The Impact of Dependence among Voters' Preferences with Partial Indifference

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Abstract

Standard Weighted Scoring Rules do not directly accommodate the possibility that some voters might have dichotomous preferences in three-candidate elections. The direct solution to this issue would be to require voters to arbitrarily break their indifference ties on candidates and report complete rankings. This option was previously found to be a poor alternative when voters have completely independent preferences. The introduction of a small degree of dependence among voters' preferences has typically been found to make a significant reduction of the impact of such negative outcomes in earlier studies. However, we find that the forced ranking option continues to be a poor choice when dependence is introduced among voters' preferences. This conclusion is reinforced by the fact that other voting options like Approval Voting and Extended Scoring Rules have been found to produce much better results. These observations are made as a result of using a significant advancement in techniques that obtain probability representations for such outcomes.

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We start by considering elections on three candidates $\{A, B, C\}$ in which voters report complete preference rankings on the candidates. There are six possible complete transitive preference rankings with three candidates, as shown in Figure 1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Figure 1. The possible complete preference rankings on three candidates.

Let $A \succ B$ denote an individual voter's preference on a given pair of candidates, such that Candidate *A* is preferred to Candidate *B*, and n_1 voters in Figure 1 have preferences with $A \succ B$, $A \succ C$ and $B \succ C$. A *voting situation* for a given number *n* voters then defines a specific set of rankings with $n = \sum_{i=1}^{6} n_i$.

A Weighted Scoring Rule (*WSR*) can be used to determine the winner of an election when voters are required to report candidate rankings. A *WSR* for a three-candidate election is defined by weights $(1, \lambda, 0)$ such that all voters give one point to their most preferred candidate, λ points to their middle ranked candidate, and zero points to their least preferred candidate. The winning candidate is the one that receives the greatest accumulated score from all voters. There are two special cases in which a WSR does not actually require that a complete ranking must be reported by voters. The first of these cases is the widely used *Plurality Rule* (PR) with weights $(1,0,0)$, so that the voters only need to report their most preferred candidate. The second special case is *Negative Plurality Rule* (NPR) with weights $(1,1,0)$, in which the voters only report their two more preferred candidates with no actual ranking being required. The use of NPR is equivalent to having voters simply identify their least preferred candidate.

The performance of voting rules is often evaluated on the basis of their *Condorcet Efficiency*. To describe this, let *AMB* denote a *majority rule* relationship on the pair of Candidates *A* and *B*, such that more voters in a voting situation have a preference ranking with $A \succ B$ than with $B \succ A$, so that $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$ from Figure 1. Candidate *A* is the *Condorcet Winner* (*CW*) if both *AMB* and *AMC*, which strongly supports the notion that Candidate *A* is the overall most preferred candidate. However, it is well known that a CW does not always exist, since majority rule cycles can exist in scenarios like *AMB*, *BMC* and *CMA* to produce an occurrence of what is known as *Condorcet's Paradox*. The *Condorcet Efficiency* of a voting rule is therefore defined by the conditional probability that it elects the CW, given that a CW does exist.

Recent research [Gehrlein et al (2015a)] indicates that significant benefits can be obtained for scenarios in which all voters have complete preference rankings when elections are being conducted with three candidates by using the well-known WSR called *Borda Rule* (BR) with weights $(1,1/2,0)$. BR is found to consistently perform very well on the basis of Condorcet Efficiency, while conditions exist for which each of the two non-ranking WSR options, PR and NPR, perform very poorly. However, all of these conclusions are based on the assumption that each voter actually has some complete preference ranking on the candidates.

The question that we address in the current study is: What happens when voters do not have complete preference rankings on candidates? We do not consider the case in which individual voters might have intransitive preferences on candidates, since transitivity is a long-held assumption of rationality for individual preferences. However, it is very plausible that voters might be indifferent between some of the candidates.

Allowing Partial Voter Indifference between Candidates

The case in which a voter is completely indifferent between all three candidates is ignored, since such a voter has absolutely no impact on the determination of how well a voting rule does at selecting a winner. There are six possible cases of partial indifference between three candidates for voters, as shown in Figure 2:

A~B	*A~C*	*B~C*	*A*	*B*	*C*
C	*B*	*A*	*B~C*	*A~C*	*A~B*
n_7	n_8	n_9	n_{10}	n_{11}	n_{12}

Figure 2. The possible preference orderings with partial indifference on three candidates.

