The Condorcet Efficiency of Voting Rules with Mutually Coherent Voter Preferences: A Borda Compromise

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Abstract

The Condorcet Efficiency of a voting rule is defined as the conditional probability that the voting rule elects the Pairwise Majority Rule Winner (PMRW), given that a PMRW exists. Five simple voting rules are considered in this paper: Plurality Rule, Negative Plurality Rule, Borda Rule, Plurality Elimination Rule and Negative Plurality Elimination Rule. In order to study the impact that the presence of degrees of group mutual coherence in voting situations will have on the probability of selecting the PMRW for each of these rules, we develop representations for their Condorcet Efficiency as a function of the proximity of voters' preferences on candidates to being perfectly single-peaked, perfectly single-troughed or perfectly polarized. The results we obtain lead us to appeal for a Borda Compromise.

1. Background and motivation

We consider elections on three candidates $\{A,B,C\}$ where each of *n* voters has some complete preference ranking on the candidates. The possibility of indifference between candidates is ignored, since each voter's preferences are complete. The possibility of cyclic, or intransitive, preferences of voters on candidates is also excluded from consideration. There are six remaining complete preference rankings that each of *n* voters might have on three candidates:

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Here, n_i denotes the number of voters that have the associated complete preference ranking on the candidates. That is, n_1 voters all have preferences with *A* being most preferred, *C* being least preferred and with *B* being ranked between them. Obviously, $\sum_{i=1}^{6} n_i = n$ and any given combination of $n_i' s$ is referred to as a voting situation. A voter preference profile, or profile, is a list of the *n* complete preference rankings on candidates that are associated with each identifiable voter in the electorate. That is, each individual voter's preferences are known in a given profile, while voters' preferences are accumulated anonymously into groups with common preference rankings in a voting situation.

One method of aggregating the preferences of a group of voters to determine the overall most preferred candidate is to use *Pairwise Majority Rule* (*PMR*). Candidate *A* beats *B* by PMR, denoted by ***AMB***, in a voting situation if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$, so that more voters rank *A* as preferred to *B* than the reverse, while ignoring the relative position of *C* in the rankings. Candidate *A* would be the *Pairwise Majority Rule Winner* (*PMRW*) if both ***AMB*** and ***AMC***. We assume that *n* is odd to avoid the possibility of PMR ties, and that voters always report their true preferences. The *Pairwise Majority Rule Loser* (*PMRL*) is defined in the obvious fashion.

The *Condorcet Criterion* suggests that the PMRW should be selected as the winner of any election. This notion is almost universally accepted, but it is well known that such a candidate does not always exist. This observation was presented in examples in Condorcet (1785) where example voting situations are given for which *AMB*, *BMC* and *CMA*, so that no candidate is the clear-cut winner. Such an outcome could lead to a very unstable election result, and many studies have considered the likelihood that this might happen. We ignore this complication for the moment and consider studies that have been performed to evaluate the propensity of various simple voting rules to elect the PMRW, given that one does exist in a voting situation.

1.1. Condorcet Efficiency of Simple Voting Rules

We consider five basic voting procedures for three-candidate elections:

***Plurality Rule* (*PR*) –** Each voter votes for their most preferred candidate, and the candidate who receives the most votes is selected as the winner.

***Negative Plurality Rule* (*NPR*) –** Each voter votes for their two more preferred candidates, and the candidate who receives the most votes is selected as the winner. This is also referred to as Anti-Plurality Rule, since it is equivalent to having each voter cast a vote against their least preferred candidate.

Borda Rule (***BR***) **–** Each voter gives one point to their most preferred candidate and one-half point to their second-ranked candidate. The total number of points that each candidate receives is accumulated over all voters, and the winner is selected as the candidate who receives the most total points.

Plurality Elimination Rule (***PER***) **–** An election is held by PR in a first stage to find the candidate who receives the fewest votes. That candidate is then eliminated from consideration and the ultimate winner is determined in a second stage of the election by using PMR on the two remaining candidates.

Negative Plurality Elimination Rule (***NPER***) **–** An election is held by NPR in a first stage to find the candidate who receives the fewest votes. That candidate is then eliminated from consideration and the ultimate winner is determined in a second stage of the election by using PMR on the two remaining candidates.

Following from the definitions, PR, NPR and BR are defined as single-stage voting rules and PER and NPER are defined as two-stage voting rules.

The *Condorcet Efficiency* of a voting rule has been defined as the conditional probability that the voting rule elects the PMRW, given that a PMRW exists. None of the voting rules that are being considered guarantee that the PMRW will be elected, but we would hope that they would have a very high probability of selecting the PMRW, given that one exists. The conditional probability that defines Condorcet Efficiency is clearly highly dependent on what is being assumed about the relative likelihood that different voting situations or voter preference profiles are being observed. A commonly used assumption in the literature is the *Impartial Culture Condition* (*IC*), which assumes that all voters have independent preferences on the candidates and that each of the six-possible preference rankings is equally likely to reflect the preferences of a randomly selected identifiable voter. This is equivalent to assuming that all voter preference profiles are equally likely to be observed. Representations have been obtained for the Condorcet Efficiency of the five Voting Rules (VR's) for three-candidate elections in the limiting case of voters $(n \to \infty)$ with IC, which is denoted as $CE_{VR}(3,\infty \mid IC)$.

Gehrlein and Fishburn (1978a, 1978b) show that $CE_{PR}(3,\infty \mid IC) = CE_{NPR}(3,\infty \mid IC)$ and find representations that can be reduced to

$$CE_{PR}(3,\infty \mid IC) = \frac{\left[\begin{array}{l} \frac{1}{4} + \frac{3}{4\pi^2}\left\{\left(Cos^{-1}\left(-\sqrt{\frac{2}{3}}\right)\right)^2 - \frac{1}{4}\left(\pi - Sin^{-1}\left(\frac{1}{3}\right)\right)^2\right\} \\ + \frac{3}{4\pi} Sin^{-1}\left(\sqrt{\frac{1}{6}}\right) + \frac{9}{8\pi^2}\int_0^{1/3} \frac{Sin^{-1}\{\theta/(1+2\theta)\}}{\sqrt{1-\theta^2}}\, d\theta \end{array}\right]}{\frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right)} \approx .7572 .$$

$$CE_{BR}(3,\infty \mid IC) = \frac{\frac{3}{2} - \frac{3}{2\pi}\left[Cos^{-1}\left(\sqrt{\frac{8}{9}}\right) + Cos^{-1}\left(\sqrt{\frac{2}{9}}\right)\right]}{\frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right)} \approx .9012 .$$

Gehrlein (1993) shows that $CE_{PER}(3,\infty \mid IC) = CE_{NPER}(3,\infty \mid IC)$ and develops a limiting representation

$$CE_{PER}(3,\infty \mid IC) = \frac{\left[\begin{array}{c} -1 - \frac{3}{4\pi^2}\left(Sin^{-1}\left(\sqrt{\frac{1}{3}}\right)\right)^2 + \frac{3}{16\pi^2}\left(3\pi + Sin^{-1}\left(\frac{1}{3}\right)\right)^2 \\ + \frac{3}{4\pi}Sin^{-1}\left(\sqrt{\frac{1}{6}}\right) - \frac{9}{8\pi^2}\int_0^{1/3}\frac{Sin^{-1}\{\theta/(1+2\theta)\}}{\sqrt{1-\theta^2}}d\theta \end{array}\right]}{\frac{3}{4} + \frac{3}{2\pi}Sin^{-1}\left(\frac{1}{3}\right)} \approx .9629.$$

The obvious conclusion is that BR is superior on the basis of Condorcet Efficiency to the other two single-stage voting rules, while both two-stage voting rules are superior to all single-stage voting rules. This observation is however completely dependent on the assumption of IC as $n \to \infty$.

