δ_i is a positive value less than 1/2. This model has been used first by Feix *et al.* (2004) under the name of Biased and Rescaled IAC (BRIAC), with the possibility of a bias in favor of one candidate that we omit in this paper. If $\Delta_i = 1/2$, for all i, the IAC is recovered, while the limit $\Delta_i \to 0$, for all i, the RIAC model tends toward the IC model if the population is finite according to Berg's interpretation [7]

RIAC can be interpreted as follows: It means that a percentage $1/2 - \Delta_i$ of the population of state i always vote for A, while the same percentage always vote B; only a fraction $2\Delta_i$ of the population judges the two alternatives. Thus, the RIAC model can be used to overcome the limitations of both the IC model (elections are too close) and the IAC model (the range of the results is too spread). For example, $\Delta_i = 0.2$ means that the results vary from 30% of the votes for A to 70% of the votes for A from elections to elections, quite a realistic pattern!

As an illustration of these different probability assumptions, we display in Table 1 the probability of the different voting situations for three voters. There are eight voting configurations. Under IC, all the configuration weights are equally likely. On the contrary, IAC puts the same weight on all identical configurations independently of the voters. Then AAA has the same weight 1/4 than AAB+ABA+BAA altogether (each having 1/4/3=1/12). RIAC with $\Delta = 1/6$ excludes the cases of unanimous vote: A and B receive at least 1/3 of the votes each. Another extreme case is displayed on the last line of Table 1 : The unanimous culture (UC), wherein all the voters always share the same preference in a given state. Trivially, in this last case, perfect proportionality for the number of mandates a_i would eliminate the referendum paradox!

3.3 Simulation techniques

In Feix *et al.* (2004), we have studied analytically, and through Monte Carlo simulations as soon as N is greater than 5, the conflict frequency between the popular and state votes. In this study, all the states were supposed to have the same population and consequently the same

[7] However, one has to be careful when Δ_i tends towards 0; the global behavior of the model changes. Resolution techniques that can be used for the RIAC case by assuming that the population is infinite cannot be applied when Δ_i becomes too small compared to n_i. In fact, the two limits $\Delta_i \to 0$ and $n_i \to \infty$ do not comute. If first $n_i \to \infty$, then $\Delta_i \to 0$, the model is a RIAC one, but if $\Delta_i \to 0$ then $n_i \to \infty$, the IC model is reached when roughly $\sqrt{n_i} \sim \Delta_i$.

number of mandates. Only small differences were found between IC and IAC models: In the limit of 101 states, the probabilities of the referendum paradox seem to stabilize around 16.5% for IAC and around 21.5% for IC. The results of these simulations are confirmed by Lepelley *et al.* (2008) with approximations based upon the law of large numbers. Here we study the conflict frequency between the direct (popular) vote and the two-tier decision when the size of the population can differ from state to state, for a given apportionment of the mandates, and under different probability assumptions; we denote this probability by $P(N, \tilde{n}, \tilde{a}, GIAC)$, where N is the number of states, $\tilde{n}$ the distribution of the population among the N states, $\tilde{a}$ the apportionment rules we study and GIAC is either IC or (R)IAC. When $a_i = n_i^\delta$, we will simply write the probability $P(N, \tilde{n}, \delta, GIAC)$. Notice that all the probabilities will be estimated for such large n_i's that the discrete nature of the population size does not play a role. Thus, $\Delta_i \to 0$ does not mean that the RIAC model converges to the IC case, as it should be if the populations were finite.

For the IC model, each voter selects a party with equal probability. When n_i is sufficiently large, the distribution of the votes follows a normal law. In each state, the excess of ballots for A or B is then given by

$$\varepsilon_i \sqrt{n_i} \tag{12}$$

where ε_i is drawn randomly according to the Gauss distribution:

$$(2\pi)^{-1/2} \exp(-\varepsilon^2/2). \tag{13}$$

The popular vote over the whole union is given by

$$\mathrm{sgn}\left(\sum_i \varepsilon_i \sqrt{n_i}\right), \tag{14}$$

while the decision taken by the representatives is given by

$$\mathrm{sgn}\left(\sum_i a_i \, \mathrm{sgn}(\varepsilon_i)\right) \tag{15}$$

The sgn function is defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{si} \quad x > 0 \\ -1 & \text{si} \quad x < 0 \end{cases} \tag{16}$$

By convention, a one value (respectively minus one value) results in the selection of candidate A (respectively B). A difference in sign between (14) and (15) means that we observe a paradox.

In the family of GIAC models, we will particularly study the RIAC models, among which the IAC assumption is a special case. We consider that p_i, the probability of voting for A in state i, are independent and are drawn from the distribution:

$$f_i(p_i) = \begin{cases} \frac{1}{2\Delta_i} & \text{if} \quad 1/2 - \Delta_i < p_i < 1/2 + \Delta_i \\ 0 & \text{elsewhere} \end{cases} \tag{17}$$

for $i = 1, \ldots, N$. Δ_i is a strictly positive value less than $1/2$. If $\Delta_i = 1/2$, for all i, the IAC is recovered. Thus, the excess of ballots for A (or B) is given by $\varepsilon_i \Delta_i n_i$ where ε_i is drawn from the uniform distribution on $[-1, 1]$. Notice that we can assume different $\Delta_i \neq 0$ for the different states. Then, we have to compare:

$$\operatorname{sgn}\left(\sum_i \varepsilon_i \Delta_i n_i\right) \tag{18}$$

for the popular vote with the vote of the representatives

$$\operatorname{sgn}\left(\sum_i a_i \operatorname{sgn}(\varepsilon_i)\right) \tag{19}$$

We can reinterpret the RIAC model with the n_i's as an IAC model with new populations given by $n'_i = \Delta_i n_i$. In this case, the RIAC distribution becomes a classical IAC one:

$$f'_i(p_i) = \begin{cases} 1 & \text{if} \quad 0 < p_i < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

Let us draw ε_i from an uniform distribution on $[-1, 1]$. Thus, we have to compare the total excess ballots, given by:

$$\operatorname{sgn}\left(\sum_i \varepsilon'_i n'_i\right) \tag{21}$$

with the vote of the representatives, given by

$$\operatorname{sgn}\left(\sum_i a'_i \operatorname{sgn}(\varepsilon'_i)\right) \tag{22}$$

Equations (18)-(19) describe the same problem as equations (21)-(22) and the two interpretations are equivalent. In words, the probability of the paradox for a RIAC model with respective weights $n_1 = 4$, $n_2 = 3$ and $n_3 = 3$ while $\Delta_1 = 0.1$, $\Delta_2 = 0.2$ and $\Delta_3 = 0.3$, is given by a simulation with the IAC model with new populations $n'_1 = 0.4$, $n'_2 = 0.6$ and $n'_3 = 0.9$. Moreover, when $\Delta_i = \Delta$ for all the states, the results are directly given by an IAC simulation.

Thus, we can focus on the two models IC and IAC to test the optimality of the different apportionment rules in various scenario using Monte Carlo techniques to simulate a large number of votes for a large number of states. The only analytical results we could provide are for three states; before turning to the simulations, we first present these results.

4 The Three-State Federation

4.1 A limited number of cases

Without loss of generality, we assume in the 3-state case that the distribution of the population is given by the vector $\tilde{n} = (n_1, n_2, n_3)$, with $\sum_{i=1}^{3} n_i = n = 1$, and $n_1 \geq n_2 \geq n_3 > 0$. Similarly, we assume that the distribution of the mandates is given by $\tilde{a} = (a_1, a_2, a_3)$, with $a_1 \geq a_2 \geq a_3 \geq 0$, and $a_1 > 0$. We consider δ-rules only, $\tilde{a} = (n_1^\delta, n_2^\delta, n_3^\delta)$. But, for a weighted majority game, it is well known that any vectors $\tilde{a} = (a_1, a_2, a_3)$ can be identified with one of these four possible cases :

Table 2: The minimal values for the referendum paradox for three states under IAC.

$n_2 \downarrow \; n_3 \rightarrow$	0^+	0.05	0.10	0.15	0.20	0.25	0.30	0.333
0^+	$\mathbf{0^+}$	−−	−−	−−	−−	−−	−−	−−
0.05	**0.0132**	**0.0185**	−−	−−	−−	−−	−−	−−
0.10	**0.0278**	**0.0319**	**0.0417**	−−	−−	−−	−−	−−
0.15	**0.0411**	**0.0486**	**0.0574**	**0.0714**	−−	−−	−−	−−
0.20	**0.0625**	**0.0681**	**0.0774**	**0.0913**	**0.1111**	−−	−−	−−
0.25	**0.0833**	**0.0904**	**0.1013**	**0.1167**	**0.1379**	**0.1666**	−−	−−
0.30	**0.1072**	**0.1165**	**0.1296**	**0.1477**	**0.1722**	0.1509	0.1242	−−
0.333	**0.1250**	**0.1361**	**0.1514**	**0.1722**	0.1646	0.1431	0.1284	0.1250
0.35	**0.1346**	**0.1468**	**0.1634**	**0.1857**	0.1614	0.1405	0.1276	−−
0.40	**0.1666**	**0.1827**	**0.2042**	0.1786	0.1563	−−	−−	−−
0.45	**0.2045**	**0.2259**	0.2006	−−	−−	−−	−−	−−
0.50^-	$\mathbf{0.25^-}$	−−	−−	−−	−−	−−	−−	−−

In bold: probabilities derived from $P(3, \tilde{n}, \tilde{a}^3, IAC)$.
Underlined: probabilities derived from $P(3, \tilde{n}, \tilde{a}^1, IAC)$.

