On The Probability of Observing Borda's Paradox

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Jean-Charles de Borda and the Marquis de Condorcet were contemporary 18th century French mathematician-philosophers who initiated the analysis of group decision-making problems with rigorous mathematical techniques. They had a mutual interest in investigating various counterintuitive or paradoxical events that could occur when a group of n individuals go through the process of selecting a best alternative from a set of available alternatives. Few arguments have been made against the use of majority rule to pick the more-favored alternative when only two alternatives are being considered, but significant problems can arise when there are three or more alternatives that are available for consideration. All group decision-making scenarios can be developed in the context of an election in which the decision makers are voters who are trying to elect the most-preferred candidate from the set of available alternatives.

Suppose that we are considering a set of three candidates $\{A,B,C\}$ and that each voter has some complete preference ranking on the candidates. Complete preference rankings do not allow voters to be indifferent between any candidates, and they do not allow voters to have intransitive, or cyclic, preferences on candidates. There are six possible complete preference rankings that voters might have in a three-candidate election, as shown in Figure 1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Figure 1. Possible complete preference rankings on three candidates.

Here, n_i denotes the number of voters that have the associated complete preference ranking on the candidates. That is, n_1 voters all have preferences with A being most preferred, C being least preferred and with B being ranked between them. Obviously, $\sum_{i=1}^{6} n_i = n$ and any given combination of $n_i's$ is referred to as a voting situation.

Borda (1784) and Condorcet (1785) both discussed the notion of considering pairwise majority rule (PMR) elections for the case of more than two candidates. Let $\boldsymbol{AMB}$ denote the election outcome that a majority of voters prefer A to B, while completely ignoring the position of *C* in their preference rankings. For example, ***AMB*** if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$. We assume throughout that *n* is odd to avoid the possibility of ties between candidates with PMR, and we also assume that voters always vote in accordance with their true preferences. Condorcet (1785) showed that voting situations can exist that result a in cyclical PMR relationship on the candidates. For the three-candidate case this would be observed in voting situations that result in outcomes like ***AMB***, ***BMC*** and ***CMA***. Numerous papers have been written on various aspects of this possible roadblock to rational group decision making, since an occurrence of Condorcet's Paradox means that no candidate can be selected as a winner without a majority of the voters preferring some other candidate to be the winner by PMR. Such an outcome would be quite disconcerting in any real election setting.

If Condorcet's Paradox is not observed in a three-candidate election, then some candidate must defeat each of the remaining two candidates by PMR. The pairwise majority rule winner (PMRW) is *A* if both ***AMB*** and ***AMC***. When there are only three candidates under consideration, the existence of a PMRW requires that a pairwise majority rule loser (PMRL) must also exist when *n* is odd. For example, *C* is the PMRL if both ***AMC*** and ***BMC***.

Borda (1785) made a particularly disturbing observation about possible election outcomes by presenting the example voting situation with *n* = 21 in Figure 2.

A	*A*	*B*	*C*
B	*C*	*C*	*B*
C	*B*	*A*	*A*
$n_1 = 1$	$n_2 = 7$	$n_5 = 7$	$n_6 = 6$

Figure 2. An example of voting situation that exhibits a Strict Borda Paradox.

Based upon PMR relationships for this voting situation, we find ***BMA*** (13-8), ***CMA*** (13-8) and ***CMB*** (13-8) to give a complete PMR ranking with ***CMBMA***. Thus, Condorcet's Paradox is not observed in this voting situation, and *C* is the PMRW while *A* is the PMRL. Borda then considers the election outcome if voting were to be done with the commonly used plurality rule, where the most preferred candidate of each voter is recorded, and the winning candidate is the one that receives the most votes. Let ***APB*** denote the outcome that voters prefer *A* to *B* by plurality rule.

With Borda's example voting situation in Figure 2, we find ***APB*** (8-7), ***APC*** (8-6) and ***BPC*** (7-6) to give a complete ranking by plurality rule, with ***APBPC***. We therefore find that plurality rule and PMR reverse the rankings on the three candidates in this example voting situation. We refer to such an outcome as an occurrence of a *Strict Borda Paradox*. Borda (1784) was primarily concerned with the less restrictive outcome that plurality rule might elect the PMRL. Such an outcome is defined as a *Strong Borda Paradox*, and it does not necessarily require that the PMRW be ranked last by plurality rule while the PMRL is chosen as the winner by plurality rule. An occurrence of either form of Borda's Paradox leads to a situation in which a candidate is elected as the winner in an election when both of the remaining candidates are preferred to the winner by a majority of voters by PMR. Such an outcome could lead to significant discontent among the electorate, despite the fact that plurality rule is the most widely used and accepted election procedure.

We also know at the onset that the possible existence of forms of Borda's Paradox is not limited to elections that use plurality rule. Consider the case of a general weighted scoring rule for a three-candidate election with weights $(1,\lambda,0)$ for which each voter's most preferred candidate is given one point, while the second most preferred candidate is given λ points. These points are then accumulated over all voters' preferences, and the candidate that receives the most points will win. Plurality rule is the special case of a weighted scoring rule with $\lambda = 0$. Daunou (1803) proved that Borda Rule, which is the weighted scoring rule with $\lambda = 1/2$, cannot rank the PMRW in last place in an election, so that a Strict Borda Paradox cannot occur. Smith (1973) and Gärdenfors (1973) then show that for sufficiently large *n*, Borda Rule is the only weighted scoring rule that has this property. Thus, all weighted scoring rules other than Borda Rule can exhibit a Strict Borda Paradox. Fishburn and Gehrlein (1976) extend these results to show that for sufficiently large *n*, Borda Rule is the only weighted scoring rule that cannot select the PMRL as the unique winner. Thus, Borda Rule is the only weighted scoring rule that cannot exhibit a Strong Borda Paradox.

The possible existence of these paradoxes poses potentially significant problems in election settings. As a result, we turn our attention to considering the probability that these paradoxical outcomes might ever be observed in real election settings.

The Probability of Actually Observing these Paradoxical Election Outcomes

Intuition suggests that the likelihood that such paradoxes will be observed should tend to be significantly reduced if the voters have preferences that are obtained by some mutually coherent process. That is, if voters' preferences become more internally consistent or coherent, then their aggregated preferences should tend to be more consistent or coherent, in the sense that the aggregated preferences should have a decreased likelihood of producing such paradoxical outcomes.