The notation *A~B* in Figure 2 indicates for example that a voter is indifferent between the selection of Candidates *A* and *B*. A voting situation which allows for the possibility of partial indifference for *n* voters then defines a specific set of voter preferences with $n = \sum_{i=1}^{12} n_i$. The voter preferences in Figure 2 represent *dichotomous preferences* in which the candidates are partitioned into a more preferred subset and a less preferred subset. The candidates within each of the two subsets are indifferent to all other candidates in the same subset, and every candidate in the more preferred subset is preferred to every candidate in the less preferred subset. The admission of partial indifference requires a modification of the majority rule relationship that is defined above. To describe this modified relationship for Candidates *A* and *B*, we only consider the preferences from Figures 1 and 2 for which an actual preference exists when comparing *A* and *B*, and the definition of a CW that is given above is revised accordingly, so that *A* is the CW when

$$n_1 + n_2 + n_4 + n_8 + n_{10} > n_3 + n_5 + n_6 + n_9 + n_{11} \; [A\boldsymbol{M}B] \quad (1)$$
$$n_1 + n_2 + n_3 + n_7 + n_{10} > n_4 + n_5 + n_6 + n_9 + n_{12} \; [A\boldsymbol{M}C]. \quad (2)$$

The definition of a WSR that is given above does not accommodate the possibility of dichotomous preferences, and there are two possible ways to proceed when applying a WSR in this scenario. The first option is the *forced ranking* approach that is the most direct solution and simply requires voters to break indifference ties on candidates and report a complete preference ranking on the candidates so that a WSR can be directly implemented as described above. The second option is to modify the definition for implementing a WSR to account for the possibility that voters might have dichotomous preferences.

Earlier research in Gehrlein (2010) strongly suggests that the forced rankings option is a poor approach to dealing with the partial indifference scenario when voters have preferences on candidates that are completely independent of the preferences of other voters. However, this assumption of independent preferences is well-known to frequently produce highly exaggerated estimates of the likelihood that bad election outcomes will be observed [see for example Gehrlein and Lepelley (2011)], so we begin our analysis by determining what happens when a degree of dependence among voters' preferences is inserted into the analysis of using the forced rankings option. The next stage of our analysis will then consider the second option that modifies WSRs to accommodate for the existence of dichotomous preferences.

The analysis in this current study follows from the development of analytical representations for the probability that various election outcomes will be observed, given different sets of assumptions regarding the likelihood that various voting situations are obtained. Attention is focused on the limiting case of large electorates as $n \to \infty$.

Two Basic Models for the Probability that a Given Voting Situation is Observed

Two basic models have served as the foundation of numerous studies that are related to the probability that various election outcomes might be observed when all voters have complete preference rankings [see e.g. Kamwa and Merlin (2015), Diss and Pérez-Asurmendi (2015) or Courtin et al. (2015) for some recent illustrations]. The first of these models is the assumption of the *Impartial Culture Condition* (IC) which is based on the notion of developing a *voter preference profile* that lists the complete preference ranking on candidates that is associated with each of the voters. Each of the n voters is randomly and independently assigned one of the six possible preference rankings in Figure 1, with an equal likelihood for the selection of each of the rankings for each voter. Every possible voter preference profile is therefore equally likely to be observed. The associated voting situation is then directly obtained from a voter preference profile by accumulating the numbers of voters with the same preference rankings. The probability that any given voting situation is observed with IC would then be obtained from a standard multinomial probability model.

The second model is the assumption of the *Impartial Anonymous Culture Condition* (IAC) which is anonymous in the sense that no voter preference profile is produced to identify the preference ranking that is associated with any given voter. It is assumed instead with IAC that all possible voting situations are equally likely to be observed. Berg and Bjurulf (1983) point out the very important fact that IAC inherently introduces a degree of dependence among voters' preferences. Both IC and IAC are unbiased with regard to introducing a CW into a voting situation since it is equally likely that *AMB* or *BMA* for any two Candidates *A* and *B* in a random voting situation that is generated by either of them. The critical difference between the two models is that voters' preferences are completely independent with IC, while a degree of dependence is introduced with IAC. It is therefore possible to observe the pure impact that the introduction of this specific dependence has on the probability that voting outcomes are observed by comparing results with IC and IAC, since nothing else is changed.