Table 3: The minimal values for the referendum paradox for three states under IC.

$n_2 \downarrow \; n_3 \rightarrow$	0^+	0.05	0.10	0.15	0.20	0.25	0.30	0.333
0^+	$\mathbf{0^+}$	−−	−−	−−	−−	−−	−−	−−
0.05	**0.0718**	**0.1024**	−−	−−	−−	−−	−−	−−
0.10	**0.1024**	**0.1266**	**0.1476**	−−	−−	−−	−−	−−
0.15	**0.1266**	**0.1476**	**0.1666**	0.1813	−−	−−	−−	−−
0.20	**0.1476**	**0.1666**	**0.1845**	0.1773	0.1727	−−	−−	−−
0.25	**0.1666**	**0.1845**	0.1824	0.1747	0.1698	0.1666	−−	−−
0.30	**0.1845**	0.1952	0.1810	0.1730	0.1679	0.1648	0.1630	−−
0.333	**0.1959**	0.1948	0.1804	0.1722	0.1671	0.1641	0.1625	0.1623
0.35	**0.2015**	0.1946	0.1801	0.1719	0.1668	0.1639	0.1624	−−
0.40	**0.2180**	0.1943	0.1796	0.1714	0.1665	−−	−−	−−
0.45	**0.2341**	0.1941	0.1994	−−	−−	−−	−−	−−
0.50^-	$\mathbf{0.25^-}$	−−	−−	−−	−−	−−	−−	−−

In bold: probabilities derived from $P(3, \tilde{n}, \tilde{a}^3, IC)$.
Underlined: probabilities derived from $P(3, \tilde{n}, \tilde{a}^1, IC)$.

- Case 1. $\tilde{a}^1 = (1, 1, 1)$. All the states have the same power. $\tilde{a}$ is equivalent to $\tilde{a}^1$ if and only if $n_1^\delta < n_2^\delta + n_3^\delta$.
- Case 2. $\tilde{a}^2 = (2, 1, 1)$. $\tilde{a}$ is equivalent to $\tilde{a}^2$ if and only if $n_1^\delta = n_2^\delta + n_3^\delta$. In case the opinion of states 2 and 3 conflicts with the choice of state 1, a tie breaking rule could be implemented.
- Case 3. $\tilde{a}^3 = (1, 0, 0)$ and state 1 is a dictator. $\tilde{a}$ is equivalent to $\tilde{a}^3$ if and only if $n_1^\delta > n_2^\delta + n_3^\delta$.
- Case 4. $\tilde{a}^4 = (2, 2, 0)$. Player 3 is a dummy player and in case of opposite opinion for the two decisive states, a tie breaking rule could be implemented. But no δ-rule can encompass this case, as $n_3 > 0$.

Throughout the study, we will always observe theses facts: the number of underlying weighted majority games that support the δ-rules is always finite, and not all weighted majority games can be represented by a δ-rule.

4.2 Case 1, $\tilde{a}^1 = (1, 1, 1)$

We focus first on $\tilde{a}^1 = (1, 1, 1)$, the most interesting case.

Proposition 1 *Let $P(3, \tilde{n}, \tilde{a}^1, IAC)$ be the likelihood of the referendum paradox for three states of large population under IAC for the distribution $\tilde{n}$ when each state gets one mandate. Then:*

$$P(3, \tilde{n}, \tilde{a}^1, IAC) = \frac{n_1^3 + n_2^3 + n_3^3 - (n_1 - n_2)^3 - (n_1 - n_3)^3 - (n_2 - n_3)^3}{24 n_1 n_2 n_3} \; if \; n_1 < \frac{1}{2} \tag{23}$$

$$P(3, \tilde{n}, \tilde{a}^1, IAC) = \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3 + 6 n_2 n_3 (n_1 - n_2 - n_3)}{24 n_1 n_2 n_3} \; if \; n_1 > \frac{1}{2} \tag{24}$$

Proof: see Appendix I.

Proposition 2 *Let $P(3, \tilde{n}, \tilde{a}^1, IC)$ be the likelihood of the referendum paradox for three states of large population under IC for the distribution $\tilde{n}$ when each state gets one mandate. Then:*

$$P(3, \tilde{n}, \tilde{a}^1, IC) = \frac{\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) + \arccos\left(\sqrt{n_3}\right)}{\pi} - 0.75 \tag{25}$$

Proof: see Appendix II.

4.3 Case 2, $\tilde{a}^2 = (2, 1, 1)$

This very specific case only occurs when $n_1^\delta = n_2^\delta + n_3^\delta$. Moreover, we have to decide how to interpret a 2:2 deadlock, when state 1 votes for A, while states 2 and 3 endorse B. A latitudinarian interpretation of the definition of the referendum paradox would be to consider this situation is never a paradox, as the popular winner is not defeated with the indirect voting rule. On the opposite, a strict version of the paradox would consider all these situations as paradoxical, if one posits that the popular winner should win with no discussion. In order to derive probabilities, we will adopt a medium term. In case of 2:2 deadlock, we assume that the election is decided by tossing a fair coin, which means that only half of these situations are considered as paradoxical, depending whether or not the popular winner wins the draw.

Proposition 3 *Let $P(3, \tilde{n}, \tilde{a}^2, IAC)$ be the likelihood of the referendum paradox for three states of large population under IAC for the distribution $\tilde{n}$ when $\tilde{a} = \tilde{a}^2$. Then:*

$$P(3, \tilde{n}, \tilde{a}^2, IAC) = P(3, \tilde{n}, \tilde{a}^1, IAC) - \frac{n_3^3}{12n_1n_2n_3} + \frac{1}{8} \tag{26}$$

Proof: see Appendix I.

Proposition 4 *Let $P(3, \tilde{n}, \tilde{a}^3, IC)$ be the likelihood of the referendum paradox for three states of large population under IC for the distribution $\tilde{n}$ when $\tilde{a} = \tilde{a}^2$. Then:*

$$P(3, \tilde{n}, \tilde{a}^3, IC) = \frac{2\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) + \arccos\left(\sqrt{n_3}\right)}{2\pi} - 0.375 \tag{27}$$

Proof: see Appendix II.

4.4 Case 3, $\tilde{a}^3 = (1, 0, 0)$

Proposition 5 *Let $P(3, \tilde{n}, \tilde{a}^3, IAC)$ be the likelihood of the referendum paradox for three states of large population under IAC for the distribution $\tilde{n}$ when state 1 is a dictator. Then:*

$$P(3, \tilde{n}, \tilde{a}^3, IAC) = \frac{(n_2+n_3)^3 - (n_2-n_3)^3 - (n_2+n_3-n_1)^3}{24n_1n_2n_3} \; if \; n_1 < \frac{1}{2} \tag{28}$$

$$P(3, \tilde{n}, \tilde{a}^3, IAC) = \frac{(n_2+n_3)^3 - (n_2-n_3)^3}{24n_1n_2n_3} \; if \; n_1 > \frac{1}{2} \tag{29}$$

Proof: see Appendix I.

Proposition 6 *Let $P(3, \tilde{n}, \tilde{a}^3, IC)$ be the likelihood of the referendum paradox for three states of large population under IC for the distribution $\tilde{n}$ when state 1 is a dictator. Then:*

$$P(3, \tilde{n}, \tilde{a}^3, IC) = \frac{\arccos\left(\sqrt{n_1}\right)}{\pi} \tag{30}$$

Proof: see Appendix II.

4.5 Comparisons

By comparing the values given by the formulas derived for the three possible apportionment cases, we are able to find the minimal value of the referendum paradox for each $\tilde{n}$. The corresponding minimal values of the paradox for IAC and IC are displayed on Table 3 and 4, respectively. First, our findings are consistent with the equal population case ($\tilde{n} = (1/3, 1/3, 1/3)$ studied previously by Feix *et al.* (2004). It is also obvious from the proofs presented in Appendix I and Appendix II that the minimal values can never been be obtained with $\tilde{a}^2$.

In Table 2 we observe that the dictatorship of state 1 is the optimal solution for IAC as soon as it gathers more than half of the population; otherwise, equal representation is the optimal solution to the minimization problem. Indeed local minima are observed when there is a perfect correspondence between the vectors of populations and the mandate vector, that is in $\tilde{n} = \tilde{a}^3 = (1, 0, 0)$ and $\tilde{n} = \tilde{a}^1 = (1/3, 1/3, 1/3)$. For a given value of n_3, the maximal value of the paradox is obtained at $n_1 = 0.5$; the same remark holds with n_2. Thus, the distance of n_1 to 0.5 is the main factor that explains the magnitude of the paradox.