Another commonly used assumption regarding the probability that various voting situations are observed is the *Impartial Anonymous Culture Condition* (*IAC*). The IAC condition assumes that all voting situations are equally likely to be observed. Gehrlein (1982) develops representations for the Condorcet Efficiency of PR, NPR, PER and NPER under the assumption of IAC for n = 9(12)…

$$CE_{PR}(3,n \mid IAC) = \frac{119n^4 + 1348n^3 + 5486n^2 + 10812n + 10395}{135(n+1)(n+3)^2(n+5)}$$

$$CE_{NPR}(3,n \mid IAC) = \frac{68n^3 + 501n^2 + 834n - 315}{108(n+1)(n+3)(n+5)}$$

$$CE_{PER}(3,n \mid IAC) = \frac{523n^4 + 6191n^3 + 25117n^2 + 40749n + 22140}{540(n+1)(n+3)^2(n+5)}$$

$$CE_{NPER}(3,n \mid IAC) = \frac{131n^4 + 1542n^3 + 6144n^2 + 9018n + 3645}{135(n+1)(n+3)^2(n+5)}.$$

Gehrlein and Lepelley (2001) develop a similar IAC based representation for $CE_{BR}(3,n \mid IAC)$ for n = 9(12)… with

$$CE_{BR}(3,n \mid IAC) = \frac{123n^4 + 1416n^3 + 5722n^2 + 10104n + 8235}{135(n+1)(n+3)^2(n+5)}.$$

In the limit as $n \to \infty$, $CE_{PR}(3,\infty \mid IAC)$ = 119/135 =.8815, $CE_{NPR}(3,\infty \mid IAC)$ = 68/108 = .6296, $CE_{BR}(3,\infty \mid IAC)$ = 123/135 = .9111, $CE_{PER}(3,\infty \mid IAC)$ = 523/540 =

.9685, and $CE_{NPER}(3,\infty \mid IAC)$ = 131/135 = .9704. We find again that BR is superior on the basis of Condorcet Efficiency to the other two single-stage voting rules, while both two-stage voting rules are superior to all single-stage voting rules. As before, this observation is completely dependent on the assumption of IAC as $n \to \infty$. Gehrlein and Lepelley (1999) find similar observations under another assumption that is referred to as the *Maximal Culture Condition* (*MC*).

Assumptions like IC, IAC and MC are described in Gehrlein (2002a) as representing voting scenarios in which voters have "balanced preferences" on the candidates, so that no candidate is assumed a priori to have any advantage in the preference rankings of the voters. It is suggested that such situations represent very general conditions under which the possibility of having any voting rule select the PMRW tends to be minimized, thereby reflecting general situations under which the selection of a voting rule that is to be used becomes most critical. Fishburn and Gehrlein (1982) also suggest that while the probabilities that are observed with IC and IAC might be exaggerated, there is little reason to expect that the relative likelihoods of various voting events would change significantly with more general models.

Others have been critical of research based on assumptions like IC and IAC. The concerns of this group focus on the fact that such assumptions place too much weight on the probability that some very unrealistic profiles and voting situations might be observed. Much of this work is summarized in Regenwetter et al (2006), who invite us to use more realistic assumptions about preference distributions.

1.2. Models with Degrees of Group Mutual Coherence

Recent advances in the development of procedures that can be used to obtain representations for the probability that various election outcomes are observed under IAC-like conditions have significantly enhanced our ability to consider the impact that the presence of degrees of group mutual coherence in given voting situations will have on the probability that election outcomes will be observed. By degree of group mutual coherence in a given voting situation, we mean the relative degree to which the preferences of voters in a voting situation seem to be mutually coherent, in the sense that they are based on some common model of rational preference formation. A group is considered to be mutually coherent if most of the voters in the group are forming their preferences according to the same rational model, despite the fact that the individual voters might arrive at having very different preferences based on this model.

Gehrlein (2006a) presents three parameters (b,t,c) that are defined for a given voting situation to consider the relative degree to which that voting situation is based on three such models of mutually coherent preference formation:

$$b = Minimum\{n_5 + n_6, n_2 + n_4, n_1 + n_3\}.$$

$$t = Minimum\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}$$

$$c = Minimum\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}.$$

By definition, b measures the minimum number of times that some candidate is bottom ranked as the least preferred in a given voting situation. If $b = 0$ for a given voting situation for three candidates, Arrow (1963) shows that the voters' preferences must then reflect the widely used condition of perfectly single-peaked preferences from Black (1958). The value of b is used to measure the relative degree to which a voting situation represents the condition of perfectly single-peaked preferences, since b is effectively the minimum number of voters whose preferences must be discarded in order to have perfectly single-peaked preferences for the remaining voters. Thus, as b increases relative to n, we have voting situations that are farther removed from the condition of perfectly single-peaked preferences.

Parameter t measures the minimum number of times that some candidate is top-ranked as the most preferred in the preference rankings of candidates, and it follows that a voting situation is perfectly single-troughed if $t = 0$. The value of t then reflects the relative proximity of a voting situation to the condition of perfectly single-troughed preferences. Single-troughed preferences are also referred to as single-dipped preferences or single-caved preferences in the literature. We use the phrase single-troughed, since that was the term that was used in the original work on the topic by Vickery (1960).

Parameter c comes from work in Ward (1965) which develops a result that leads to the conclusion that a PMRW must exist in a three-candidate election if some candidate is perfectly polarizing, in the sense that this candidate is never middle ranked, or ranked at the center, of any voter's preference ranking. That is, each voter considers this candidate to be either the most preferred or the least preferred, with $c = 0$. By definition, $0 \le b,t,c \le n/3$ and the proportion of voters who have preferences that reflect the proximity of a voting situation to perfectly single-peaked preferences, perfectly single-troughed preferences and perfectly polarized preferences decreases as the relative values of *b, c* and *t* increase for a given *n*.

We are interested in determining the impact that these measures of group coherence have on the Condorcet Efficiency of the five voting rules that we are considering for three-candidate elections. This is done by developing representations for the Condorcet Efficiency of voting rules, $CE_{VR}(3,n \mid IAC_X(k))$, under the assumption that all voting situations for which Parameter *X* has a specified value of *k* are equally likely to be observed, for $X \in \{b,t,c\}$. We now return to the previously expressed concern about the possibility that a PMRW might not exist. If there is a significant probability that a PMRW does not exist, the conditional nature of the definition of Condorcet Efficiency makes it a meaningless measure for evaluating voting rules. Gehrlein (2006a) shows that if we have a large group of voters whose preferences reflect voting situations are at all close to the scenarios of perfectly single-peaked, perfectly single-troughed or perfectly polarized preferences in three-candidate elections, then it is very likely that a PMRW will exist.