Gehrlein (2006a) provides extensive analysis to support this notion, where a strong relationship is shown to exist between the probability that a PMRW exists and the propensity of voters' preferences to reflect situations that suggest that these preferences are formed by some mutually coherent process. Several measures that reflect the degree of mutual coherence among voters' preferences in a given voting situation are developed in that study. One of these measures is

$$b = Minimum\{n_5 + n_6, n_2 + n_4, n_1 + n_3\}. \qquad (1)$$

Parameter b measures the minimum number of times that some candidate is bottom ranked in a given voting situation.

Arrow (1963) shows that if b = 0 for a three-candidate voting situation, then the voters' preferences must meet the well known condition of single-peaked preferences, as developed in Black (1958). A PMRW must exist in such cases to preclude the existence of Condorcet's Paradox. Niemi (1969) introduces the concept of the proximity of voting situations to the condition of perfectly single-peaked preferences. Niemi suggests that voting situations that are reasonably 'close' to meeting the condition of perfectly single-peaked should have a high probability of having a PMRW. Gehrlein (2006a) uses parameter b as a measure of the proximity of a voting situation to the condition of perfectly single-peaked preferences, and a strong relationship is found between parameter b and the probability that a PMRW exists.

This was done by using a procedure called EUPIA2 that is developed in detail in Gehrlein (2005, 2006b) to obtain an algebraic representation for the conditional probability $P_{PMRW}(3,n \mid IAC_b(k))$ that a PMRW will exist in a randomly selected voting situation, given that attention is restricted to voting situations for which parameter *b* has some specified value *k*. This closed form representation is based on the assumption, $IAC_b(k)$, that all voting situations for which $b = k$ are equally likely to be observed for the specified value of *k*. Then, $P_{PMRW}(3,n \mid IAC_b(k))$ is found to remain large for all *k* values that represent values of parameter *b* that are at all close to representing perfectly single-peaked preferences.

A second parameter that measures the degree of group coherence among voters' preferences in a voting situation that is of interest to the current study is

$$t = Minimum\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{2}$$

Here, parameter *t* measures the minimum number of times that some candidate is top-ranked in a voting situation. Following the logic of earlier discussion, parameter *t* is used to measure the proximity of a voting situation to the well-known condition of perfectly single-troughed preferences, as developed by Vickery (1960). A strong relationship is also found to exist between parameter *t* and the probability that a PMRW exists, since it is found that

$$P_{PMRW}(3,n \mid IAC_b(k)) = P_{PMRW}(3,n \mid IAC_t(k)) . \tag{3}$$

This confirms an assertion in Vickery (1960) that the condition of single-troughed preferences is equivalent to the condition of single-peaked preferences, since every single-peaked voting situation corresponds to a single troughed-voting situation in which all voters' preference rankings are inverted. At least, this assertion is correct when we are considering the probability that a PMRW exists.

All of this, along with other findings, leads to the general conclusion that it is quite unlikely that Condorcet's Paradox would ever be observed in a real voting situation with a small number of candidates with large electorates if voters' preferences reflect any significant degree of group coherence. This provides strong support to explain why so few examples of Condorcet's Paradox have been observed in practice, despite the many attempts that have been made to find them in empirical election results. The purpose of the current study is to investigate the possibility that the same conclusion will be reached if attention is turned to the probability that Borda's Paradox will be observed in real elections.

We begin with a brief summary of empirical studies that have been conducted to find evidence of the existence of Borda's Paradox. Weber (1978) presents a widely cited example of a Strong Borda Paradox in the 1970 U.S. Senate election in New York State. Riker (1982) performs an analysis of the 1912 U.S. Presidential election to find an example of a Strong Borda Paradox. Van Newenhizen (1992) considers an example to suggest the possible existence of a Strong Borda Paradox in the 1988 national elections for Prime Minister of Canada. Colman and Poutney (1978) perform an analysis of survey results of voters' preferences in 261 different three-candidate contests in British General Elections. No occurrences of a Strong Borda Paradox were observed, but 14 of the 261 elections exhibited the occurrence of a *Weak Borda Paradox*. A Weak Borda Paradox occurs when the PMRL is not ranked last by plurality rule. Bezembinder (1996) considers the possibility that a Strict Borda Paradox occurs when it is assumed that voters' preferences are perfectly single-peaked. Statistical analysis is performed on actual voting results during the era of Weimar Germany. The general conclusion is that there were a large number of disagreements between plurality rule rankings and PMR rankings. However, no examples of a Strict Borda Paradox were found.

These empirical studies lead to the conclusion that while examples of various forms of Borda's Paradox might not be widespread, there is evidence to suggest that this paradox does occasionally occur in real elections. The lack of a significant number of examples that display Borda's Paradox might result from the fact that most recorded examples of voters' preferences from real elections are likely to reflect some significant degree of mutual coherence, which would make them unlikely suspects for producing such paradoxical results. We continue with an examination of the question as to whether or not increased mutual coherence among voters' preferences tends to reduce the probability that Borda's Paradox is observed.

The Probability of Observing a Strict Borda Paradox

Our objective is to develop a closed form representation for the conditional probability that a Strict Borda Paradox occurs for a randomly selected voting situation with *n* voters in a three-candidate election with plurality rule, given that attention is restricted to voting situations for which parameter *b* has some specified value *k*. Following the discussion that was presented above, such a development could mirror the development that led to the representation for $P_{PMRW}(3,n\,|\,IAC_b(k))$ in previous studies. The associated probability that a Strict Borda Paradox is observed in a randomly selected voting situation is denoted by $P_{StBP}^{PR}(3,n\,|\,IAC_b(k))$. Since any form of Borda's Paradox can only be observed in voting situations in which a PMRW exists, it obviously follows that $P_{StBP}^{PR}(3,n\,|\,IAC_b(k)) \le P_{PMRW}(3,n\,|\,IAC_b(k))$.

It is clear in advance that it will be impossible to show that the conditional probability $P_{StBP}^{PR}(3,n\,|\,IAC_b(k))$ consistently increases as *k* increases, as a result of the following observation.

Theorem 1. $P_{StBP}^{PR}(3,n\,|\,IAC_b(n/3)) = 0$ **for *n* a multiple of 3.**

Proof. Assume without a loss of generality that *A* is both the PMRL and the winner by plurality rule, which are necessary, but not sufficient, requirements for a Strict Borda Paradox to be observed. If *A* is the PMRL, then

$$n_1 + n_2 + n_3 < n_4 + n_5 + n_6 \tag{4}$$

$$n_1 + n_2 + n_4 < n_3 + n_5 + n_6\,. \tag{5}$$

If *A* is the winner by plurality rule, then it is easy to show that $n_1 + n_2 > n/3$. If $b = n/3$, it follows from definition that $n_5 + n_6 = n/3$. Using both of these facts with (4) leads to $n_4 > n_3$, while using both of these facts with (5) leads to $n_3 > n_4$. Because of this contradiction, all of these conditions can not hold simultaneously. **QED**

In order to investigate the general tendency for $P_{StBP}^{PR}(3,n\,|\,IAC_b(k))$ to change as *k* increases, a representation for $P_{StBP}^{PR}(3,n\,|\,IAC_b(k))$ was obtained with the EUPIA2 procedure that was mentioned above. This procedure is based on some relatively simple concepts, but it is very cumbersome to implement on a computational basis and it also requires significant algebraic manipulation. As a result, specific details that are involved in the derivation of our representation for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ are not included, but they are available from the authors upon request. It is also possible to verify values that are obtained from the representation for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ by using computer enumeration procedures.