At last, by comparing the formulas for $P(3, \tilde{n}, \tilde{a}^1, IAC)$ with $P(3, \tilde{n}, \tilde{a}^3, IAC)$ we can verify that $\delta = 1$ is always the optimal δ-rule. Recall that $n_1 = 1 - n_2 - n_3$. If $n_1 \leq 1/2$,

$$P(3, \tilde{n}, \tilde{a}^1, IAC) - P(3, \tilde{n}, \tilde{a}^3, IAC) = \frac{(2\,n_2 - 1 + 2\,n_3)\,(4\,n_2{}^2 - 4\,n_2 + 5\,n_2\,n_3 + 1 - 4\,n_3 + 4\,n_3{}^2)}{12 n_1 n_2 n_3} \tag{31}$$

The roots for this equation are :

$$n_2^* = \frac{1}{2} - n_3 \tag{32}$$

$$n_2^{**} = \frac{1}{2} - \frac{5}{8} n_3 + \frac{1}{8}\sqrt{24 n_3 - 39 n_3^2} \tag{33}$$

$$n_2^{***} = \frac{1}{2} - \frac{5}{8} n_3 - \frac{1}{8}\sqrt{24 n_3 - 39 n_3^2} \tag{34}$$

The domain where $n_1 \geq n_2 \geq n_3 > 0$ and $n_1 + n_2 + n_3 = 1$ is the triangle in dot lines depicted on Figure 1a. The dashed line isolate situations for which $n_1 < 0.5$ (above the line). It also corresponds to the equality $n_2^* = 0$. As the plain curve describes the points such as $n_2^{**} = 0$ and $n_2^{***} = 0$, it is now obvious that equation (31) is negative when $n_1 < 0.5$, meaning that $\tilde{a}^1 = (1, 1, 1)$ is optimal. Incidentally, it corresponds to $\delta = 1$. When $n_1 \geq 1/2$, we get:

$$P(3, \tilde{n}, \tilde{a}^1, IAC) - P(3, \tilde{n}, \tilde{a}^3, IAC) = \frac{n_2 n_3 (n_1 - n_2 - n_3)}{4 n_1 n_2 n_3} \tag{35}$$

This value is positive as soon as $n_1 > n/2$, which means that $P(3, \tilde{n}, \tilde{a}^3, IAC)$ always gives the optimal value. But then, $\tilde{a}^3 = (1, 0, 0)$ is also implemented with $\delta = 1$.

Proposition 7 *For three-state federations, the proportional representation ($\delta = 1$) is a δ-rule that always minimizes the likelihood of the referendum paradox under IAC.*

In Table 3, we first observe that the likelihood of the referendum paradox with IC is always superior to the probabilities derived for IAC. This fact is not surprising, as the IC model describes more heterogenous societies than the IAC assumption. Local minima are still observed at $\tilde{n} = (1, 0, 0)$ and $\tilde{n} = (1/3, 1/3, 1/3)$, but with IC, the region for which dictatorship of 1 is optimal is much more reduced. This can be explained by the fact that fluctuations in state 1 go as $\sqrt{n_1}$ for the IC model, which relatively reduces its influence on the determination of the popular winner. Hence, $a^1 = (1, 1, 1)$ is more often the optimal apportionment of the mandates.

By comparing $P(3, \tilde{n}, \tilde{a}^1, IC)$ and $P(3, \tilde{n}, \tilde{a}^3, IC)$, we observe that the former is inferior to the latter (and $a^1 = (1, 1, 1)$ is optimal) if and only if:

$$n_2 \geq \cos^2\left(\frac{\pi}{4} + \arccos\left(\sqrt{n_3}\right)\right) \tag{36}$$

Then, one may notice that the square root rule, characterized by $\delta = 1/2$, points toward the majority game whenever $\sqrt{n_1} \leq \sqrt{n_2} + \sqrt{n_3}$, and to the dictatorship otherwise. Solving this inequality leads to :

$$n_2 \geq \frac{1}{2} - \frac{n_3}{3} - \frac{1}{2}\sqrt{2\,n_3 - 3\,n_3{}^2} \tag{37}$$

Equation (36) is displayed in bold on Figure 1 for values n_2 and n_3 compatible with our constraints (the interior of the triangle); above it, a^1 is the optimal game, while a^3 enjoys this status below the line. By drawing equation (37), on the same figure, we identify below the dashed line the $\tilde{n}$ that the square root rule associates with the dictatorship of state 1. Clearly, the square root rule fails to be optimal, as the games in between the two curves should be associated with $\tilde{a}^1$. To give an example, consider $\tilde{n} = (0.65, 0.30, 0.05)$ in Table 3. Using the square root rule leads to the weights $(0.806, 0.547, 0, 224)$ and the dictatorship of state 1, while $\tilde{a}^1$ is optimal. By integrating the volumes between the two lines, we derive that the square root rule fails to be optimal for 10.38% the federations. This graphic interpretation suggest that a value slightly smaller than 0.5 would enable us to move the dashed line closer to the optimal curve described by equation (36).

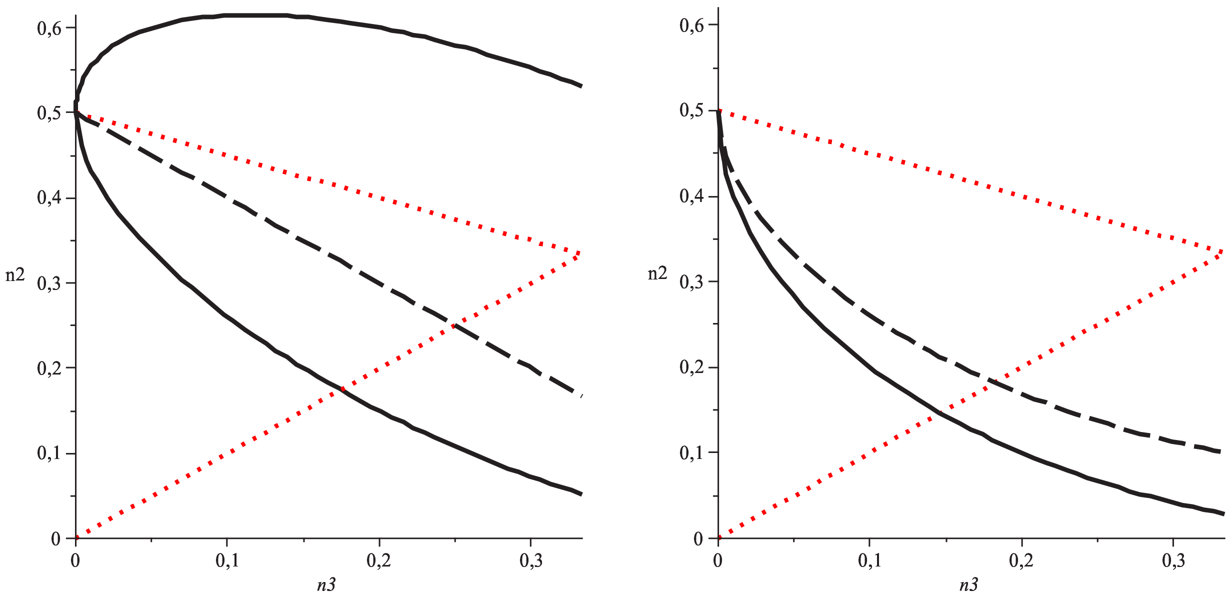

Figure 1: The boundaries between the majority game and dictatorship for the minimization of the referendum paradox. a) The optimality of the proportional rule under IAC b) Domains for the square root rule and the optimal δ rule under IC

For $N > 3$, the number of states becomes too important to give a complete enumeration for all the cases. Moreover, obtaining formulas for more than three states, though technically possible for $N = 4$ and $N = 5$ (as in Feix *et al.* (2004)), would be cumbersome. So, we turn on to the search for an optimal allocation rule among the family of δ-rules with computer simulations.

5 The general study of the δ-rules via simulations

5.1 Staircase curves

We now extend our study to the general case of $N \geq 3$. More precisely, we assume that $a_i = n_i^\alpha$ for the IC case and $a_i = (\Delta_i n_i)^\beta$ for the RIAC case. Our objective is to check whether the recommendations we should adopt when we wish to minimize the likelihood of the referendum paradox are compatible with approximate solutions that have been suggested in the power index literature ($\beta = 1$ for IAC, $\alpha = 0.5$ for IC). But before getting to the heart of the matter, we first study two cases of unequal populations in detail, to highlight several facts about the shapes of the curves we obtain.

Let us consider first the populations $(\Delta_i n_i) = \tilde{n}^1 = (3.2, 2.3, 1.8, 1.4, 1)$ for a RIAC simulation, and the populations equal to the square of the previous ones for a IC simulation (i.e. $(n_i) = \tilde{n}^2 = (10.24, 5.29, 3.24, 1.96, 1)$).

For each value of α and β, represented by a point, 1,000,000 elections have been performed. α (resp β) is incremented by step of 0.005 (resp 0.01) on the range $[0, 1.5]$ (resp $[0, 3]$). Figure 2 and 3 display the results of a Monte Carlo simulation, with all the points being connected for clarity.

First, as in the case $N = 3$, the δ-rules can only be associated to a limited number of underlying games. We immediately recognize on the figures 2 and 3 six plateaus, each one corresponding to an underlying weighted majority game.

More surprising is the fact that we obtain very similar shapes for both cases, though the magnitude along the horizontal and vertical axis are different. As we have chosen $\tilde{n}^2$ such that $n_i^2 = (n_i^1)^2$, if we assume furthermore that $\alpha = \beta/2$, equations (14) and (15) becomes equivalent to equations (18) and (19).

$$\operatorname{sgn}\left(\sum_i \varepsilon_i \sqrt{(n_i^2)}\right) = \operatorname{sgn}\left(\sum_i \varepsilon_i \sqrt{(n_i^1)^2}\right) = \operatorname{sgn}\left(\sum_i \varepsilon_i (n_i^1)\right) \tag{38}$$

$$\operatorname{sgn}\left(\sum_i a_i \operatorname{sgn}(\varepsilon_i)\right) = \operatorname{sgn}\left(\sum_i (n_i^2)^\alpha \operatorname{sgn}(\varepsilon_i)\right) = \operatorname{sgn}\left(\sum_i (n_i^1)^\beta \operatorname{sgn}(\varepsilon_i)\right) \tag{39}$$

In other words, the games that we will encounter for an IAC simulation with a population (n_i) as β varies are the same ones that will encounter for an IC simulation with $\alpha = \beta/2$ and populations $(n_i)^2$. This explain why we recover a very similar staircase structure in both cases. However, one difference remains: on the one hand the ε_i are drawn from a normal law, and on the other hand, they are drawn from a uniform law. As a consequence, the minimal values of the paradox may not appear for the same plateau.