While it is not possible to know a priori how a group of potential voters will be forming their individual preferences on candidates in any given election, it is quite plausible to assume that most voters will be mutually forming their preferences according to one of the three models that have been discussed, depending upon the particular situation at hand. Moreover, it really does not matter which of these models might apply in any particular case in terms of the ultimate result that there is a very high probability that a PMRW will exist for large electorates. As a result, the use of Condorcet Efficiency to evaluate the effectiveness of voting rules is very reasonable for three-candidate elections, since very few people object to the basic logic that leads to the Condorcet Criterion.

2. Representations for $CE_{VR}(3,n \mid IAC_X(k))$

Gehrlein (2005, 2006b) develops a procedure (EUPIA2) that can be used to obtain representations for $CE_{VR}(3,n \mid IAC_X(k))$. The logic behind this procedure is quite straightforward to develop, but it can become extremely cumbersome to implement. Gehrlein and Lepelley (2007a, 22007b) use the EUPIA2 procedure to develop a number of related representations under the $IAC_X(k)$ assumption. The representations that are obtained for general *n* become so complex that analysis is restricted to the limiting case as $n \to \infty$. It is also found in these studies that while it is theoretically possible for EUPIA2 to obtain these types of representations, there are some cases for which it is not practically possible to obtain them with a reasonable degree of effort.

When it was not feasible to use EUPIA2 in Gehrlein and Lepelley (2007a, 2007b), a procedure from Lepelley et al (2006) was used. Lepelley et al (2006) point out that the EUPIA2 procedure is a special case of a more general problem, which is equivalent to counting the integral points in polyhedra, that was first studied by Ehrhart (1962). Some new and efficient algorithms have been recently proposed by computer scientists to solve this problem. In Gehrlein and Lepelley (2007a, 2007b), an algorithm that was developed by Barvinok and Pommershein (1999) (see Lepelley et al, 2006, for a presentation of this procedure) was used to obtain limiting representations for probabilities as $n \to \infty$.

We adopt a similar strategy in the current study, and use EUPIA2 to develop representations for $CE_{VR}(3,n \mid IAC_X(k))$ whenever it is possible to do so, since these representations for finite *n* can be verified by computer enumeration. However, due to the extreme complexity of the resulting representations, we only present the resulting limiting representations in the current study. The general representations for finite *n* are available from the authors on request. When EUPIA2 was not able to obtain the general representation with reasonable effort, the algorithm due to Barvinok and Pommershein (1999) was used to obtain the limiting representation.

The definition of the $IAC_X(k)$ assumption in the $CE_{VR}(3,n \mid IAC_X(k))$ representations is quite direct, but it needs some elaboration when we consider the limiting distribution as $n \to \infty$. In this limit, we do not define specific values of *k* for

Parameter *X*, but instead consider the proportion $\alpha_k = k/n$ as $n \to \infty$. The limiting Condorcet Efficiency representations are therefore given by $CE_{VR}(3,\infty \mid IAC_X(\alpha_k))$. It follows that $0 \le \alpha_k \le 1/3$ and that α_k is the minimum proportion of all voters whose preferences must be ignored to make the remaining voters' preferences perfectly single-peaked (when *X* = *b*), perfectly single-troughed (when *X* = *t*) or perfectly polarized (when (*X* = *c*).

2.1. Single-Stage Representations

We summarize the limiting representations for $CE_{VR}(3,\infty \mid IAC_X(\alpha_k))$ that are obtained in the current study and then go on to analyze the results. The representations for PR are taken directly from Gehrlein and Lepelley (2007a).

PR for Parameter *b*:

$$CE_{PR}(3,\infty \mid IAC_b(\alpha_k)) = \frac{432\alpha_k^3 - 198\alpha_k^2 - 81\alpha_k + 31}{36(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 0 \le \alpha_k \le 1/6$$

$$= \frac{6480\alpha_k^4 - 3348\alpha_k^3 - 54\alpha_k^2 + 150\alpha_k + 1}{216\alpha_k(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{3024\alpha_k^3 - 2700\alpha_k^2 + 792\alpha_k - 109}{108(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k < 1/3$$

PR for Parameter *t*:

$$CE_{PR}(3,\infty \mid IAC_t(\alpha_k)) = \frac{72\alpha_k^3 - 22\alpha_k^2 - 27\alpha_k + 8}{8(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 0 \le \alpha_k \le 1/4$$

$$= \frac{(2\alpha_k - 1)(12\alpha_k^2 - 15\alpha_k + 5)}{4(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k < 1/3.$$

PR for Parameter *c*:

$$CE_{PR}(3,\infty \mid IAC_c(\alpha_k)) = \frac{2(931\alpha_k^3 - 276\alpha_k^2 - 189\alpha_k + 62)}{9(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{for } 0 \le \alpha_k \le 1/6.$$

$$= \frac{(-44736\alpha_k^4 + 48096\alpha_k^3 - 24840\alpha_k^2 + 4776\alpha_k - 77)}{216\alpha_k(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{\left(104640\alpha_k^4 - 140256\alpha_k^3 + 71496\alpha_k^2 - 14568\alpha_k + 815\right)}{216\left(117\alpha_k^4 - 228\alpha_k^3 + 150\alpha_k^2 - 32\alpha_k + 1\right)}, \text{ for } 1/4 \le \alpha_k \le 3/10$$

$$= \frac{2\left(-945\alpha_k^3 + 513\alpha_k^2 - 27\alpha_k + 25\right)}{27\left(39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1\right)}, \text{ for } 3/10 \le \alpha_k \le 1/3.$$

The remaining single-stage voting rule representations that have been developed for the current study are:

NPR for Parameter *b*:

$$CE_{NPR}(3,\infty \mid IAC_b(\alpha_k)) = \frac{3(2\alpha_k - 1)\left(4\alpha_k^2 + 2\alpha_k - 1\right)}{4\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{for } 0 \le \alpha_k \le 1/4$$

$$= \frac{-3\alpha_k(2\alpha_k - 1)^2}{18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1}, \text{for } 1/4 \le \alpha_k \le 1/3.$$

NPR for Parameter *t*:

$$CE_{NPR}(3,\infty \mid IAC_t(\alpha_k)) = \frac{1179a_k^3 - 252\alpha_k^2 - 162\alpha_k + 47}{72\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{for } 0 \le \alpha_k \le 1/6$$

$$= \frac{-4590a_k^4 + 5832a_k^3 - 2700\alpha_k^2 + 462\alpha_k - 7}{432a_k\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{3267a_k^3 - 1971\alpha_k^2 + 225\alpha_k - 10}{108\left(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1\right)}, \text{for } 1/4 \le \alpha_k \le 1/3.$$