We note that there recently has been a significantly increased interest in developing procedures that can be used for obtaining representations for the probability that various election outcomes will be observed. Recent studies by Lepelley et al. (2007) and Wilson and Pritchard (2007) are examples of these procedures. Either of these two approaches could have been used to obtain the representations that will follow. The representation that was obtained for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ is shown in Figure 3. It is assumed in this representation that there are no ties between candidates by plurality rule.

The notation that is used in the representation for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ is such that $[p/q]^{+}$ is the smallest integer value that is greater than or equal to the ratio p/q for integer *p* and *q*. Similarly, $[p/q]^{-}$ is the largest integer value that is less than or equal to the ratio p/q for integer *p* and *q*. The term $\delta_p^q = 1$ if *p* is an integer multiple of *q*, and $\delta_p^q = 0$ otherwise.

While the representation for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ in Figure 3 is correct and can be verified by computer enumeration, it is so complex that it is of little practical use. However, it does provide an avenue to obtaining a much simpler representation for the limiting probability for $P_{StBP}^{PR}(3,n \mid IAC_b(k))$ as $n \to \infty$. Using a finite value of *k* in any such representation is not appropriate. Instead, *k* is replaced by $\alpha_k n$, such that α_k is the minimum proportion of profiles for which some candidate is ranked as least preferred, with $\alpha_k = k/n$. The limiting representation is then obtained by letting $n \to \infty$ after this substitution is made.

Figure 3. A representation for $P_{StBP}^{PR}(3,n\mid IAC_b(k))$.

$$P_{StBP}^{PR}(3,n\mid IAC_b(k))$$

$$= \frac{(k+1)\left\{\begin{array}{c} 27(3k^3+11k^2+9k-1)-9(4k^2+8k-1)n+n^3-4\delta_{n+1}^{12}(54\delta_k^2+12n-31)-16\delta_{n+11}^{12}(3n+1) \\ -108\delta_{n+9}^{12}(2\delta_k^2-1)-16\delta_{n+7}^{12}(3n-1)-4\delta_{n+5}^{12}(54\delta_k^2+12n-23) \end{array}\right\} + 27\delta_k^2(2k+3)}{72(k+1)(n-3k)\{(n+1)(n+5)-3k(2+k)\}},$$

for $0 \le k \le [(n-1)/6]^-$

$$= \frac{\begin{array}{c} -162(21k^4+14k^3+6k^2+k+1)+27(104k^3+60k^2+10k-1)n-9(96k^2+42k-1)n^2+3(38k+9)n^3-5n^4+162\delta_k^2(6k+1-2n) \\ +2\delta_{n+1}^{12}\{2(864k^2+564k+141-(288k+89)n+12n^2)-324\delta_k^2(4k+1-n)\}+16\delta_{n+11}^{12}\{3(72k^2+16k+5)-2(36k-1)n+3n^2\} \\ -324\delta_{n+9}^{12}(2\delta_k^2-1)(4k+1-n)+16\delta_{n+7}^{12}\{3(72k^2+20k+5)-2(36k+1)n+3n^2\} \\ +4\delta_{n+5}^{12}\{(864k^2+516k+141)-(288k+73)n+12n^2-162\delta_k^2(4k+1-n)\} \end{array}}{432(k+1)(n-3k)\{(n+1)(n+5)-3k(2+k)\}},$$

for $[(n+1)/6]^+ \le k \le [(n-1)/4]^-$

$$= \frac{\begin{array}{c} (n+2-3k)(n-4-3k)\{99k^2+96k+12-2(33k+16)n+11n^2\}+81\delta_k^2(6k+1-2n) \\ +32(\delta_{n+3}^6)\{3(18k^2+14k+1)-2(18k+7)n+6n^2\}+32\delta_{n+1}^6(3k-n) \end{array}}{216(k+1)(n-3k)\{(n+1)(n+5)-3k(2+k)\}},$$

for $[(n+1)/4]^+ \le k \le [(n-1)/3]^-$.

The limiting representation $P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$ is found to be:

$$P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$$
$$= \frac{(27\alpha_k^2 - 3\alpha_k - 1)}{72(3\alpha_k^2 - 1)}, \text{ for } 0 \le \alpha_k \le 1/6 \tag{6}$$
$$= \frac{-3402\alpha_k^4 + 2808\alpha_k^3 - 864\alpha_k^2 + 114\alpha_k - 5}{432\alpha_k(3\alpha_k - 1)(3\alpha_k^2 - 1)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$
$$= \frac{11(3\alpha_k - 1)^3}{216\alpha_k(3\alpha_k^2 - 1)}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

The representation in (6) is clearly much more tractable than the representation in Figure 3. An observation follows directly from (6) for the special case of perfectly single-peaked preferences, with $\alpha_k = 0$. It is easily seen that $P_{StBP}^{PR}(3,\infty \mid IAC_b(0)) = 1/72$, which verifies a result from Bezembinder (1996) and is in disagreement with a result from Saari and Valognes (1999). Calculated values of $P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$ are given in Figure 4 for each $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$.

Figure 4. Computed values of $P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$, $P_{StBP}^{PR}(3,\infty \mid IAC_b^*(\alpha_k))$ and $P_{StBP}^{PR}(3,\infty \mid IAC_t^*(\alpha_k))$.