Let us now comment more precisely Figure 3. For the RIAC case, we encounter 6 plateaus. Easy computation shows they correspond to 6 of the 7 possible 5-person weighted majority games with an odd number of mandates[8]. The vectors of weights are $a^1 = (1, 1, 1, 1, 1)$, $a^2 = (2, 2, 1, 1, 1)$, $a^3 = (3, 2, 2, 1, 1)$, $a^4 = (4, 2, 2, 2, 1)$, $a^5 = (5, 2, 2, 2, 2)$ and $a^6 = (1, 0, 0, 0, 0)$. We first meet a majority game with equal weights, and then, progressively move towards the dictatorship of state 1. The optimal value, which leads to a probability of about 16%, is obtained for values of β in between 0.9 and 1.3. It is superior to the exact value of 14.32 % that Feix *et al.* (2004) derived for five state with equal population. The pure federal case leads to a paradox in about 19% of the simulations, and the dictatorial case in about 24%.

It is also possible to encounter games with an even number of mandates. For example, in between a^1 and a^2, the δ-rule is equivalent to the extra game $a^7 = (3, 3, 2, 2, 2)$ for a unique value of $\delta \approx 0.595420$. The reader will immediately realize that such games exist in between games with an odd number of mandates, but that they are only realized at specific values of $\delta \approx 0.862767$, 1.307213, 1.822085, 2.209592. Then, the computer simulations, with finite steps for α or β, will never be able to catch them. Moreover, intuition from the case $N = 3$ tells

[8] The only "missing" game is defined by $\tilde{a} = (3, 3, 3, 0, 0)$.

us that games with an even number of mandates are likely to increase the probability of the referendum paradox. For all these reasons, we should not consider these specific games in our search for the optimal two-tier voting rules.

The picture for the IC case is very similar as we assumed $(n_i)^2 = (\Delta_i n_i^1)^2$ and $2\alpha = \beta$. We recover the six plateaus, corresponding to the same six different voting games. However, the magnitudes are different. We start with a value of 21.5% in the federal, and next obtain a minimum slightly lower than 20% for α in between 0.43 and 0.65 approximately, and then progressively go up to 25.5% for the dictatorial case.

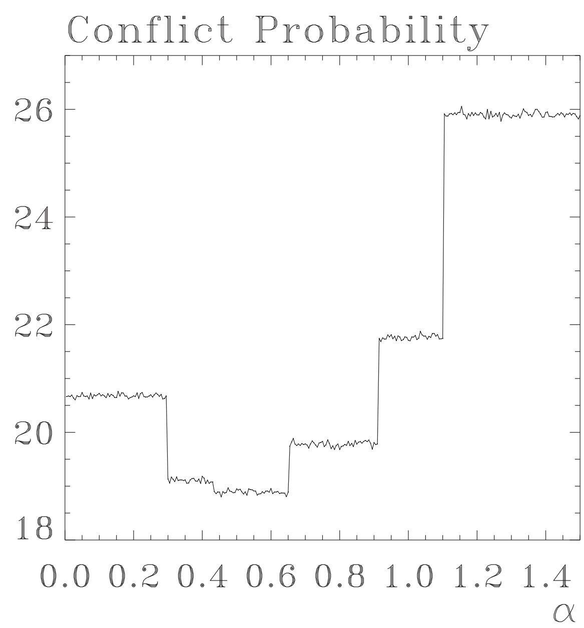

Figure 2: IC model : Percentage of conflicts between states and population decision for $N = 5$ states of population $\tilde{n} = (10.24, 5.29, 3.24, 1.96, 1)$ respectively.

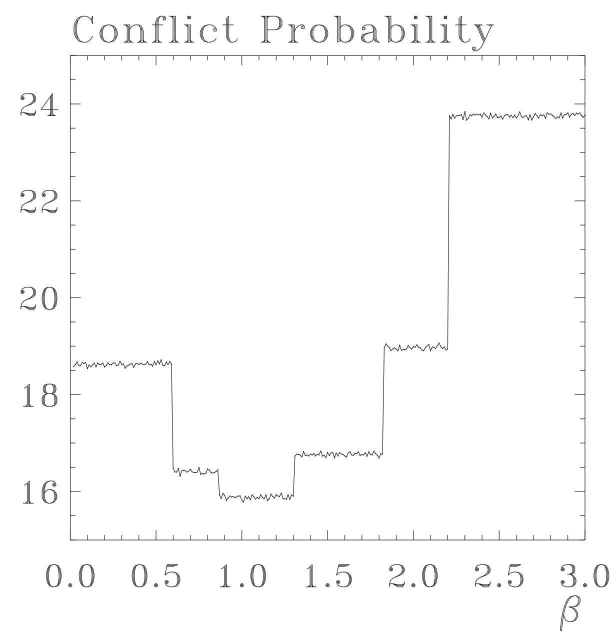

Figure 3: RIAC model : same as figure 2. $2\Delta_i N_i$ =3.2, 2.3, 1.8, 1.4,1, for $i = 1, \dots, 5$ (that is the square root of the values taken for figure 2).

Figure 4 and 5 present a much more complicated situation with 20 states. Due to the numerous number of possible majority weighted games with 20 players, the curves become almost continuous. The pattern of the two curves are similar : first a plateau around the federal case, then a decline till the optimal value (around $\beta = 1$ in the IAC case, and around $\alpha = 0.5$ in the IC model) and then a regular increase, at least in the range of values of α and β shown figures 4 and 5. Notice that we do not reach the dictatorial case for both simulations[9].

5.2 Toward a general result under IAC

Till now, we have observed with the formulas for $N = 3$ states and two specific cases of unequal population (one for $N = 5$ states and one for $N = 20$) that $\beta = 1$ seems to be the optimal choice for a δ-rule under IAC in order to minimize the occurrence of the referendum paradox. In this section, we try to obtain general conclusions by drawing systematically several distributions $\tilde{n}$ for a given number of states.

[9]Some simulations, not presented in this paper, have shown that a second local minima, above the first one, may exist for higher values of α and β, before reaching the dictatorial case

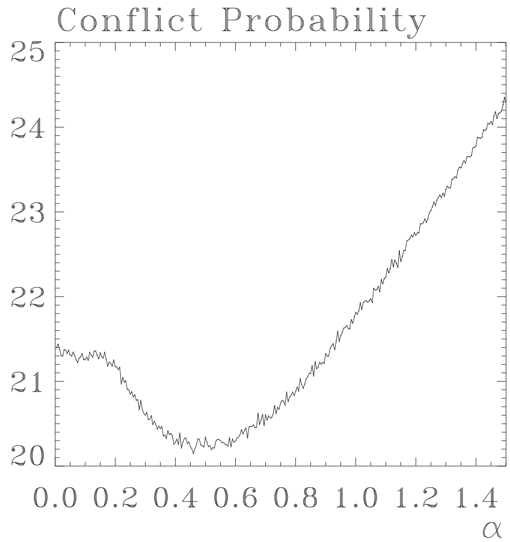

Figure 4: IC model : same as figure 2 but for $N = 20$ states of population 35.88, 32.72, 30.69, 23.52, 18.84, 17.72, 17.31, 15.44, 14.06, 11.83, 11.49, 9.92, 8.82, 6.45, 5.81, 4.49, 3.13, 2.34, 1.72, 1, for $i = 1, \ldots, 20$.

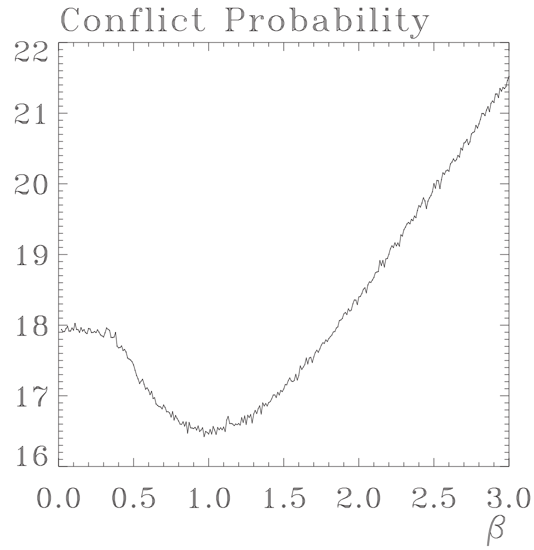

Figure 5: RIAC model : same as figure 3 but for $N = 20$ states with n_i =5.99, 5.72, 5.54, 4.85, 4.34, 4.21, 4.16, 3.93, 3.75, 3.44, 3.39, 3.15, 2.97, 2.54, 2.41, 2.12, 1.77, 1.53, 1.31, 1.,for $i = 1, \ldots, 20$.

Conjecture 1 *For the RIAC model, the minimum of conflicts between the popular and states among the class of δ-rules is obtained by taking* $a_i = \Delta_i n_i$ *(*$\beta = 1$*) .*

We will test systematically the conjecture by simulation for $N = 3$ to $N = 33$, with an extra simulation for $N = 50$. For each value of N, we will first draw randomly 1,000 different federations from the uniform distribution on the unit simplex. Next, we will consider 100,000 elections for each federation[10], in order to estimate the likelihood of the paradox for different values of β, as explained in section 3.3. β will vary from 0 to 2, with a step of 0.05, and an increased precision of 0.01 in between 0.9 and 1.1.