NPR for Parameter *c*:

$$CE_{NPR}(3,\infty \mid IAC_c(\alpha_k)) = \frac{-1404\alpha_k^3 + 252\alpha_k^2 - 135\alpha_k + 61}{9\left(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16\right)}, \text{for } 0 \le \alpha_k \le 1/6$$

$$= \frac{94608\alpha_k^4 - 88128\alpha_k^3 + 22464\alpha_k^2 - 1632\alpha_k + 139}{216\alpha_k\left(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16\right)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{-2\left(2403\alpha_k^3 - 1323\alpha_k^2 + 81\alpha_k - 7\right)}{27\left(39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1\right)}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

BR for Parameter *b*:

$$CE_{BR}(3,\infty \mid IAC_b(\alpha_k)) = \frac{507\alpha_k^3 - 184\alpha_k^2 - 132\alpha_k + 44}{48\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{ for } 0 \le \alpha_k \le 1/6$$

$$= \frac{5352\alpha_k^4 - 2336\alpha_k^3 - 840\alpha_k^2 + 328\alpha_k + 1}{384\alpha_k\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{60696\alpha_k^4 - 63712\alpha_k^3 + 24744\alpha_k^2 - 4024\alpha_k + 203}{192(1-3\alpha_k)(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k \le 2/7$$

$$= \frac{73760\alpha_k^4 - 89952\alpha_k^3 + 41112\alpha_k^2 - 8520\alpha_k + 693}{192(3\alpha_k - 1)(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 2/7 \le \alpha_k \le 3/10$$

$$= \frac{165\alpha_k^3 - 153\alpha_k^2 + 47\alpha_k - 6}{4(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 3/10 \le \alpha_k < 1/3.$$

BR for Parameter *t*:

$$CE_{BR}(3,\infty \mid IAC_t(\alpha_k)) = \frac{195\alpha_k^3 - 80\alpha_k^2 - 66\alpha_k + 22}{24(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 0 \le \alpha_k \le 1/6$$

$$= \frac{4416\alpha_k^4 - 2144\alpha_k^3 - 840\alpha_k^2 + 328\alpha_k + 1}{384\alpha_k(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 1/6 \le \alpha_k \le 1/5$$

$$= \frac{54416\alpha_k^4 - 42144\alpha_k^3 + 11160\alpha_k^2 - 1272\alpha_k + 81}{384(1-3\alpha_k)(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 1/5 \le \alpha_k \le 1/4$$

$$= \frac{62096\alpha_k^4 - 74400\alpha_k^3 + 33336\alpha_k^2 - 6792\alpha_k + 549}{192(3\alpha_k - 1)(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k \le 3/10$$

$$= \frac{84\alpha_k^3 - 72\alpha_k^2 + 20\alpha_k - 3}{4(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 3/10 \le \alpha_k < 1/3.$$

BR for Parameter *c*:

$$CE_{BR}(3,\infty \mid IAC_c(\alpha_t)) = \frac{2379\alpha_k^3 - 648\alpha_k^2 - 564\alpha_k + 176}{12(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{ for } 0 \le \alpha_k \le 1/6$$

$$= \frac{9303\alpha_k^4 - 1512\alpha_k^3 - 3252\alpha_k^2 + 928\alpha_k - 2}{60\alpha_k(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{ for } 1/6 \le \alpha_k \le 1/5$$

$$= \frac{-6322\alpha_k^4 + 10988\alpha_k^3 - 7002\alpha_k^2 + 1428\alpha_k - 27}{60\alpha_k(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{ for } 1/5 \le \alpha_k \le 1/4$$

$$= \frac{6322\alpha_k^4 - 10988\alpha_k^3 + 7002\alpha_k^2 - 1428\alpha_k + 27}{60(3\alpha_k - 1)(39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k \le 2/7$$

$$= \frac{-699\alpha_k^3 + 315\alpha_k^2 + 23\alpha_k + 17}{12(39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1)}, \text{ for } 2/7 \le \alpha_k < 1/3.$$

The representations for $CE_{VR}(3,\infty \mid IAC_b(0))$ verify limiting efficiency values for $VR \in \{PR, NPR, BR\}$ in Lepelley (1995). We also note the easily proved result that $CE_{PR}(3,\infty \mid IAC_t(0)) = 1$. These representations were used to obtain values of $CE_{VR}(3,\infty \mid IAC_X(\alpha_k))$ for each $\alpha_k = 0.00(.01).33$, with $X \in \{b,t,c\}$ and $VR \in \{PR, NPR, BR\}$. The results are shown graphically in Figure 1 for Parameter *b*, in Figure 2 for Parameter *c* and in Figure 3 for Parameter *t*.

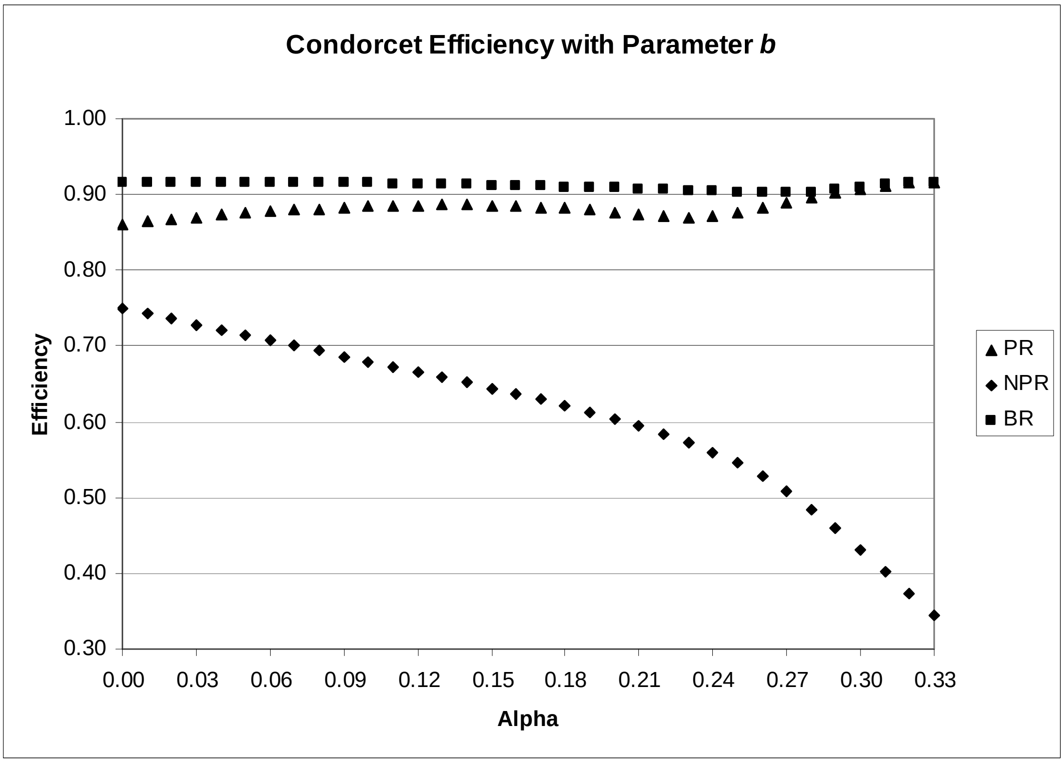

Figure 1. Graphical values of $CE_{VR}(3,\infty \mid IAC_b(\alpha_k))$ for PR, NPR and BR.