α_k	$P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$	$P_{StBP}^{PR}(3,\infty \mid IAC_b^*(\alpha_k))$	$P_{StBP}^{PR}(3,\infty \mid IAC_t^*(\alpha_k))$
0	.0139	.0139	0
.01	.0143	.0143	.0000
.03	.0148	.0149	.0000
.05	.0151	.0152	.0000
.07	.0152	.0153	.0001
.09	.0150	.0151	.0003
.11	.0145	.0146	.0006
.13	.0137	.0140	.0012
.15	.0125	.0129	.0021
.17	.0111	.0116	.0036
.19	.0094	.0100	.0059
.21	.0078	.0084	.0098
.23	.0061	.0069	.0164
.25	.0039	.0046	.0284
.27	.0017	.0021	.0498
.29	.0005	.0007	.0799
.31	.0001	.0001	.1170
.33	.0000	.0000	.1592
1/3	0	0	1/6

The calculated values of $P_{StBP}^{PR}\left(3,\infty \mid IAC_b\left(\alpha_k\right)\right)$ show that these probabilities follow our intuition and increase as α_k increases over the range $0 \le \alpha_k \le .07$. That is, as α_k increases we have voting situations that are more removed from the condition of perfect single-peakedness. Since this reflects reduced levels of mutual coherence among voters' preferences, it sounds quite logical that the associated probability of observing a Strict Borda Paradox would increase. However, a very counterintuitive result is observed in these computed values since $P_{StBP}^{PR}\left(3,\infty \mid IAC_b\left(\alpha_k\right)\right)$ decreases over the range of values $.07 \le \alpha_k \le 1/3$, to indicate that the probability of observing a Strict Borda Paradox with Plurality Rule decreases as the mutual coherence of voters' preferences decreases.

Gehrlein (2007) develops a representation for the cumulative proportion of all possible voting situations, $CP_{VS}\left(3,\infty \mid \alpha_k\right)$, which have a specified value of α_k or less when $n \to \infty$, with

$$P_{VS}\left(3,\infty \mid \alpha_k\right) = 3\alpha_k^{\ 2}\left(10 - 20\alpha_k - 15\alpha_k^{\ 2} + 36\alpha_k^{\ 3}\right), \text{ for } 0 \le \alpha_k \le 1/3. \qquad (7)$$

The representation in (7) can be used to show that only 12.6 percent of all voting situations have α_k in the range $0 \le \alpha_k \le .07$ as $n \to \infty$, so the counterintuitive observation is valid over a large majority of possible voting situations.

The observed impact that changing *k* has on $P_{StBP}^{PR}\left(3,n \mid IAC_b(k)\right)$ could come from two possible components. First, we already know that changing *k* has a significant impact on the probability that a PMRW exists. In addition, changing *k* also has an impact on the propensity of the candidate ranking by plurality rule to be the reverse of the candidate ranking by PMR. This first component has already been studied in detail, with the results being summarized in earlier discussion. It is possible to focus on the second component, and develop a representation for the conditional probability, $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$, that is based only on voting situations for which a PMRW exists. This closed form representation is based on the assumption, $IAC_b^*(k)$, that all voting situations that have a PMRW and for which $b = k$ are equally likely to be observed for the specified value of *k*. The conditional probability $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$ is therefore conditional both on the assumption that a PMRW exists and that $b = k$.

As in the analysis with $P_{StBP}^{PR}\left(3,n \mid IAC_b(k)\right)$, it will be impossible to show that $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$ consistently increases as k increases, as a result of the following observation that follows directly from Theorem 1.

Corollary 1. $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(n/3)\right)=0$ **for *n* a multiple of 3.**

The EUPIA2 procedure was used to obtain a representation for the probability $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$, and the result is shown in Figure 5. The limiting representation, $P_{StBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$, as $n \to \infty$ is obtained following earlier discussion and

$$P_{StBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$$

$$= \frac{(3\alpha_k - 1)(27\alpha_k^2 - 3\alpha_k - 1)}{72(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 0 \le \alpha_k \le 1/6 \qquad (8)$$

$$= \frac{-3402\alpha_k^4 + 2808\alpha_k^3 - 864\alpha_k^2 + 114\alpha_k - 5}{432\alpha_k(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{(3\alpha_k - 1)(9\alpha_k^2 - 6\alpha_k + 1)}{108(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

Calculated values of $P_{StBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ are given in Figure 4 for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$. The same counterintuitive results regarding a generally decreasing probability that a Strict Borda Paradox will be observed as voters' preferences become more removed from the condition of perfect single-peakedness is therefore obtained regardless of whether or not we have a conditional requirement for a PMRW to exist. There are two explanations that could be presented to explain the existence of this counterintuitive result. First, the use of parameter *b* to measure the degree of mutual coherence among voters' preferences might be an inadequate measure, despite earlier observations regarding the expected relationship between parameter *b* and the probability that a PMRW exists. The second possible explanation is that there might be something unusual about the behavior of plurality rule that causes our unexpected observation. We begin by considering what happens if parameter *t* is used instead of parameter *b* to measure the degree of mutual coherence among voters' preferences.

Figure 5. A representation for $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$.

$$P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$$

$$= \frac{(k+1)\left\{\begin{array}{c} 27\left(3k^3+11k^2+9k-1\right)-9\left(4k^2+8k-1\right)n+n^3-4\delta_{n+1}^{12}\left(54\delta_k^2+12n-31\right)-16\delta_{n+11}^{12}(3n+1) \\ -108\delta_{n+9}^{12}\left(2\delta_k^2-1\right)-16\delta_{n+7}^{12}(3n-1)-4\delta_{n+5}^{12}\left(54\delta_k^2+12n-23\right) \end{array}\right\} + 27\delta_k^2(2k+3)}{72(b+1)(n-3b)\{(n+1)(n+5)-3b(2+b)\}},$$

for $0 \le k \le [(n-1)/6]^-$

$$= \frac{\begin{array}{c} -162\left(21k^4+14k^3+6k^2+k+1\right)+27\left(104k^3+60k^2+10k-1\right)n-9\left(96k^2+42k-1\right)n^2+3(38k+9)n^3-5n^4+162\delta_k^2(6k+1-2n) \\ +2\delta_{n+1}^{12}\left\{2\left(864k^2+564k+141-(288k+89)n+12n^2\right)-324\delta_k^2(4k+1-n)\right\}+16\delta_{n+11}^{12}\left\{3\left(72k^2+16k+5\right)-2(36k-1)n+3n^2\right\} \\ -324\delta_{n+9}^{12}\left(2\delta_k^2-1\right)(4k+1-n)+16\delta_{n+7}^{12}\left\{3\left(72k^2+20k+5\right)-2(36k+1)n+3n^2\right\} \\ +4\delta_{n+5}^{12}\left\{\left(864k^2+516k+141\right)-(288k+73)n+12n^2-162\delta_k^2(4k+1-n)\right\} \end{array}}{432(k+1)\left\{k\left(-17+21k+11k^2\right)+\left(5-26k-4k^2\right)n+3(2-k)n^2+n^3\right\}},$$

for $[(n+1)/6]^+ \le k \le [(n-1)/4]^-$

$$= \frac{\begin{array}{c} (n+2-3k)(n-4-3k)\left\{99k^2+96k+12-2(33k+16)n+11n^2\right\}+81\delta_k^2(6k+1-2n) \\ +32\left(\delta_{n+3}^6\right)\left\{3\left(18k^2+14k+1\right)-2(18k+7)n+6n^2\right\}+32\delta_{n+1}^6(3k-n) \end{array}}{108(n-3k)\left\{(n+1)\left(n^2+2n+9\right)-6\left(n^2+1\right)k+18nk^2-18k^3\right\}},$$

for $[(n+1)/4]^+ \le k \le [(n-1)/3]^-$.