In Table 4, we summarize the results of these simulations. First, for each federation, we will check whether $\beta = 1$ leads to the minimal value of the paradox. Let $F(N, \beta, IAC)$ be the percentage of federations of size N for which β gives the minimal probability of the paradox. Column 2 gives the value of this frequency for $\beta = 1$. Column 3 and 4 gives the optimal $\beta^\star$ and the corresponding value $F(N, \beta^\star, IAC)$. $\beta = 1$ is almost all the time the value that maximizes the probability of getting the minimal occurrence of the referendum paradox. If not, this statute is occupied either by $\beta = 0.99$ or $\beta = 1.01$! For $N = 3$, we observe 4 cases out of 1,000 where $\beta = 1$ is not optimal. A closer look reveals that these four cases all concern situation where n_1 is slightly inferior to 0.5, where the equations (23) and (28) lead to very close values for the paradox. Thus, this value inferior to one is just a consequence of the fluctuations of the simulations. However, as N grows, $F(N, 1, IAC)$ plunges. But in fact, the number of possible underlying weighted majority games also increases with N, and now each value of β around 1 leads to a specific game; due to fluctuations of the random trials, values of β close to 1 can

[10]Ideally, we should have drawn a new set of elections for each value of β, as we did in the previous section. This would have allowed us to define a clear statistical test about the optimality of $\beta = 1$ compared to other values of β. However, one immediately realizes that drawing a new batch of data for each β would have enormously increased the computation time.

also frequently be designated as the optimal game. Thus, to check the optimality of $\beta = 1$, we consider column 5, which reports the maximal deviation for the optimal value of the paradox for the 1,000 federations of size N. With the exception of $N = 3$, this maximal deviation is always inferior to 0.15%.

At last, at the aggregated level, we will report in column 6 $P(N, \beta, IAC)$, the mean value of the $P(N, \tilde{n}, \beta, IAC)$ among the 1,000 federations. $\beta = 1$ has always produced the minimal value of the paradox, except for $N = 3$ where we observe a value of $P(N, \tilde{n}, 1.01, IAC) = 11.873858\%$, which can hardly be considered as significatively different from $P(N, \tilde{n}, 1, IAC)$

To give a broader perspective on the behavior of $P(N, \beta, IAC)$, we display it for several values of N as β varies on Figure 6. All the curves display the same pattern. We encounter the maximal value of the paradox for the pure federalism case ($\beta = 0$), and, after a plateau, the probabilities decline to their optimal values obtained for $\beta = 1$. After this point, the occurrence of the referendum paradox increases again. As N grows, the curves move up progressively and seem to reach a limit. In fact, in Table 4, the values of $P(N, 1, IAC)$ appear to converge to 16.5%, the limit value of the paradox for equally populated states that has been observed in Feix *et al.* (2004).

5.3 How far from optimality is the square root rule?

Testing whether $\alpha = 0.5$ is the value that minimizes the referendum paradox in the class of δ-rules under IC is vain as we noticed already that this conjecture is false for $N = 3$. Thus, the question is to evaluate how far from optimality is the square root rule.

The method we use to simulate results under IC is similar to the one we use for the IAC assumption. For $N = 3$ to $N = 33$, we still draw 1,000 population vectors from the simplex, and then generate randomly 100,000 voting situations. We will add a simulation at $N = 50$ to have a glance at the large federation case. The only difference is that now ε_i is drawn from the gaussian:

$$(2\pi)^{-1/2} \exp(-\varepsilon^2/2).$$

Different values of α are tested in between 0 and 1, with an increment 0.05, reduced to 0.005 in between 0.4 and 0.6.

Table 5 summarizes our findings. Let $F(N, \alpha, IC)$ be the percentage of federations of size N for which α gives the minimal probability of the paradox. Columns 2 and 3 display the optimal $\alpha^\star$ and the corresponding value, while Columns 4 gives the value of $F(N, 0.5, IC)$. As one may have guessed, the optimal α tends to be slightly smaller than 0.5, ranging from 0.435 to 0.505 according to N. Again, the values for $F(N, \alpha, IC)$ decline quickly as N grows, as each specific value of α becomes associated with a specific game. Contrarily to the IAC case, we really observe in column 5 deviations from the optimum which are quite significative for small values of N [11]. However, the maximal deviations stay below 0.15 from $N = 10$.

At the aggregated level, we study the mean value of the paradox over the 1,000 federations, $P(N, \alpha, IC)$. Values for $P(N, 0.5, IC)$ are displayed on column 8. They are almost never the optimal values $P(N, \alpha^\star, IC)$ for small values of N, though the differences are tiny. The optimal value is $\alpha^\star = 0.44$ for $N = 3$, and it seems to converge toward 0.5. *Thus, the fact that the square root rule is "on average" the optimal two-tier voting rule under the IC assumption may be true for $N > 20$.*

[11] Again, we cannot propose a proper test as we use the same batch of elections to compute the $P(N, \tilde{n}, \alpha, IC)$ for different values of α.

Table 4: Testing the optimality of $\beta = 1$ under the IAC assumption

N	$F(N,1,IAC)$ (in %)	$\beta^\star$	$F(N,\beta^\star,IAC)$ (in %)	Max Dev (in %)	$P(N,1,IAC)$ (in %)
3	99.6	1.00	99.6	0.252	11.874026
4	99.3	1.00	99.3	0.103	13.491412
5	97.4	1.00	97.4	0.113	14.205342
6	91.0	0.99	91.0	0.148	14.468412
7	78.8	1.00	78.8	0.104	14.925542
8	60.7	1.01	61.5	0.122	15.001071
9	37.0	1.00	37.0	0.106	15.341858
10	22.9	1.01	23.1	0.094	15.363265
11	18.8	1.00	18.8	0.111	15.589958
12	15.3	1.00	15.3	0.150	15.665654
13	12.6	1.00	12.6	0.088	15.819551
14	14.6	1.00	14.6	0.078	15.825305
15	14.1	1.00	14.1	0.094	15.862417
16	14.2	1.00	14.2	0.080	15.941952
17	13.0	0.99	13.5	0.096	15.976371
18	12.2	1.01	12.2	0.102	15.996521
19	14.8	1.00	14.8	0.080	16.084561
20	14.7	1.00	14.7	0.110	16.091198
21	13.0	1.00	13.0	0.079	16.134544
22	14.0	1.00	14.0	0.096	16.111878
23	13.4	1.00	13.4	0.106	16.194433
24	13.3	1.00	13.3	0.086	16.196593
25	12.5	1.00	12.5	0.091	16.229020
26	13.0	0.99	14.3	0.092	16.248753
27	14.9	1.00	14.9	0.108	16.222186
28	13.3	1.00	13.3	0.087	16.284212
29	15.5	1.00	15.5	0.093	16.278161
30	15.6	1.00	15.6	0.092	16.279040
31	14.0	1.00	14.0	0.093	16.303384
32	15.0	1.00	15.0	0.086	16.294782
33	12.7	0.99	13.6	0.092	16.336437
50	15.1	1.00	15.1	0.092	16.437010

As we did for the IAC case, we display on Figure 7 the evolution of the paradox as α varies for different values of N. The curves exhibit a similar pattern: after a plateau around the value $\alpha = 0$, they decline continuously to their minimal value reached just before $\alpha = 0.5$. Then, the values of the paradox increase again as α increases.

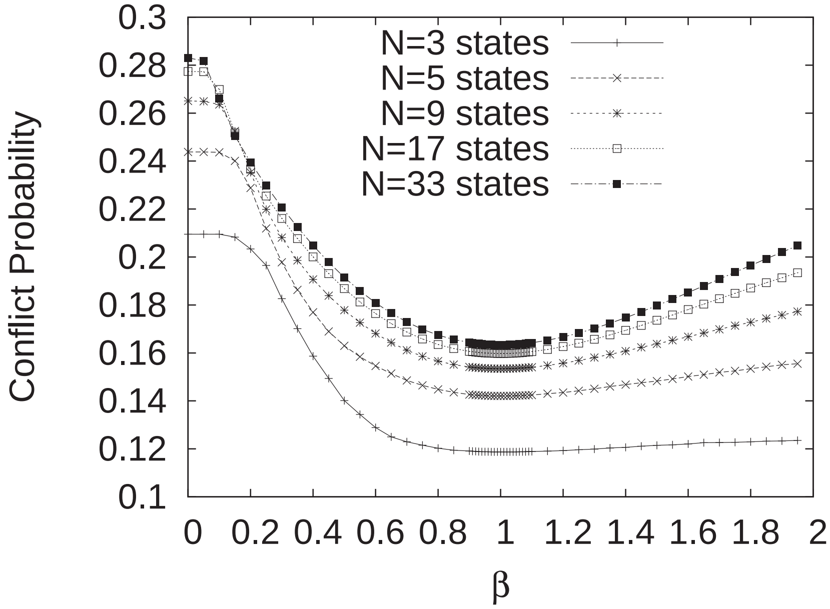

Figure 6: The Probability of the referendum paradox as a function of β under the IAC assumption for an odd number of states.