The results in Figure 1 show that the overall results from IAC representations that accumulate the $IAC_b(k)$ probabilities over the range of all possible *b* values are completely consistent with the results that are observed at every level of possible *b* values. That is, BR is superior to PR on the basis of Condorcet Efficiency, and both are consistently superior to NPR. While the numerical values of $CE_{VR}(3,\infty \mid IAC_c(\alpha_k))$ are quite different in Figure 2 than the $CE_{VR}(3,\infty \mid IAC_b(\alpha_k))$ values in Figure 1, the same general pattern of the relative performance of VR, NPR and BR on the basis of Condorcet Efficiency is observed in both.

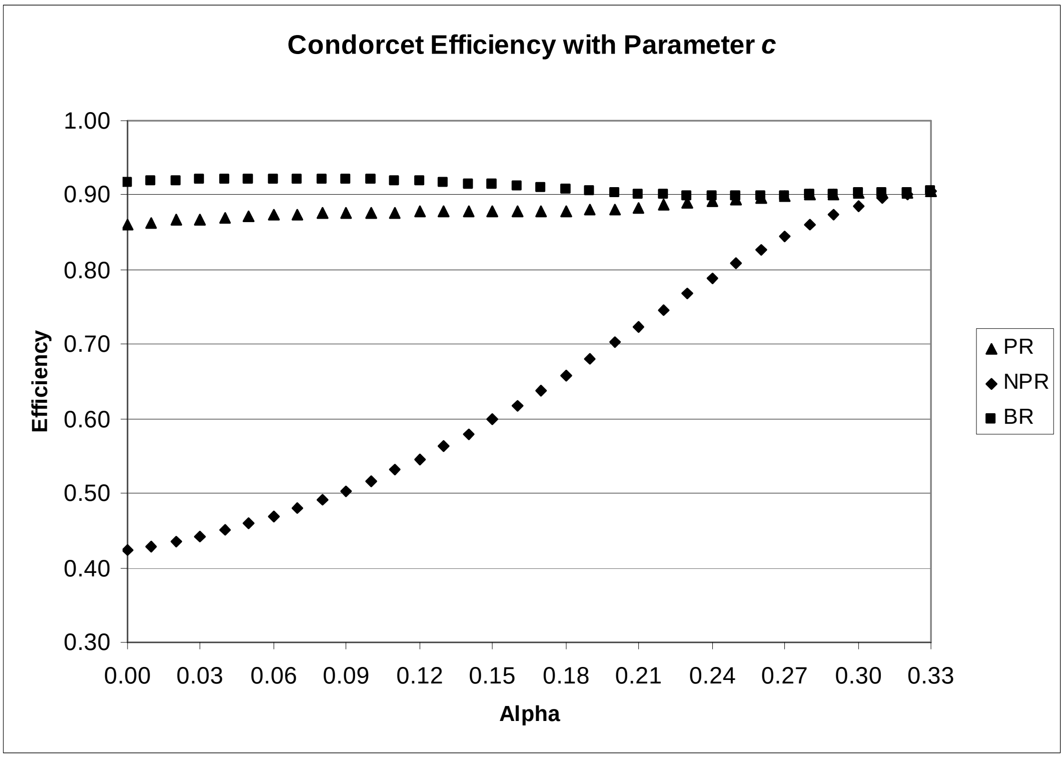

Figure 2. Graphical values of $CE_{VR}(3,\infty \mid IAC_c(\alpha_k))$ for PR, NPR and BR.

The observations in Figure 3 show that the relative performance of PR, NPR and BR on the basis of Condorcet Efficiency is not as consistent over the range of α_k values for Parameter *c* as it is for Parameters *b* and *c*. BR is consistently superior to NPR over the entire range of α_k values. PR is the most Condorcet Efficient voting rule for the range of values with $0 \le \alpha_k \le .17$, while it is the least Condorcet Efficient voting rule for the range of values with $.25 \le \alpha_k \le .33$. The Condorcet Efficiency of PR can be very poor in some cases, since $CE_{VR}(3,\infty \mid IAC_t(1/3)) \to 1/3$. This effectively makes PR a random chooser of an election winner when $\alpha_k \to 1/3$ with Parameter *t*. However, this would reflect a very non-intuitive situation in which the voters were the least mutually coherent with a model driven by single-troughed preferences on candidates.

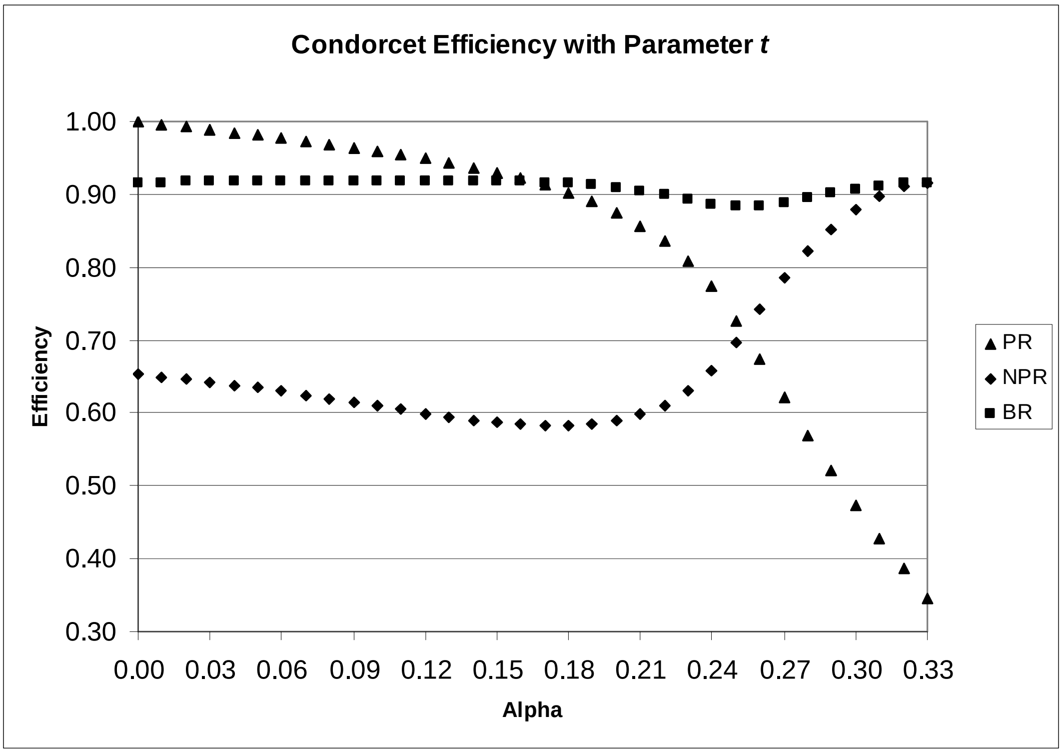

Figure 3. Graphical values of $CE_{VR}(3,\infty \mid IAC_t(\alpha_k))$ for PR, NPR and BR.