The Impact of Using Parameter *t* to Measure Mutual Coherence

It was noted previously that Vickery (1960) asserted that the condition of single-troughed preferences is equivalent to the condition of single-peaked preferences, since every single-peaked voting situation corresponds to a single troughed-voting situation in which all voters' preference rankings are inverted. This assertion has been seen to be correct when we are considering the probability that a PMRW exists. However, a very different outcome is observed when attention is changed from the consideration of Condorcet's Paradox to considering variations of Borda's Paradox.

It was seen above that values of $P_{StBP}^{PR}(3,\infty \mid IAC_b(\alpha_k))$ and $P_{StBP}^{PR}(3,\infty \mid IAC_b^*(\alpha_k))$ are very similar. As a result, we restrict attention to probability representations like those with the assumption of $IAC_b^*(k)$. In the case of parameter *t*, this assumption becomes $IAC_t^*(k)$, as defined in the obvious way. Something very different happens with $P_{StBP}^{PR}(3,n \mid IAC_t^*(k))$ as *t* is increased than we observed above with the assumption of $IAC_b^*(k)$. A strong indication of this observation comes from the following theorem.

Theorem 2. $P_{StBP}^{PR}(3,n \mid IAC_t^*(0)) = 0$ **for odd *n*.**

Proof. If $t = 0$ in a given voting situation, some candidate is never ranked as most preferred by any voter. A given candidate of the other two candidates must therefore be ranked as most preferred by at least $(n+1)/2$ voters. That given candidate must therefore be both the PMRW and the winner by plurality rule. Thus, a Strict Borda Paradox cannot occur for the given voting situation by definition. **QED**

General representations for $P_{StBP}^{PR}(3,n \mid IAC_t^*(k))$ were obtained as above, with results being summarized in (9), with

$$P_{StBP}^{PR}(3,n \mid IAC_t^*(k)) \tag{9}$$

$$= \frac{k(k+1)(k+2)}{k(11k^2+21k-17)-(4k^2+26k-5)n-3(k-2)n^2+n^3},$$
$$\text{for } 0 \le k \le [(n-1)/4]^-$$

$$= \frac{(1+k)(n-1-3k)\{6k(n-k)-(n+1)(n-3)\}}{(n-3k)\{3(3-2k-6k^3)+(11+18k^2)n+3(1-2k)n^2+n^3\}},$$
$$\text{for } [(n+1)/4]^+ \le k \le [(n-1)/3]^- .$$

The representations for $P_{StBP}^{PR}\left(3,n \mid IAC_t^*(k)\right)$ in (9) are clearly much simpler than the representations for $P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$ in Figure 5, but we restrict attention to the limiting representations as $n \to \infty$. Following previous arguments, we find

$$P_{StBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right) \tag{10}$$

$$= \frac{5\alpha_k^3}{16(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1)}, \text{ for } 0 \le \alpha_k \le 1/4$$

$$= \frac{87\alpha_k^3 - 99\alpha_k^2 + 31\alpha_k - 3}{8(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1)}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

Calculated values of $P_{StBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ are listed in Figure 4 for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$.

The calculated values in Figure 4 show results that are completely in line with our intuition. The probability that a Strict Borda Paradox is observed consistently increases as voters' preferences reflect situations that are more removed from perfectly single-troughed preferences, as measured by parameter *t*. When parameter *t* is used as a measure of the degree of mutual coherence among voters' preferences, we find results that are completely in accord with intuition, as opposed to our observations when parameter *b* is used. We now turn attention to the probability that a Strict Borda Paradox is observed with another voting rule. In particular, we consider what happens when negative plurality rule is used as an election procedure.

Strict Borda Paradox with Negative Plurality Rule

Negative plurality rule requires each voter to cast a vote against their least preferred candidate. The winner (loser) is the candidate that receives the fewest (most) of these last place votes. Let *A**R**B* denote the outcome that *A* beats *B* by negative plurality rule in a voting situation, and *A**R**B* if and only if $n_5 + n_6 < n_2 + n_4$. A Strict Borda Paradox can occur with negative plurality voting in the same way that it can happen with plurality voting. Let $P_{StBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right)$ denote the conditional probability that a Strict Borda Paradox is observed when elections are held with negative plurality rule under the assumption of $IAC_b^*(k)$. The behavior that is exhibited as *b* increases is easy to determine as a result of the following two theorems.

Theorem 3. $P_{StBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right) = P_{StBP}^{PR}\left(3,n \mid IAC_t^*(k)\right)$ **for odd *n*.**

Proof. Let *VS* denote a given voting situation with *b* = *k* that displays a Strict Borda Paradox with negative plurality rule, with a complete PMR relationship ***AMBMC*** with a complete ranking on candidates by negative plurality rule ***CRBCA***. There is no loss of generality with the assumption that ***AMBMC***. For every such *VS*, there is a unique voting situation *VS** that it obtained from *VS* by the transformation: $n_1 \leftrightarrow n_6$, $n_2 \leftrightarrow n_5$ and $n_3 \leftrightarrow n_4$. This transformation effectively creates *VS** by inverting the preferences of all voters in *VS*. It is then easily shown that *VS** has *t* = *k*, that the complete ranking is ***CMBMA*** by PMR and that the complete ranking is ***ARBRC*** by negative plurality rule. Since the mapping that transforms *VS* to *VS** is 1-1, it then follows directly from definitions that $P_{StBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right) \geq P_{StBP}^{PR}\left(3,n \mid IAC_t^*(k)\right)$. The same arguments can also be used to prove that $P_{StBP}^{PR}\left(3,n \mid IAC_t^*(k)\right) \geq P_{StBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right)$. **QED**

Theorem 4. $P_{StBP}^{NPR}\left(3,n \mid IAC_t^*(k)\right) = P_{StBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$ **for odd *n*.**

Proof. The proof follows identical arguments to those used in the proof of Theorem 3.

The combined results of Theorem 2 and Theorem 3 lead to the observation:

Corollary 2. $P_{StBP}^{NPR}\left(3,n \mid IAC_b^*(0)\right) = 0$.