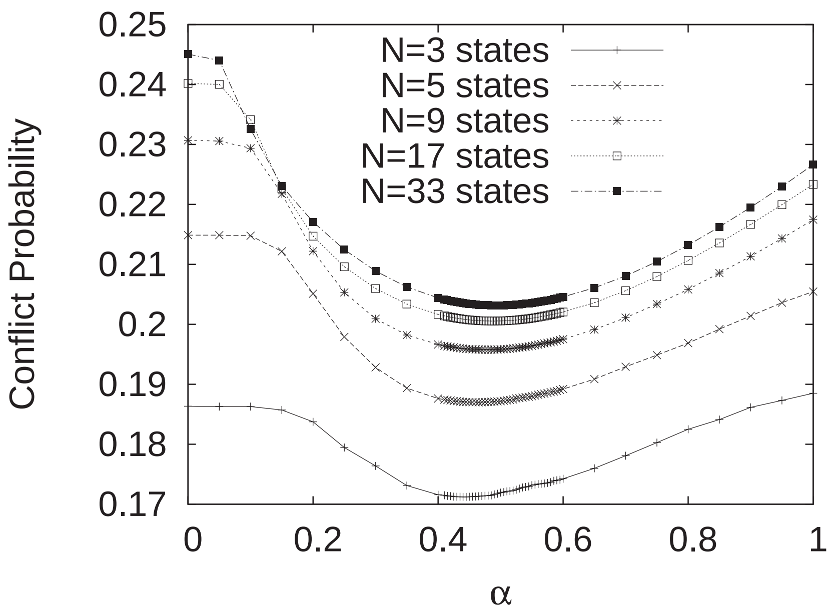

Figure 7: The Probability of the referendum paradox as a function of α under the IC assumption for an odd number of states.

Table 5: Testing the optimality of $\alpha = 0.5$ under the IC assumption

N	$\alpha^\star$	$F(N, \alpha^\star, IC)$ (in %)	$F(N, 0.5, IC)$ (in %)	Max Dev (in %)	$\alpha^\star$	$P(N, \alpha^\star, IC)$ (in %)	$P(N, 0.5, IC)$ (in %)
3	0.435; 0.445	98.9	91.8	1.750	0.440	17.122054	17.192503
4	0.440; 0.445	97.9	92.8	1.130	0.440	18.286668	18.313228
5	0.465	94.0	85.4	0.736	0.465	18.700525	18.718918
6	0.465; 0.470	86.0	76.4	0.509	0.470	19.043587	19.056982
7	0.47	67.7	59.3	0.338	0.475	19.277502	19.285892
8	0.48	50.5	44.4	0.220	0.480	19.448020	19.454457
9	0.475; 0.490	27.8	22.4	0.183	0.480	19.576784	19.583048
10	0.485	18.6	15.1	0.158	0.480	19.667515	19.673367
11	0.475	11.8	6.7	0.145	0.480	19.751666	19.756572
12	0.495	9.6	7.5	0.124	0.480	19.832513	19.836304
13	0.48	8.6	6.7	0.119	0.485	19.887741	19.891258
14	0.475	8.6	5.8	0.130	0.490	19.934268	19.935717
15	0.485	8.5	5.9	0.144	0.485	19.967989	19.970677
16	0.475	7.7	6.4	0.100	0.485	20.027822	20.030398
17	0.48	8.5	6.4	0.088	0.490	20.055471	20.057235
18	0.49	8.9	7.3	0.092	0.490	20.083991	20.085902
19	0.485	7.9	6.2	0.106	0.490	20.109915	20.113009
20	0.5	8.5	8.5	0.111	0.490	20.139596	20.140905
21	0.485	8.3	7.4	0.104	0.495	20.152353	20.153762
22	0.485; 0.495	8.1	6.2	0.101	0.495	20.179569	20.180349
23	0.5	8.1	8.1	0.097	0.490	20.201968	20.202698
24	0.5	8.4	8.4	0.107	0.490	20.219576	20.220962
25	0.495	8.5	6.2	0.084	0.495	20.224132	20.224593
26	0.49	9.0	7.3	0.109	0.490	20.236001	20.236878
27	0.505	8.5	7.3	0.091	0.500	20.261662	20.261662
28	0.49	8.3	7.8	0.096	0.490	20.259041	20.259843
29	0.49	7.9	7.0	0.098	0.490	20.275846	20.276625
30	0.475	8.9	8.5	0.111	0.490	20.290999	20.291985
31	0.485	8.3	7.3	0.084	0.490	20.295823	20.296438
32	0.485	8.9	8.1	0.098	0.495	20.306279	20.306543
33	0.48	8.4	7.7	0.108	0.495	20.316317	20.316557
50	0.49	8.7	7.8	0.094	0.495	20.404686	20.405353

6 Conclusion

In this paper, we have first discussed the literature about the optimal weight a state should receive in a two-tier majority voting rule. We emphasized the fact that most of the criteria which are used to compare the voting rules are all based upon abstract concepts like "pivotal players" or "utility". A simpler concept, like "equal opportunity of success" is more appealing but does not really discriminate among the voting rules as long as they treat equally the candidates. The concept of *majority efficiency* that we proposed in this paper is, in our opinion, easier to defend. First, it is just the application of the well known Condorcet criterion to a two candidate election. Secondly, a referendum paradox can be observed by all the citizens when it occurs, as shows the 2000 US presidential election. As a consequence, it can be studied not only from a theoretical point of view, as we did in this paper, but could also be the subject of empirical studies, as soon as one possesses a sufficiently large and consistent electoral database. To some extent, studying this paradox would be a good way to build a bridge between formal *a priori* models used in game theory, economics, and social choice theory, and empirical facts described in political sciences.

However, the main objective of this paper was to check whether the majority efficiency was compatible with classic normative recommendations from game theory based upon *a priori* voting models. More precisely, we studied the δ-rules, which allocate a number of mandates proportional to n_i^δ to state i. Using the IAC assumption, which is equivalent to the underlying probabilistic model defining the Shapley Shubik power index, we obtained a very clear answer: with exact formula, we prove that the proportional rule is optimal for $N = 3$, and our simulations extend the result till $N = 50$. The picture is more fuzzy when we turn to the IC assumption, which assumes that each voter chooses between A and B by tossing a fair coin. Exact formulas for $N = 3$ demonstrate that $\alpha = 0.5$ is not optimal; the computer simulations suggest that, for N large, the square root rule is either the optimal solution "on average" or extremely close to it. But this last comment has to be taken with caution. By drawing the populations of the federations from an uniform distribution on the unit simplex, we miss situations where n_1 is extremely large as N grows. To give an example, for $N = 50$, the maximal size that we drew for n_1 is 0.2038. Thus, as we only drew 1,000 federations, some parts of the simplex are no longer explored for large values of N. We would need a different method for the generation of the populations to state more robustly our findings. Such a method would also be necessary if one wants to study precisely the impact of the dispersion of the population on the magnitude of the paradox.

At last, one may wonder what happens when we consider rules which are different from the δ-rules. At this stage, we cannot guarantee that other two-tier voting schemes may beat the proportional rule or the square root rule. But we can already compare directly three famous voting rules, the federal rule ($\delta = 0$), the square root ($\delta = 0.5$) and the proportional rule ($\delta = 1$). The results, for the IC assumption, are displayed on Figure 9. We recover that the square root rule does better than the proportional rule or the federal rule, but the margin are not impressive: As N grows, its advantage over the proportional rule (resp. the federal rule) stabilizes around 2.5% (resp. 4%). Such differences may not be perceived empirically unless someone gathers a huge set of data. Moreover, Gelman *et al.* have noticed that the IC model does not fit with electoral data, so it seems imprudent to set normative recommendations on this sole basis. On the other hand, by describing more homogeneous societies, the IAC could be more reliable. The results, displayed in Figure 8 are quite shocking for the federalism[12]. With

[12] To compute the values for an even number of states, we draw a fair coin in case of a tie in terms of mandates.

33 states, the probability of paradoxes reaches 28% with the federal rule, while it stays below 16.5% for the proportional rule. Even the cube rule ($\beta = 2$) performs better! Over a long series of elections, such a difference could be detected, and unless the "one state – one mandate" principle enjoys a strong support in the society, our study suggests it should be abandoned in federations.

This explains the discontinuities we observe for $\alpha = \beta = 0$ between odd and even values of N.

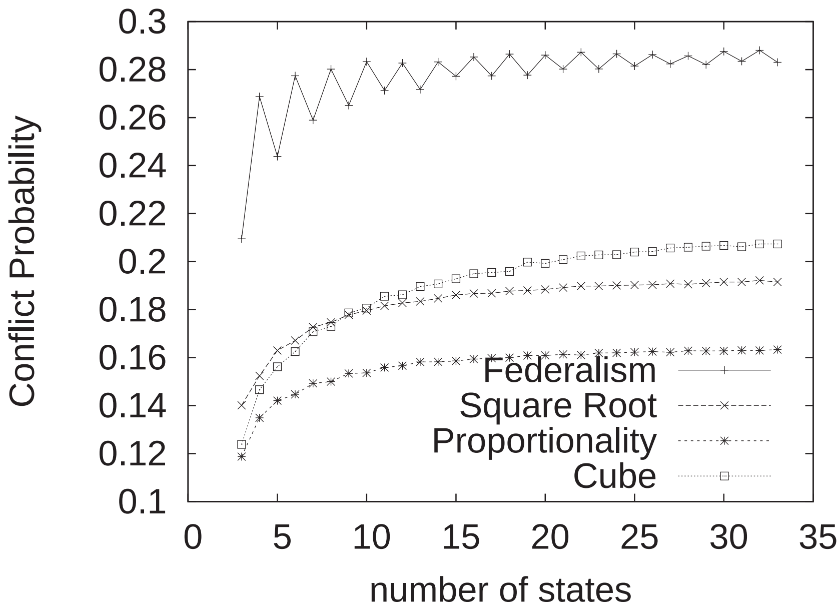

Figure 8: Comparing different voting rules on their ability to avoid the referendum paradox under the IAC assumption