The extension of IC to the consideration of partial indifference was addressed in Fishburn and Gehrlein (1980) where the definition of the Impartial Weak Order Condition was developed, and we refer to that condition as the *Extended Impartial Culture Condition* (EIC) in the current study. To describe EIC, let α denote the probability that a given voter will have a complete preference ranking in the form of one of the six preference scenarios in Figure 1, so that there is a probability $1-\alpha$ that this voter will have one of the six possible rankings with partial indifference in Figure 2. A random voter preference profile with EIC then determines preferences for each voter such that each complete preference ranking from Figure 1 has a probability of $\alpha/6$ and each preference scenario in Figure 2 has a probability of $(1-\alpha)/6$ of being observed for that voter.

The limiting representation as $n \to \infty$ for the probability that a CW exists is denoted by $P_{CW}^{\infty}(\alpha, EIC)$ for a specified α, and it is obtained in Fishburn and Gehrlein (1980) as

$$P_{CW}^{\infty}(\alpha, EIC) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{2+\alpha}\right). \quad (3)$$

The representation in (3) directly matches the well-known results of Guilbaud (1952) for $P_{CW}^{\infty}(1, EIC) \approx .91226$ and Sen (1964) for $P_{CW}^{\infty}(0, EIC) = 1$. Computed values of $P_{CW}^{\infty}(\alpha, EIC)$ from (3) are shown graphically in Figure 3 for each $\alpha = 0(.05)1$.

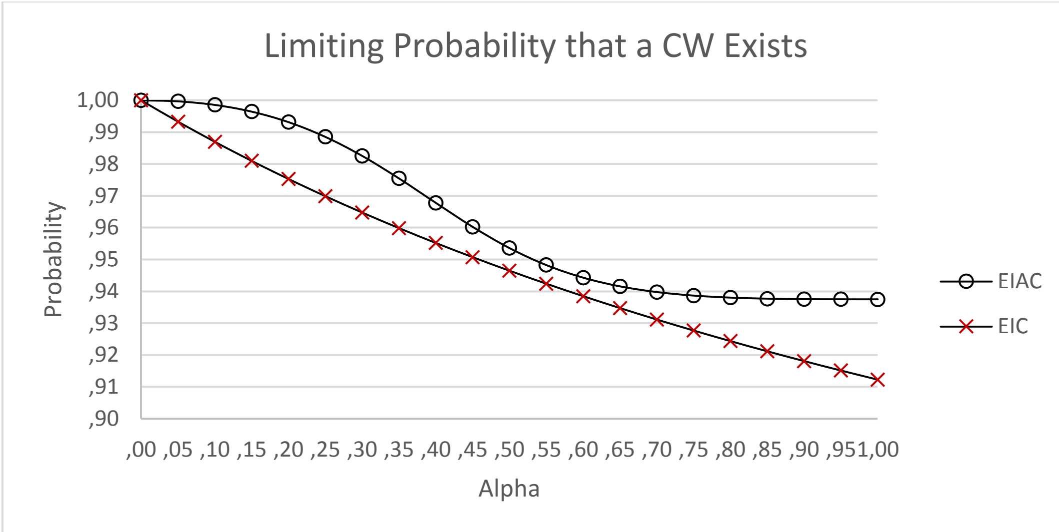

Figure 3. Computed values of $P_{CW}^{\infty}(\alpha, EIC)$ and $P_{CW}^{\infty}(\alpha, EIAC)$.

The results in Figure 3 show that $P_{CW}^{\infty}(\alpha, EIC)$ consistently decreases as α increases, so that an increased level of partial indifference has a consistent impact to increase the probability that a CW will exist.

The notion of extending IAC to account for the possibility of partial indifference was developed in Gehrlein and Lepelley (2015). To describe the *Extended IAC Condition* (EIAC), let k be the number of the n voters in a voting situation who have complete preference rankings on the candidates like those in Figure 1, with

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = k \quad (4)$$
$$n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12} = n - k. \quad (5)$$

EIAC then assumes that all voting situations with any specified value of k are equally likely to be observed. We note for emphasis that this does not lead to the conclusion that all values of k are equally likely to be observed in voting situations, but all voting situations with the same value of k are equally likely to occur.