Given previous results that were observed in this study with Theorem 3, we can conclude that $P_{StBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ increases as α_k increases, following our intuition. Theorem 4 implies that $P_{StBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ will generally decrease as α_k increases, contrary to our intuition. This is the opposite of result the we found when plurality rule was used to determine the winner, and no consistent relationships are found between the probability that a Strict Borda Paradox is observed and the degree of mutual coherence among voters' preferences when this degree of coherence is measured by parameters *b* or *t*. As a next step, we investigate the possible existence of this relationship when attention moves to the probability that a Strong Borda Paradox is observed.

The Probability of Observing a Strong Borda Paradox

A Strong Borda Paradox is observed when the PMRL is selected as the winner of an election. This condition is less restrictive than the requirements for a Strict Borda Paradox to occur, so the probability that a Strong Borda Paradox is observed is greater than the probability that a Strict Borda Paradox will exist. Let $P_{SgBP}^{PR}\left(3,n\mid IAC_b^*(k)\right)$ denote the probability that a Strong Borda Paradox is observed with plurality rule under the assumption $IAC_b^*(k)$. It is impossible to show that $P_{StBP}^{PR}\left(3,n\mid IAC_b^*(k)\right)$ consistently increases as *k* increases, because of a result that follows directly from Theorem 1.

Corollary 3. $P_{SgBP}^{PR}\left(3,n\mid IAC_b^*(n/3)\right)=0$ **for *n* a multiple of 3.**

A representation was obtained for $P_{SgBP}^{PR}\left(3,n\mid IAC_b^*(k)\right)$ in order to determine the general nature of how it changes as *k* increases, and the results are shown in Figure 6. While the representation for $P_{SgBP}^{PR}\left(3,n\mid IAC_b^*(k)\right)$ in Figure 6 is less complicated than the representation for $P_{StBP}^{PR}\left(3,n\mid IAC_b^*(k)\right)$ in Figure 5, it is still intractable for analysis. The limiting representation for $P_{SgBP}^{PR}\left(3,\infty\mid IAC_b^*(\alpha_k)\right)$ as $n\rightarrow\infty$ is obtained following previous analysis, with

$$P_{SgBP}^{PR}\left(3,\infty\mid IAC_b^*(\alpha_k)\right) \tag{11}$$

$$=\frac{108\alpha_k^3-36\alpha_k^2+1}{36\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)},\ \text{for } 0\le\alpha_k\le 1/6$$

$$=\frac{-10368\alpha_k^4+6912\alpha_k^3-1728\alpha_k^2+192\alpha_k-7}{432\alpha_k\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)},\ \text{for } 1/6\le\alpha_k\le 1/4$$

$$=\frac{16\left(3\alpha_k-1\right)^3}{27\left(18\alpha_k^3-18\alpha_k^2+6\alpha_k-1\right)},\ \text{for } 1/4\le\alpha_k\le 1/3.$$

Figure 6. A representation for $P_{SgBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$.

$$P_{SgBP}^{PR}\left(3,n \mid IAC_b^*(k)\right)$$

$$=\frac{108k^3+288k^2+144k+5-9\left(4k^2+8k+1\right)n+3n^2+n^3-8\delta_{n+1}^6\left(5+3n\right)-32\delta_{n+3}^6}{36\left\{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3\left(k-2\right)n^2+n^3\right\}},$$
$$\text{for } 0\le k\le\left[\left(n-1\right)/6\right]^-$$

$$=\frac{\begin{array}{c}-10368k^4-13824k^3-4320k^2+192k+125+4\left(12k+5\right)\left(144k^2+84k-5\right)n-6\left(288k^2+192k+25\right)n^2+4\left(48k+17\right)n^3-7n^4\\-16\delta_{n+5}^6\left\{216k^2+168k+31-2\left(36k+19\right)n+3n^2\right\}-16\delta_{n+3}^6\left\{216k^2+120k+23-2\left(36k+11\right)n+3n^2\right\}\end{array}}{432\left(k+1\right)\left\{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3\left(k-2\right)n^2+n^3\right\}},$$
$$\text{for } \left[\left(n+1\right)/6\right]^+\le k\le\left[\left(n-1\right)/4\right]^-$$

$$=\frac{4\left(n-2-3k\right)\left(n+1-3k\right)\left(2n-1-6k\right)^2+2\delta_{n+3}^6\left\{54k^2+24k+1-4\left(9k+2\right)n+6n^2\right\}+2\delta_{n+5}^6\left\{54k^2+12k-1-4\left(9k+1\right)n+6n^2\right\}}{27\left(n-3k\right)\left\{3\left(3-2k-6k^3\right)+\left(11+18k^2\right)n+3\left(1-2k\right)n^2+n^3\right\}},$$
$$\text{for } \left[\left(n+1\right)/4\right]^+\le k\le\left[\left(n-1\right)/3\right]^- .$$

Based on the representation in (11) we note that $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(0)\right)=1/36$, which is in agreement with a result proved in Lepelley (1993). The representation in (11) was used to compute values of $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$, and the resulting values are listed in Figure 7.

Figure 7. Computed values of $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ **,** $P_{SgBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ **,** $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ **and** $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ **.**

α_k	$P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$	$P_{SgBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$	$P_{SgBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$	$P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$
0	.0278	0	0	.0486
.01	.0285	.0000	.0000	.0482
.03	.0297	.0002	.0000	.0472
.05	.0305	.0007	.0001	.0460
.07	.0309	.0015	.0004	.0446
.09	.0310	.0027	.0010	.0430
.11	.0309	.0044	.0021	.0411
.13	.0308	.0067	.0039	.0387
.15	.0310	.0100	.0068	.0358
.17	.0318	.0147	.0115	.0322
.19	.0336	.0214	.0190	.0279
.21	.0359	.0314	.0313	.0233
.23	.0361	.0471	.0524	.0183
.25	.0269	.0739	.0909	.0118
.27	.0120	.1185	.1459	.0053
.29	.0039	.1778	.2039	.0017
.31	.0006	.2469	.2634	.0003
.33	.0000	.3208	.3233	.0000
1/3	0	1/3	1/3	0

The computed values in Figure 7 show that $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ increases, according to intuition, over the range of values $0 \le \alpha_k \le .23$. This increase is clearly not monotonic, since there is some very minor inconsistent variation in $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ values in the range $.09 \le \alpha_k \le .15$. Values of $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ then decrease dramatically, contrary to intuition, over the range $.23 \le \alpha_k \le 1/3$. By using (7), it is found that approximately 80 percent of all voting situations fall into the range $0 \le \alpha_k \le .23$. But, the representation for the conditional probability $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ is based on the condition that a PMRW must exist. Results that are given in Gehrlein

(2006a) indicate that 97 percent of all voting situations in the range $0 \le \alpha_k \le .23$ have a PMRW, so it is safe to conclude that $P_{SgBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ generally behaves according to our intuition as α_k increases for voting situations with a PMRW over a wide range of α_k values that are at all close to being perfectly single-peaked. The generally counterintuitive results that were observed when considering the probability that a Strict Borda Paradox in this same situation are therefore eliminated when considering the probability that a Strong Borda Paradox will be observed.