It is definitely of interest to gain some insight into what proportion of all voting situations fall into the region with $0 \le \alpha_k \le .17$ for Parameter *t*, so we have a better idea of how significant the range of dominance of PR over BR actually is in Figure 3. In order to determine this, EUPIA2 was used to obtain representations for $CP_{VS*}^{X}(3,\infty \mid \alpha_k^{-})$, the limiting cumulative proportion as $n \to \infty$ of all possible voting situations that have a PMRW which have a specified value of α_k or less for parameter *X*.

The resulting representations are given by:

For Parameters *b* and *t*:

$$CP_{VS*}^{b}(3,\infty \mid \alpha_k^{-}) = CP_{VS*}^{t}(3,\infty \mid \alpha_k^{-})$$

$$= \frac{32}{5}\alpha_k^2\left(22\alpha_k^3 - 10\alpha_k^2 - 10\alpha_k + 5\right), \text{ for } 0 \le \alpha_k \le 1/4$$

$$= \frac{1728\alpha_k^5 - 2880\alpha_k^4 + 1920\alpha_k^3 - 720\alpha_k^2 + 160\alpha_k - 11}{5}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

For Parameter *c*:

$$CP_{VS*}^{c}\left(3,\infty\,|\,\alpha_k^{-}\right)=\frac{4}{5}\alpha_k^2\left(139\alpha_k^3-35\alpha_k^2-90\alpha_k+40\right),\ \text{for } 0\le\alpha_k\le 1/4$$

$$=\frac{-468\alpha_k^5+1140\alpha_k^4-1000\alpha_k^3+320\alpha_k^2-20\alpha_k+1}{5},\ \text{for } 1/4\le\alpha_k\le 1/3.$$

Since $CP_{VS*}^{b}\left(3,\infty\,|\,.17^{-}\right)=.5769$, the range over which PR has a greater Condorcet Efficiency than BR covers a significant proportion of all possible voting situations, based on our measure of proximity to single troughed preferences. At the other extreme, the observation that $CP_{VS*}^{b}\left(3,\infty\,|\,.25^{-}\right)=.8875$ indicates that PR is only dominated by both of the other single-stage voting rules in the 11.25% of voting situations that are the farthest removed from the condition of perfectly single-troughed preferences.

2.2. A Borda Compromise

Since we do not know a priori which type of model will be the basis by which voters preferences will be formed, there is no absolute answer as to which single-stage voting rule will result in the greatest Condorcet Efficiency. BR is the obvious choice if preferences are based on single-peaked preferences or polarized preferences. However, the answer is mixed when preferences are based on single-troughed preferences.

The calculated values show us that any tendency towards single-peaked preferences, single-troughed preferences or polarized preferences has a weak impact on the Condorcet Efficiency of BR. It also turns out that $CE_{BR}\left(3,\infty\,|\,IAC_X\left(\alpha_k\right)\right)\ge .88$ for $X\in\{b,t,m\}$. By contrast, both PR and NPR can have values of Condorcet Efficiency values that can fall to .33. It is clearly not a feasible option to obtain the votes from the electorate before the decision is made as to how the winner of an election will be determined. As a consequence, an appeal to a “maximin” argument could be used to support the use of BR when nothing is known a priori about the type of model that is most likely to reflect the preferences that an electorate might typically have. Since we also know that BR can not elect the PMRL, while both PR and NPR can do so, BR can be supported as being a good voting rule to use in a single stage voting situation. It is noted that this assertion is based on the assumption that voters will report their true preferences, and that voters must report more information in an election with BR than with PR or NPR.

2.3. Two-Stage Representations

As in the analysis of single-stage voting rules, we summarize the limiting representations for $CE_{VR}(3,\infty \mid IAC_X(\alpha_k))$ that are obtained in the current study for two-stage voting rules and then go on to analyze the results.

PER for Parameter *b*:

$$CE_{PER}(3,\infty \mid IAC_b(\alpha_k)) = \frac{1458a_k^3 - 540\alpha_k^2 - 405\alpha_k + 137}{144(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{for } 0 \le \alpha_k \le 1/6$$

$$= \frac{77760a_k^4 - 48384a_k^3 + 864\alpha_k^2 + 1920\alpha_k + 65}{3456a_k(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{297a_k^3 - 297\alpha_k^2 + 99\alpha_k - 20}{27(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{for } 1/4 \le \alpha_k \le 1/3$$

PER for Parameter *t*:

$$CE_{PER}(3,\infty \mid IAC_t(\alpha_k)) = \frac{10a_k^3 - 4\alpha_k^2 - 3\alpha_k + 1}{11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1}, \text{for } 0 \le \alpha_k \le 1/4$$

$$= \frac{(2\alpha_k - 1)(6\alpha_k^2 - 3\alpha_k + 1)}{18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1}, \text{for } 1/4 \le \alpha_k \le 1/3$$

PER for Parameter *c*:

$$CE_{PER}(3,\infty \mid IAC_c(\alpha_k)) = \frac{1557a_k^3 - 288\alpha_k^2 - 459\alpha_k + 137}{9(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{for } 0 \le \alpha_k \le 1/8$$

$$= \frac{-6060a_k^4 + 4992\alpha_k^3 - 2988\alpha_k^2 + 644\alpha_k - 3}{36\alpha_k(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{for } 1/8 \le \alpha_k \le 1/6$$

$$= \frac{53064a_k^4 - 31392\alpha_k^3 - 2160\alpha_k^2 + 2064\alpha_k + 59}{216\alpha_k(139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16)}, \text{for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{1917a_k^3 - 2565\alpha_k^2 + 1071\alpha_k - 59}{27(39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1)}, \text{for } 1/4 \le \alpha_k \le 1/3$$

NPER for Parameter *b*:

$$CE_{NPER}\left(3,\infty \mid IAC_b\left(\alpha_k\right)\right)=\frac{179a_k^3-68\alpha_k^2-48\alpha_k+16}{16\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)}, \text{for } 0\le\alpha_k\le 1/4$$

$$=\frac{-111\alpha_k^3+111\alpha_k^2-27\alpha_k-1}{8\left(18\alpha_k^3-18\alpha_k^2+6\alpha_k-1\right)}, \text{for } 1/4\le\alpha_k\le 1/3$$

NPER for Parameter *t*:

$$CE_{NPER}\left(3,\infty \mid IAC_t\left(\alpha_k\right)\right)=\frac{288a_k^3-108\alpha_k^2-108\alpha_k+35}{36\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)}, \text{for } 0\le\alpha_k\le 1/6$$

$$=\frac{15120a_k^4-8640a_k^3+432\alpha_k^2+240\alpha_k+7}{432a_k\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)}, \text{for } 1/6\le\alpha_k\le 1/4$$

$$=\frac{54a_k^3-54\alpha_k^2+18\alpha_k-11}{27\left(18\alpha_k^3-18\alpha_k^2+6\alpha_k-1\right)}, \text{for } 1/4\le\alpha_k\le 1/3$$

NPER for Parameter *c*:

$$CE_{NPER}\left(3,\infty \mid IAC_c\left(\alpha_k\right)\right)=\frac{1653a_k^3-756\alpha_k^2-324\alpha_k+128}{9\left(139\alpha_k^3-28\alpha_k^2-54\alpha_k+16\right)}, \text{for } 0\le\alpha_k\le 1/6$$

$$=\frac{8172a_k^4+3024\alpha_k^3-7992\alpha_k^2+2112\alpha_k-29}{108\alpha_k\left(139\alpha_k^3-28\alpha_k^2-54\alpha_k+16\right)}, \text{for } 1/6\le\alpha_k\le 1/4$$

$$=\frac{1431a_k^3-2079\alpha_k^2+909\alpha_k-41}{27\left(39\alpha_k^3-63\alpha_k^2+29\alpha_k-1\right)}, \text{for } 1/4\le\alpha_k\le 1/3$$

Some results follow directly from these representations. First, we observe the very easily proved result that $CE_{PER}\left(3,\infty \mid IAC_t(0)\right)$ = $CE_{NPER}\left(3,\infty \mid IAC_b(0)\right)$ = 1. In addition, as $\alpha_k \to 1/3$ we note that $CE_{PER}\left(3,\infty \mid IAC_t(1/3)\right)$ = $CE_{NPER}\left(3,\infty \mid IAC_b(1/3)\right)$ = 2/3. However, slightly different values are obtained from these two representations over the rest of the range of α_k values.