So, Candidate A is the CW for voting situations with a specified k when (1), (2), (4) and (5) hold simultaneously. Limiting probability representations as $n \rightarrow \infty$ that are based on EIAC are obtained from the relative volumes of subsets of a reference polytope. In case of just one parameter n this is the unit simplex, in the case of two parameters it is a Cartesian product of unit simplices (sometimes called simplotope). We begin with a representation for $P_{CW}^{\infty}(\alpha, EIAC)$. The development of this particular probability representation and the others that are obtained in this study begin by using Ehrhart Polynomial Theory to produce representations for the associated probabilities as a function of general n and k. These general representations are soon found to become far too complex to be of any practical use, so they are therefore made manageable by reducing them to account only for the limiting case as $n \rightarrow \infty$. This procedure becomes quite complicated to implement, and it is made even more difficult both by the fact that the parameter *k* is being specified and by the fact that 12 variable dimensions for the n_i terms must be accounted for in this case. Even 18 variable dimensions will be used later in the study. We wish to focus primarily in this study on the conclusions that can be reached from these representations, and an explanation of the technical details of how these representations are obtained is therefore given in the appendix to this paper. The results that are obtained in this study mark a highly significant advancement in the level of sophistication that has been considered in previous studies of this type by appealing to symmetries that exist within the definitions of election outcomes for voting situations.

The resulting limiting representation for $P_{CW}^{\infty}(\alpha, EIAC)$ is given by:

$$P_{CW}^{\infty}(\alpha, EIAC) = \frac{[1573\alpha^5 - 6480\alpha^4 + 12960\alpha^3 - 13280\alpha^2 + 6720\alpha - 1344]}{1344(\alpha-1)^5}, \text{ for } 0 \leq \alpha \leq \frac{1}{4} \quad (6)$$

$$= \frac{\left[\begin{matrix} 822064\alpha^{10} - 1688320\alpha^9 + 1082880\alpha^8 - 53760\alpha^7 - 107520\alpha^6 - 64512\alpha^5 \\ +53760\alpha^4 - 7680\alpha^3 + 720\alpha^2 - 40\alpha + 1 \end{matrix}\right]}{193536\alpha^5(1-\alpha)^5}, \text{ for } \frac{1}{4} \leq \alpha \leq \frac{1}{3}$$

$$= \frac{\left[\begin{matrix} 890357\alpha^{10} - 3626090\alpha^9 + 6298245\alpha^8 - 5982360\alpha^7 + 3322410\alpha^6 - 1098972\alpha^5 \\ +235410\alpha^4 - 40920\alpha^3 + 4545\alpha^2 - 290\alpha + 8 \end{matrix}\right]}{193536\alpha^5(\alpha-1)^5}, \text{ for } \frac{1}{3} \leq \alpha \leq \frac{1}{2}$$

$$= \frac{5[33073\alpha^5 + 11539\alpha^4 - 15626\alpha^3 + 9794\alpha^2 - 2791\alpha + 299]}{193536\alpha^5}, \text{ for } \frac{1}{2} \leq \alpha \leq 1.$$

The representation in (6) gives results to directly verify observations of Gehrlein and Fishburn (1976) with $P_{CW}^{\infty}(1, EIAC) = 15/16$ and Sen (1964) with $P_{CW}^{\infty}(0, EIAC) = 1$. This representation also verifies the same result that was developed for use in Gehrlein and Lepelley (2015), but where it was not directly reported. The representation from (6) is used to compute values of $P_{CW}^{\infty}(\alpha, EIAC)$ for each $\alpha = 0(.05)1$ are the results are shown graphically in Figure 3.

The results in Figure 3 show that the introduction of a degree of dependence among voters' preference with EIAC does indeed consistently increase the probability that a CW exists compared to the case of complete independence with EIC for all $0 \leq \alpha \leq 1$. But, we must consider the relative impact that IAC and EIAC have on introducing a degree of dependence among voter's preferences before we proceed with our analysis. To do this, we follow Berg and Bjurulf (1983) that considered IC and IAC in the context of a Polya-Eggenberger urn model to describe how random voting situations can be generated.

To generate a random voting situation with IC, we start with an urn that contains six balls of different colors to represent the six possible voter preference rankings in Figure 1. A ball is selected at random to represent the preference ranking of the first voter and the ball is placed back in the urn. The process is repeated *n* times to get a voter preference profile to then obtain its corresponding voting situation. The probability that the second ball has the same color as the first ball remains 1/6 with IC and there is complete independence among voters' preferences.