When parameter t is used to measure the degree of mutual coherence among voters' preferences, there is reason to assume that $P_{SgBP}^{PR}\left(3,n \mid IAC_t^*(k)\right)$ should tend to increase as k increases, since that same behavior was observed previously with $P_{StBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$. This notion is further reinforced by an observation that follows directly from Theorem 2.

Corollary 4. $P_{StBP}^{PR}\left(3,n \mid IAC_t^*(0)\right) = 0$ for odd n.

A representation for $P_{SgBP}^{PR}\left(3,n \mid IAC_t^*(k)\right)$ is obtained as:

$$P_{SgBP}^{PR}\left(3,n \mid IAC_t^*(k)\right) \tag{12}$$

$$= \frac{(k-1)(k+1)\left\{-3\left(k^2-4k-1\right)+4kn\right\}-3\delta_k^2\left\{4k^3-6k^2-12k-1-4k(k+1)n\right\}}{16(k+1)\left\{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3(k-2)n^2+n^3\right\}},$$
$$\text{for } 0 \le k \le [(n-1)/4]^-$$

$$= \frac{\begin{array}{c}3\left(255k^4+4k^3-30k^2-36k-8\right)-6\left(170k^3+32k^2+6k-2\right)n+2\left(240k^2+48k+7\right)n^2 \\ -12(8k+1)n^3+7n^4-3\delta_k^2\left\{4k^3-6k^2-12k-1-4k(k+1)n\right\}\end{array}}{8(n-3k)\left\{(n+1)\left(n^2+2n+9\right)-6\left(n^2+1\right)k+18nk^2-18k^3\right\}},$$
$$\text{for } [(n+1)/4]^+ \le k \le [(n-1)/3]^- .$$

The limiting representation $P_{STBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ is then found as

$$P_{STBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right) \tag{13}$$

$$= \frac{\alpha_k^2(4-3\alpha_k)}{16\left(11\alpha_k^3-4\alpha_k^2-3\alpha_k+1\right)}, \text{ for } 0 \le \alpha_k \le 1/4$$

$$= \frac{765\alpha_k^4-1020\alpha_k^3+480\alpha_k^2-96\alpha_k+7}{8(3\alpha_k-1)\left(18\alpha_k^3-18\alpha_k^2+6\alpha_k-1\right)}, \text{ for } 1/4 \le \alpha_k \le 1/3 .$$

The representation in (13) was used to compute values of $P_{SgBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$, and the resulting values are listed in Figure 7. These calculated values show that values of $P_{SgBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ consistently increase as α_k increases, according to our intuition. Thus, when plurality rule is used for elections we find that the probability of observing a Strong Borda Paradox generally tends to increase as both parameters *b* and *t* increase. The obvious question then becomes, does this same behavior continue when negative plurality rule is used in elections.

The Probability of Observing a Strong Borda Paradox with Negative Plurality Rule

There is good reason to assume that $P_{SgBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right)$ will increase as *k* increases, since that same behavior was observed previously with $P_{StBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$. This notion is further reinforced by the following observation that is also proved by Lepelley (1993).

Theorem 5. $P_{SgBP}^{NPR}\left(3,n \mid IAC_b^*(0)\right) = 0$ **for odd** ***n*****.**

Proof. If $b = 0$, then some candidate is never ranked as least preferred by any voter, so one of the other two candidates must be ranked in last place by at least $(n+1)/2$ voters. By definition, that candidate must be both the PMRL and the bottom ranked candidate by negative plurality rule, so a Strong Borda Paradox cannot be observed. **QED**

A representation for $P_{SgBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right)$ is found to be given by

$$P_{SgBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right) \tag{14}$$

$$= \frac{k(k+1)(k+2)}{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3(k-2)n^2+n^3}, \text{ for } 0 \le k \le [(n-1)/4]^-$$

$$= \frac{(1+k)(n-1-3k)\{6k(n-k)-(n+1)(n-3)\}}{(n-3k)\left\{3\left(3-2k-6k^3\right)+\left(11+18k^2\right)n+3(1-2k)n^2+n^3\right\}},$$
$$\text{for } [(n+1)/4]^+ \le k \le [(n-1)/3]^- .$$

The resulting limiting representation for $P_{SgBP}^{NPR}\left(3,n \mid IAC_b^*(k)\right)$ as $n \to \infty$ is given by $P_{STBP}^{NP}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$, with

$$P_{SgBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right) \tag{15}$$

$$= \frac{\alpha_k^3}{11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1}, \text{ for } 0 \le \alpha_k \le 1/4$$

$$= \frac{\alpha_k\left(6\alpha_k^2 - 6\alpha_k + 1\right)}{18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

The representation in (15) was used to compute values of $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$, and the resulting values are listed in Figure 7. These calculated results show that calculated values of $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ behave exactly as intuition suggests by increasing as α_k increases.

Unfortunately, we can not anticipate the same intuitively appealing result for $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ because of the observation in Theorem 6.

Theorem 6. $P_{SgBP}^{NPR}\left(3,n \mid IAC_t(n/3)\right) = 0$ **for *n* a multiple of 3.**

Proof. Assume without a loss of generality that *A* is both the PMRL and the winner by negative plurality rule, which are necessary and sufficient conditions for a Strong Borda Paradox to be observed. If *A* is the PMRL, then

$$n_1 + n_2 + n_3 < n_4 + n_5 + n_6 \tag{16}$$

$$n_1 + n_2 + n_4 < n_3 + n_5 + n_6 . \tag{17}$$

If *A* is the winner by negative plurality rule, then it is easy to show that $n_5 + n_6 < n/3$. If $t = n/3$, it follows from definition that $n_1 + n_2 = n/3$. Using both of these facts with (16) leads to $n_4 > n_3$, while using both of these facts with (17) leads to $n_3 > n_4$. Because of this contradiction, all of these conditions can not hold simultaneously. **QED**

A representation for $P_{SgBP}^{NPR}\left(3,n \mid IAC_t^*(k)\right)$ is shown in Figure 8 and it is very complex, but it has been verified by computer enumeration.