These representations were used to obtain values of $CE_{VR}\left(3,\infty \mid IAC_X\left(\alpha_k\right)\right)$ for each $\alpha_k = 0.00(.01).33$, with $X \in \{b,t,c\}$ and $VR \in \{PER, NPER\}$. The results are shown graphically in Figure 4 for Parameter *b*, in Figure 5 for Parameter *t* and in Figure 6 for Parameter *c*. Values of $CE_{BR}\left(3,\infty \mid IAC_X\left(\alpha_k\right)\right)$ are also included in each of these figures for comparison purposes.

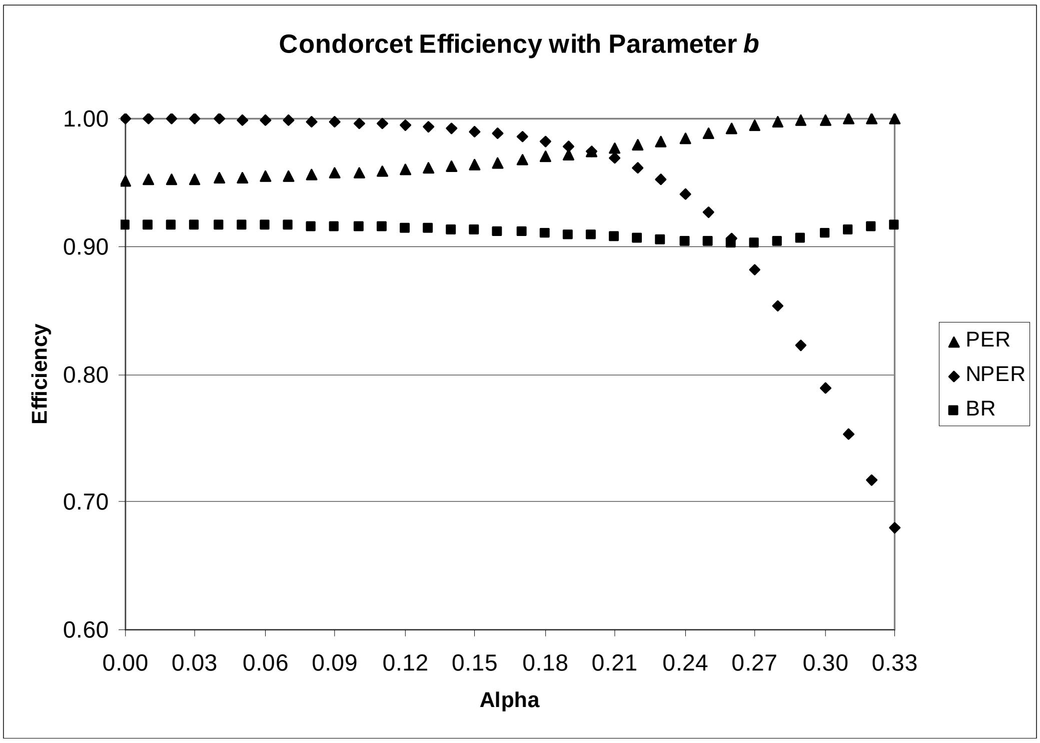

Figure 4. Graphical values of $CE_{VR}(3,\infty \mid IAC_b(\alpha_k))$ **for PER, NPER and BR.**

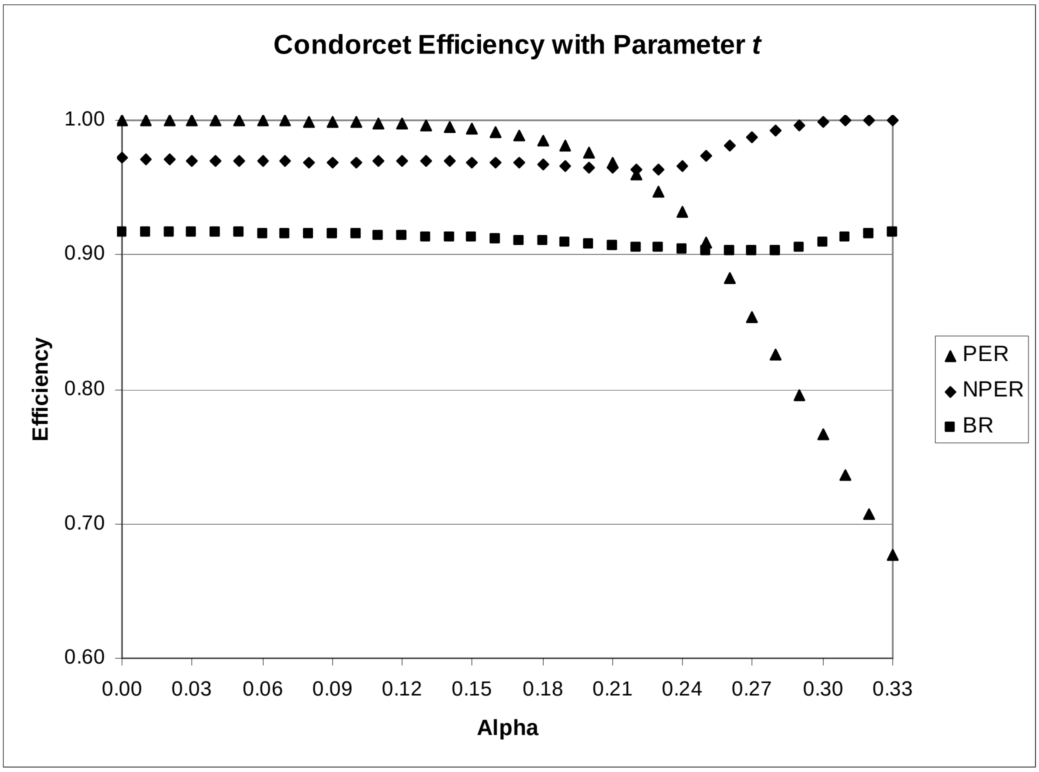

Figure 5. Graphical values of $CE_{VR}(3,\infty \mid IAC_t(\alpha_k))$ **for PER, NPER and BR.**