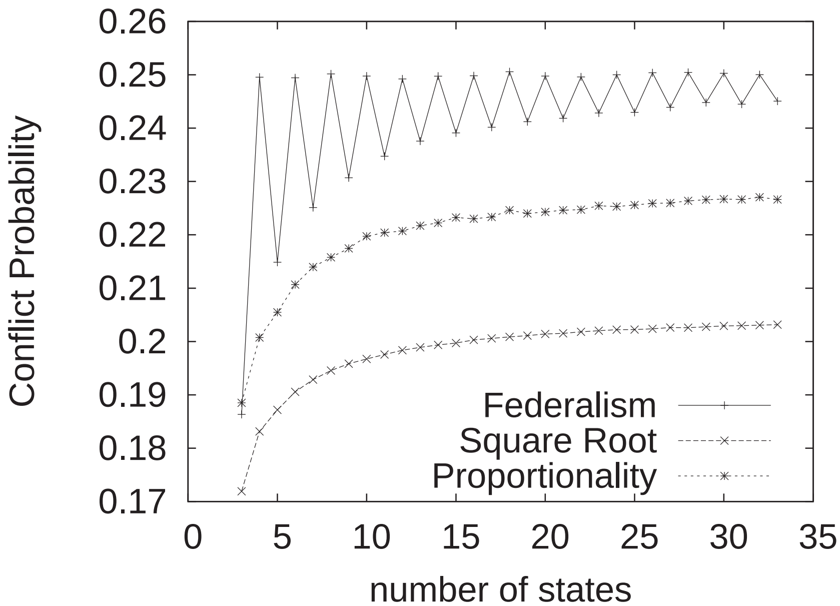

Figure 9: Comparing different voting rules on their ability to avoid the referendum paradox under the IC assumption

Appendix I: Formulas for the IAC case and three states.

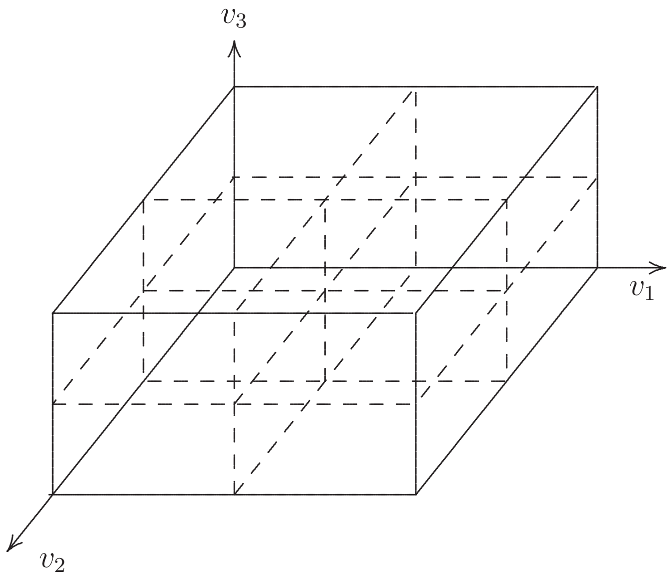

Figure 10: An example of the space of voting profiles under IAC, with $n_1 = 4$, $n_2 = 3$ and $n_3 = 2$

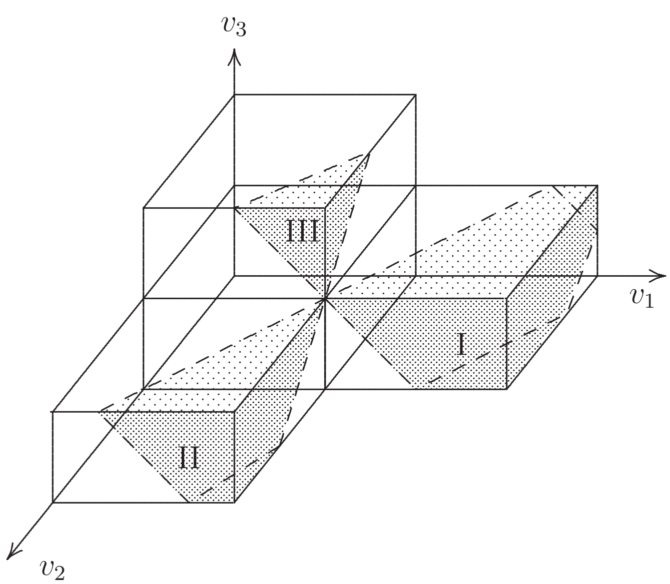

Figure 11: The volumes for the referendum paradox under IAC, with $n_1 = 4$, $n_2 = 3$ and $n_3 = 2$

Let v_i be number of votes for A in state i. Under IAC assumption, we draw a value for ε_i, $i = 1, 2, 3$ in the uniform distribution over $[0, 1]$, and multiply them by n_i to get a vector $\tilde{v} = (v_1, v_2, v_3)$. Thus, each point in the parallelepiped depicted on Figure 10 is a possible voting outcome; each of them is equally likely and the total volume of the parallelepiped is $n_1 n_2 n_3$. The parallelepiped is cut into height zones by the hyperplanes $v_1 = n_1/2$, $v_2 = n_2/2$ and $v_3 = n_3/2$.

Case 1: $\tilde{a} = \tilde{a}^1 = (1, 1, 1)$

Subcase $\mathbf{n}_1 < \mathbf{n/2}$. We depict on Figure 11 the four cases where A get the support of at least two states. These regions are cut by the hyperplane $v_1 + v_2 + v_3 = n/2$. This defines three regions (labeled I, II and III) where the paradox occurs. Let V_t be the volume of region t. We can easily derive that:

$$V_I = \frac{1}{6}\left[\left(\frac{n_1}{2}\right)^3 - \left(\frac{n_1 - n_3}{2}\right)^3 - \left(\frac{n_1 - n_2}{2}\right)^3\right] \tag{40}$$

$$V_{II} = \frac{1}{6}\left[\left(\frac{n_2}{2}\right)^3 - \left(\frac{n_2 - n_3}{2}\right)^3\right] \tag{41}$$

$$V_{III} = \frac{1}{6}\left(\frac{n_3}{2}\right)^3 \tag{42}$$

As the cases where B wins at least two states are symmetric, we can deduce that:

$$P(3, \tilde{n}, \tilde{a}, IAC) = \frac{2(V_I + V_{II} + V_{III})}{n_1 n_2 n_3} \tag{43}$$

$$= \frac{n_1^3 + n_2^3 + n_3^3 - (n_1 - n_2)^3 - (n_1 - n_3)^3 - (n_2 - n_3)^3}{24 n_1 n_2 n_3} \tag{44}$$

Subcase $\mathbf{n}_1 > \mathbf{n/2}$. In Figure 12, we observe that the volumes are the same as in the previous subcase, except for V_I:

$$V_I = \frac{1}{6}\left[\left(\frac{n_2 + n_3}{2}\right)^3 - \left(\frac{n_2}{2}\right)^3 - \left(\frac{n_3}{2}\right)^3\right] + \frac{n_2 n_3 (n_1 - n_2 - n_3)}{8} \tag{45}$$

As the cases where B wins at least two states are symmetric, we can deduce that:

$$P(3, \tilde{n}, \tilde{a}, IAC) = \frac{2(V_I + V_{II} + V_{III})}{n_1 n_2 n_3} \tag{46}$$

$$= \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3 + 6 n_2 n_3 (n_1 - n_2 - n_3)}{24 n_1 n_2 n_3} \tag{47}$$

Case 2: $\tilde{a} = \tilde{a}^2 = (2, 1, 1)$

The only difference with Case 1 comes from the regions III displayed on Figures 11 and 12. The whole region is a tossup in terms of mandates, and it accounts to 1/16 to the magnitude of the paradox, as there is a 50% chance that the draw picks the wrong candidate. Thus, to obtain formula (26), we discard expression (42) and add 1/16 instead. As 1/16 is always superior to V_{III}, the new probability is always superior to $P(3, \tilde{n}, a^1, IAC)$.

Case 3: $\tilde{a} = \tilde{a}^1 = (1, 0, 0)$

As state 1 is a dictator, we just have to identify the regions for which $v_1 < n/2$ while $v_1 + v_2 + v_3 > n/2$. Figure 13 displays the paradoxical regions for the subcase $n_1 < n/2$ and easy computations lead to equation (28). The same process applies for subcase $n_1 > n/2$.