Figure 8. A representation for $P_{SgBP}^{NPR}\left(3,n \mid IAC_t^*(k)\right)$ **.**

$$0 \le k \le [(n-1)/6]^-$$

$$\frac{(k+1)\left[126k^3+306k^2+81k+145-9\left(4k^2+20k+1\right)n-3(9k-5)n^2+7n^3-4\left\{8\delta_{n+5}^{12}(3n+7)+43\delta_{n+3}^{12}+\delta_{n+11}^{12}(83+24n)+16\delta_{n+9}^{12}+27\delta_{n+7}^{12}\right\}\right]+54\delta_k^2\left(\delta_{n+3}^4-\delta_{n+1}^4\right)\{2(k+2)+n\}}{144(k+1)\left\{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3(k-2)n^2+n^3\right\}}$$

$$[(n+1)/6]^+ \le k \le [(n-1)/4]^-$$

$$\frac{\begin{array}{l}27\left(-1472k^4-640k^3+992k^2+544k+33\right)+288\left(120k^3+60k^2-36k-13\right)n+18\left(-624k^2-304k+47\right)n^2+(1536k+536)n^3\\ -65n^4-1296\delta_{n+9}^{12}(n+2)-16\delta_{n+5}^{12}\left\{-1728k^2-768k-22+(576k+193)n-24n^2\right\}-128\delta_{n+11}^{12}\left\{-216k^2-96k-23+(14+72k)n-3n^2\right\}\\ -16\delta_{n+1}^{12}(480k+226-79n)-256\delta_{n+7}^{12}(30k+4-10n)\end{array}}{3456(k+1)\left\{k\left(11k^2+21k-17\right)-\left(4k^2+26k-5\right)n-3(k-2)n^2+n^3\right\}}$$

$$[(n+1)/4]^+ \le k \le [(n-1)/3]^-$$

$$\frac{(21k+1-7n)(3k-2-n)(3k+1-n)(3k+4-n)+4\delta_{n+3}^6\left(108k^2+60k+2-4(18k+5)n+12n^2\right)+8\delta_{n+1}^6(3k+1-n)(18k-1-6n)}{27(n-3k)\left\{(n+1)\left(n^2+2n+9\right)-6\left(n^2+1\right)k+18nk^2-18k^3\right\}}$$

The limiting probability representation for $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ is obtained following previous discussion, with:

$$P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right) = \frac{126\alpha_k^3 - 36\alpha_k^2 - 27\alpha_k + 7}{144\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{ for } 0 \le \alpha_k \le 1/6 \qquad (18)$$

$$= \frac{-39744\alpha_k^4 + 34560\alpha_k^3 - 11232\alpha_k^2 + 1536\alpha_k - 65}{3456\alpha_k\left(11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1\right)}, \text{ for } 1/6 \le \alpha_k \le 1/4$$

$$= \frac{7\left(3\alpha_k - 1\right)^3}{27\left(18\alpha_k^3 - 18\alpha_k^2 + 6\alpha_k - 1\right)}, \text{ for } 1/4 \le \alpha_k \le 1/3.$$

The representation in (18) was used to compute values of $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ for each value of $\alpha_k = .01(.02).33$, along with values for $\alpha_k = 0$ and $\alpha_k = 1/3$, and the resulting values are listed in Figure 7. These computed values show the counter-intuitive result that $P_{SgBP}^{NPR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ consistently decreases as α_k increases.

Overall Probabilities

The computed probabilities for $P_{StBP}^{PR}\left(3,\infty \mid IAC_b^*(\alpha_k)\right)$ and $P_{StBP}^{PR}\left(3,\infty \mid IAC_t^*(\alpha_k)\right)$ are relatively large for some specific values of α_k, and they are relatively small for other specific values of α_k. It is therefore of interest to obtain some overall, or expected, estimate of the probability that a Strict Borda Paradox will be observed with plurality rule over the range of all possible α_k. This was done by accumulating the total number of voting situations that exhibit a Strict Borda Paradox with plurality rule, based on the numerator of the representation in (9), over the range of all possible $0 \le t \le n/3$. This term is then divided by the total number of all possible voting situations with n voters, $\prod_{i=1}^{5}(n+i)/120$, to obtain a representation for the probability $P_{StBP}^{PR}(3,n \mid IAC)$ that a Strict Borda Paradox is observed with plurality rule, given the condition, IAC, that all possible voting situations are equally likely to be observed. This is a cumbersome, but straightforward, process. In the limiting case as $n \rightarrow \infty$, it is found in conjunction with Theorem 3, that

$$P_{StBP}^{PR}(3,n \mid IAC) = P_{StBP}^{NPR}(3,n \mid IAC) = 1/90 = .0111. \qquad (19)$$

This suggests that it is quite unlikely that an occurrence of a Strict Borda Paradox will ever be observed with either plurality rule or with negative plurality rule under the IAC assumption.

The same general logic was then used to obtain estimates of the probability that a Strong Borda Paradox will be observed with the assumption of IAC, and

$$P_{SgBP}^{PR}(3,n \mid IAC) = 4/135 = .0296. \qquad (20)$$

$$P_{SgBP}^{NPR}(3,n \mid IAC) = 17/540 = .0315.$$

As a result, we find a significantly greater probability that a Strong Borda Paradox will be observed, but this likelihood is still not very large under the IAC assumption. The results in (20) verify results in Lepelley (1993).

Conclusion

The most salient results that we have obtained in this study can be summarized as follows.

1) In some specific circumstances that we have identified, Borda's Paradox can occur with a probability that is far from being negligible: when voting situations are more and more removed from perfect single-troughedness, the strict version of the Paradox has a probability of occurrence that can be higher than 15% when plurality rule is used to rank the candidates, and the strong version can occur with a more than 30% probability when voters' preferences are far from perfect single-peakedness and when the voting rule is negative plurality rule.
2) It remains however that the overall probability that Borda's Paradox will be observed is rather low (about 1% for the strict version and about 3% for the strong version under the IAC assumption) and significantly lower than the likelihood of Condorcet's Paradox (6.25% under the same assumption). As a consequence,

Borda's Paradox could be considered as less problematic than Condorcet's Paradox in real election settings.

3) This assertion should be balanced by the following observation, which certainly constitutes our main finding. We know from previous studies that, when voters' preferences become more internally consistent, Condorcet's Paradox probability is reduced and tends to 0, in accordance with our intuition. The results we have obtained show that the impact of an increasing degree of mutual coherence among voters' preferences on the likelihood of Borda's Paradox is much more subtle and more difficult to analyze: it turns out that this impact depends both on the measure of mutual coherence that is considered and on the voting rule that is used. In some circumstances, the probability that Borda's Paradox will occur increases when voters' preferences become more internally consistent.

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