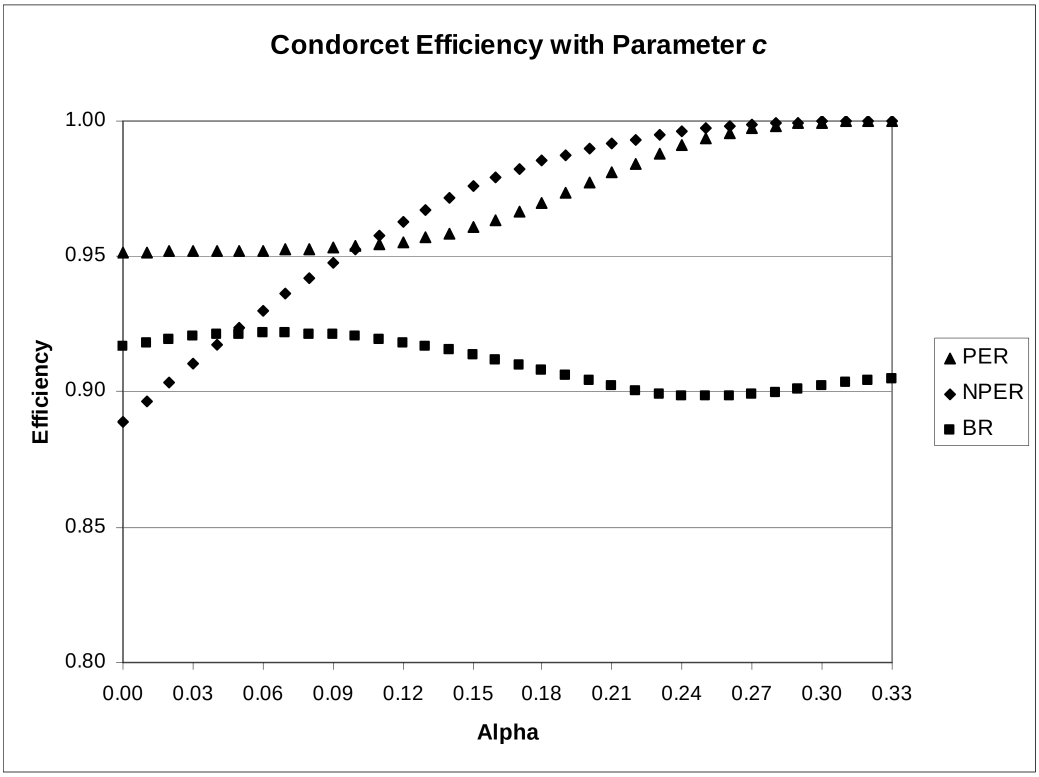

Figure 6. Graphical values of $CE_{VR}\left(3,\infty \mid IAC_c\left(\alpha_k\right)\right)$ for PER, NPER and BR.

The results in Figure 4 indicate that both two-stage election rules have greater values of Condorcet efficiency than BR does over the range $0 \le \alpha_k \le .26$ with the model based on Parameter *b*. Since $CP_{VS*}^{b}\left(3,\infty \mid .26^{-}\right) = .9132$, there is only a very small range over which BR is more Condorcet Efficient than NPER, and that range represents the less than 10% of voting situations that have a PMRW that are farthest removed from being perfectly single-peaked. The same general observations are made with regard to the results in Figure 5 with Condorcet Efficiency values that are based on Parameter *t*. The difference in this case is that it is the less than 10% of voting situations that have a PMRW that are farthest removed from being perfectly single-troughed in which BR is more Condorcet Efficient than PER.

Different results are observed in Figure 6, where Condorcet Efficiencies are based on Parameter *c*. Here, both two stage voting rules have greater values of Condorcet Efficiency that BR does over most of the region of possible α_k values. The only exception is that BR has a greater Condorcet Efficiency that NPER in the range

$0 \le \alpha_k \le .05$. Since $CP_{VS*}^{c}\left(3,\infty \mid .05^{-}\right) = .0709$, both two-stage voting rule have greater Condorcet Efficiency that BR, except for the 7% of voting situations that have a PMRW that are closest to having perfectly polarized preferences, as measure by Parameter *c*.

If we accept the hypothesis that we would generally expect to have large electorates that are neither highly mutually coherent nor highly mutually incoherent in their preferences, we would conclude that both two-stage voting rules have greater Condorcet Efficiency than BR in the realm of realistic voting situations for models that are based on Parameters *b*, *t* and *c*. However, the previous discussion of the Borda Compromise still applies in general; particularly since the use of two-stage voting systems reflect the use of more complicated voting procedures.

3. Concluding remark and future research

Gehrlein and Lepelley (2007a) introduced the general hypothesis, denoted by H2, that the Condorcet efficiency of voting rules should tend to increase as voters' preferences show an increased degree of mutual coherence in the way in which the individual voters form their preferences on the candidates. According to this rather intuitively appealing assumption, the curves shown in Figures 1-6 should be decreasing as the Parameters *b*, *t* and *c* increase. A quick look at these Figures shows that, on the overall, our results **do not support** H2. Among the fifteen curves that we have considered, only four exhibit the expected shape. These exceptions are NPR and single-peakedness, PR and single-troughedness, NPER and single-peakedness, PER and single-troughedness. It is worthwhile to mention that the proximity to perfectly-polarized preferences, where Parameter *c* tends to zero, seems to increase (and not decrease) the Condorcet efficiency of each of the five voting rules. Based on the results that have been obtained in this study, strong support is given for the compromise position of using BR when nothing is known a priori about the type of model that is most likely to reflect the preferences that an electorate might typically have.

Suggestions for future research include the pursuit of answers to the following questions:

- Do the same results hold up when stronger forms to measuring group coherence are considered? The measures of group coherence that are used in the current study are based on the widely used assumptions of single-peaked preferences, single-troughed preferences and polarized preferences in the literature. Gehrlein (2008) introduces some additional stronger measures of group coherence that have an even greater correlation with the probability that a PMRW exists. Different results could possibly be observed when such other measures of group coherence are considered.

- What is the likelihood that all single-stage voting rules will elect the same winner? This question is partially addressed in Gehrlein (2006b) for Parameter *b*, where a representation is obtained for the probability that all weighted scoring rules will elect the PMRW, when there is one for a specified value of *b*. A weighted scoring rule assigns one point to each voter's most preferred candidate and λ points to each voter's middle ranked candidate in a three-candidate election. The winner is determined as the candidate that receives the most total points. PR, BR and NPR are all weighted scoring rules, with λ values of 0, 1/2 and 1 respectively. Gehrlein (2006b) shows that the probability that all weighted scoring rules will elect the PMRW tends to increase as voting situations become closer to being perfectly single-peaked.

- What does empirical research tell us about the regions of α_k values that most typically reflect real voting situations for these different models? There has been a significant amount of research related to the structure of voters' preferences in actual election data. If we were to know a priori that voters' preferences were going to be close to being single-peaked, single-troughed or perfectly polarized in a given election scenario, then we would be able to select a voting rule that would tend to maximize the Condorcet Efficiency accordingly. However, as mentioned above, it would not be feasible to wait until after the election results were obtained to announce which voting rule that was going to be used to determine the winner.

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