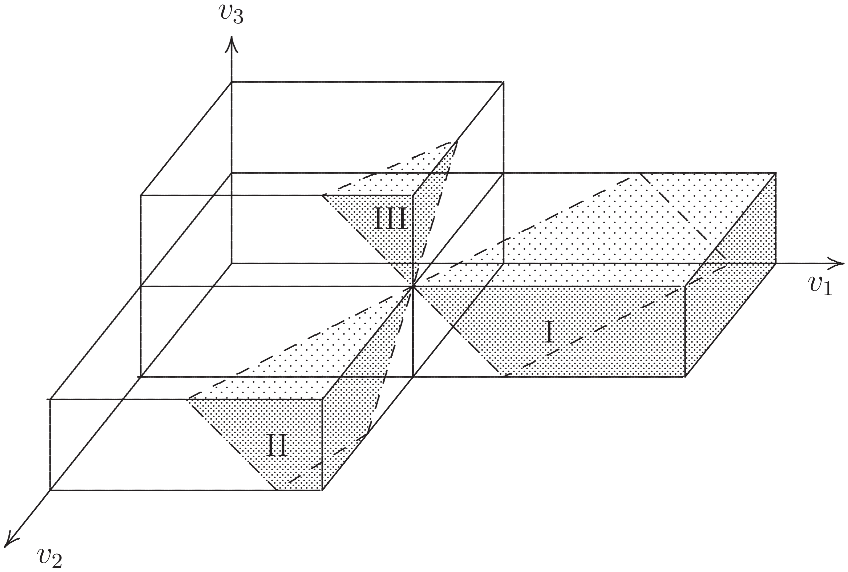

Figure 12: The referendum paradox under IAC, an example when $n_1 = 6$, $n_2 = 3$ and $n_3 = 2$

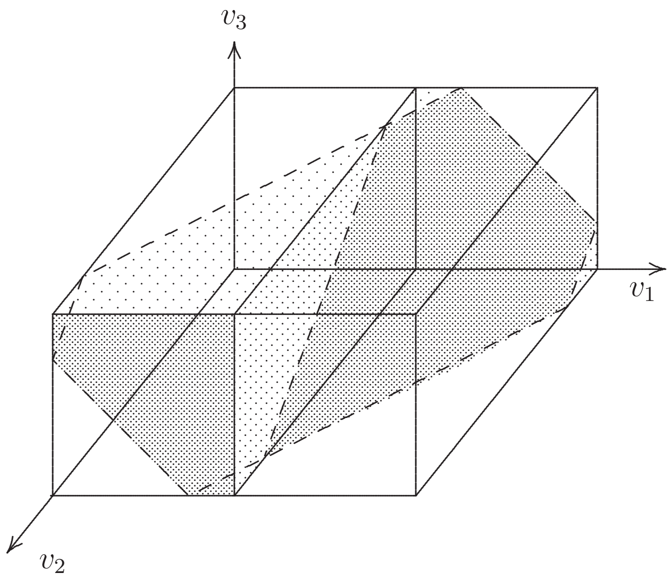

Figure 13: The volume for the referendum paradox under IAC, when state 1 is a dictator, for $n_1 < n/2$.

Appendix II: Formulas for three states under IC.

Case 1: $\tilde{a} = \tilde{a}^1 = (1, 1, 1)$

The proof is similar to the one proposed in the section 2 of Feix *et al.* (2004) for the case of equal population. The Impartial Culture assumes that each voter picks his preference randomly among the possible preference types according to an uniform probability distribution. In our case, each voter in each state has a probability $\frac{1}{2}$ to cast his vote in favor of candidate A, and a probability $\frac{1}{2}$ to cast his vote for candidate B. Let v_i be number of vote for A in state i. A referendum paradox occurs if and only if two different states i and k do not select the popular winner. There are six cases:

$$v_1 > n_1/2, \quad v_2 > n_2/2, \quad v_1 + v_2 + v_3 < n/2 \tag{48}$$
$$v_1 < n_1/2, \quad v_2 < n_2/2, \quad v_1 + v_2 + v_3 > n/2 \tag{49}$$
$$v_1 > n_1/2, \quad v_3 > n_2/2, \quad v_1 + v_2 + v_3 < n/2 \tag{50}$$
$$v_1 < n_1/2, \quad v_3 < n_2/2, \quad v_1 + v_2 + v_3 > n/2 \tag{51}$$
$$v_2 > n_1/2, \quad v_3 > n_2/2, \quad v_1 + v_2 + v_3 < n/2 \tag{52}$$
$$v_2 < n_1/2, \quad v_3 < n_2/2, \quad v_1 + v_2 + v_3 > n/2 \tag{53}$$

When the number of voters is large in each state, the distribution of v_i tends to a normal law, of mean $n_i/2$ and variance $\sigma_i = \sqrt{n_i}/2$. For each n_i, $i = 1, \ldots, N$, let

$$x_i = \frac{1}{\sigma_i}\left(v_i - \frac{n_i}{2}\right) \tag{54}$$

The Central Limit Theorem implies the following convergence for the density function as $n \to \infty$:

$$f(x_i) \mapsto \frac{1}{\sqrt{2\pi}} e^{\frac{-x_i^2}{2}}.$$

For the three-state case, the joint distribution of $\mathbf{x} = (x_1, x_2, x_3)$ as $n \to \infty$ is given by:

$$f(\mathbf{x}) \mapsto \frac{1}{(\sqrt{2\pi})^3} e^{\frac{-|\mathbf{x}|^2}{2}}$$

where $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2$. By subtracting or dividing the number of voters by the same constant, the quantities change but the comparison between them is unchanged, therefore inequations (48) to (53) are satisfied iff inequations (55) to (60) are satisfied too:

$$x_1 > 0, \quad x_2 > 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} < 0 \tag{55}$$
$$x_1 < 0, \quad x_2 < 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} > 0 \tag{56}$$
$$x_1 > 0, \quad x_3 > 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} < 0 \tag{57}$$
$$x_1 < 0, \quad x_3 < 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} > 0 \tag{58}$$
$$x_2 > 0, \quad x_3 > 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} < 0 \tag{59}$$
$$x_2 < 0, \quad x_3 < 0, \quad x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} > 0 \tag{60}$$

Let $P(X)$ be the probability of the referendum paradox for three states of population $n \to \infty$ under the IC condition for the system of inequations (X). Thus,

$$P(X) = \frac{1}{(\sqrt{2\pi})^3} \int_{C_X} e^{\frac{-|\mathbf{x}|^2}{2}} dx_1 dx_2 dx_3$$

where $C_X = \{x \in \mathbb{R}^3 : x \text{ satisfies inqualities in (X) }\}$. We will now detail the computation for the system (55). We must integrate $P(55)$ in the triangular cone delimited by the three straight lines Oa, Ob and Oc (See Figure 14). We write as $r^2 d\Omega dr$ the volume element $dx_1 dx_2 dx_3$ where $d\Omega$ is the element of solid angle and $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ and the integration on r is straightforward. We observe that

$$P(55) = \frac{1}{4\pi} \int d\Omega$$

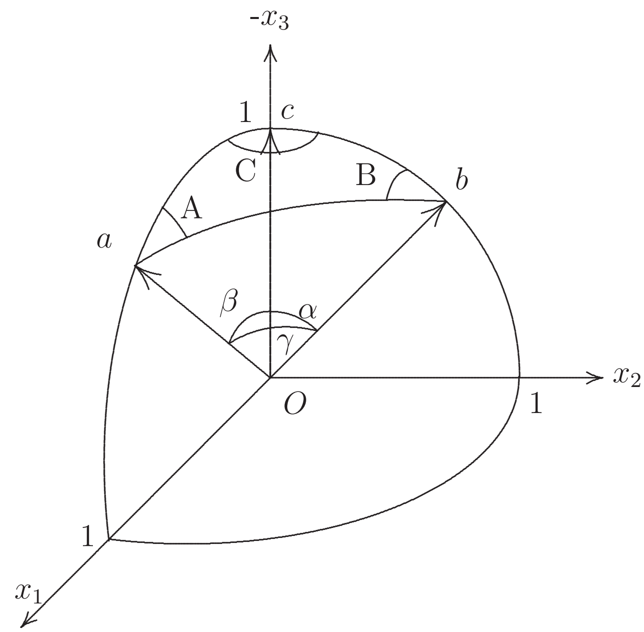

Figure 14: The cone C_{55} of conflicts with the IC model

Hence, computing the desired probability reduces to find the measure of the cone C_1 and to divide it by the surface of the sphere, 4π. Therefore, the measure of the cone is exactly the solid angle of this cone. In fact, C_{55} defines a spherical triangle on the surface of the unit sphere in $\mathbb{R}^3$ (see Figure 14). The area of this surface is given by $S = A+B+C-\pi$ while the area of the sphere is 4π. To perform this surface, notice that each hyperplane defined by the inequalities in 55 defines a normal vector pointing into the half space where the condition is satisfied. We get here:

$$W_1 = (1,0,0),\ W_2 = (0,1,0),\ W_3 = (-\sqrt{n_1}, -\sqrt{n_2}, -\sqrt{n_3})$$

Thus,

$$\begin{aligned}
\cos A &= \tfrac{-W_2.W_3}{||W_2||\,||W_3||} &= \sqrt{n_2} \quad (61)\\
\cos B &= \tfrac{-W_1.W_3}{||W_1||\,||W_3||} &= \sqrt{n_1} \quad (62)\\
\cos C &= \tfrac{-W_1.W_2}{||W_1||\,||W_2||} &= 0 \quad (63)
\end{aligned}$$

Thus

$$\begin{aligned}
S &= A+B+C-\pi \quad (64)\\
&= \arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) - \frac{\pi}{2} \quad (65)\\
P(55) &= \frac{\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right)}{4\pi} - \frac{1}{8} \quad (66)
\end{aligned}$$

We compute the other five probabilities $P(X)$ in a similar way, and then we get:

$$P(3, \tilde{n}, \tilde{a}, IC) = \frac{\arccos\left(\sqrt{n_1}\right) + \arccos\left(\sqrt{n_2}\right) + \arccos\left(\sqrt{n_3}\right)}{\pi} - \frac{3}{4}$$

Case 2: $\tilde{a} = \tilde{a}^2 = (2, 1, 1)$

As for the IAC case, the only modification in the computations come from the 2:2 deadlocks, which are described by equations (59) and (60). We have to replace each of the corresponding volumes by 1/16 instead. Easy computation leads to equation (27). Again, the volumes we have discarded are smaller than the one we added. Hence $P(3, \tilde{n}, a^2, IC) > P(3, \tilde{n}, a^1, IC)$.

Case 3: $\tilde{a} = \tilde{a}^1 = (1, 0, 0)$

The volume of paradoxical events is given by twice the angle between the plane $x_1 > 0$ and $x_1\sqrt{n_1} + x_2\sqrt{n_2} + x_3\sqrt{n_3} < 0$, divided by 2π. Easy computations lead to formula (30).

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