The case with IAC is quite different. Balls are drawn at each of the *n* steps to determine the preferences of the associated voter in a voter preference profile, except that the ball that is drawn at each step is now placed back into the urn along with one additional ball of the same color that was drawn for that step. The probability that the second ball that is drawn is the same color as the first is now 2/7 with IAC, and not 1/6 as with IC. Some dependence has clearly been introduced among voters' preferences.

When we consider EIC and EIAC, there are twelve balls of different colors at the start of the experiment, to represent the twelve possible voter preference rankings in Figures 1 and 2. Everything else remains the same in obtaining a random voting situation. The probability that the second ball drawn has the same color as the first ball drawn is now 1/12 with EIC and 2/13 with EIAC. There are clear differences in the amount of dependence that is being introduced with EIAC relative to IAC. But, the basic notion that all voting situations are equally likely to be observed with both the IAC and EIAC scenarios holds, while all voting profiles remain equally likely to be observed with the IC and EIC scenarios. The next step is to determine what happens when voters are required to use the forced ranking option

The Impact of the Forced Ranking Option on the Probability that a CW Exists

When the voters with dichotomous preferences like those in Figure 2 are required to break their indifference ties on candidates, there are 12 possible corresponding forced ranking options in Figure 4. The complete rankings in Figure 4 are obtained from the dichotomous preferences in Figure 2 as follows. Figure 2 showed n_7 voters with $A{\sim}B$ in the more preferred subset, and Figure 4 states instead that there are n'_7 voters with $A{\sim}B$ in the more preferred subset who broke the indifference tie with $A \succ$ B and n^*_7who broke the indifference tie with $B \succ A$.

A	A	B	A	B	C
B	C	C	B	A	A
C	B	A	C	C	B
n'_7	n'_8	n'_9	n'_{10}	n'_{11}	n'_{12}
B	C	C	A	B	C
A	A	B	C	C	B
C	B	A	B	A	A
n^*_7	n^*_8	n^*_9	n^*_{10}	n^*_{11}	n^*_{12}

Figure 4. Forced ranking options from dichotomous preferences in Figure 2.

We note for example that there are $n_1 + n'_7 + n'_{10}$ complete rankings with $A \succ B \succ C$ after the forced preference ranking option is employed, but these are not accumulated to a single value for the common ranking since this common ranking is being obtained from three different sources.

The total number of the *n* voters with complete preference rankings is still defined by *k* from (4), but we now have

$$n'_7 + n^*_7 + n'_8 + n^*_8 + n'_9 + n^*_9 + n'_{10} + n^*_{10} + n'_{11} + n^*_{11} + n'_{12} + n^*_{12} = n - k \quad (7)$$

A different majority rule relationship must be defined for the forced ranking option, and it is denoted as $\boldsymbol{M}^*$. The *Forced Condorcet Winner* (FCW) is Candidate *A* based on the preference rankings of voters from Figures 1 and 4 for the forced ranking option, when:

$$n_1 + n_2 + n_4 + n'_7 + n'_8 + n^*_8 + n'_{10} + n^*_{10} + n'_{12} > \quad (8)$$
$$n_3 + n_5 + n_6 + n^*_7 + n'_9 + n^*_9 + n'_{11} + n^*_{11} + n^*_{12} \quad [A\boldsymbol{M}^*B]$$
$$n_1 + n_2 + n_3 + n'_7 + n^*_7 + n'_8 + n'_{10} + n^*_{10} + n'_{11} > \quad (9)$$
$$n_4 + n_5 + n_6 + n^*_8 + n'_9 + n^*_9 + n^*_{11} + n'_{12} + n^*_{12} \quad [A\boldsymbol{M}^*C]$$

The concept of EIC was further extended to consider the forced ranking option in Gehrlein and Valognes (2001), and we refer to that model as the assumption of the *Force Ranking Impartial Culture Condition* (*FIC*). To describe this model, let α denote the probability that a voter will have one of the six complete preference rankings in Figure 1, and it follows that $(1-\alpha)$ is the probability that a voter will have one of the 12 forced complete rankings in Figure 4. When a voter preference profile is being obtained with FIC, it is assumed for each voter that each ranking in Figure 1 has a probability $\frac{\alpha}{6}$ of being selected and each forced ranking in Figure 4 has a probability $\frac{1-\alpha}{12}$ of being selected as that voter's preference. Gehrlein and Valognes (2001) prove that

$$P^{\infty}_{FCW}(\alpha, FIC) = P^{\infty}_{CW}(1, EIC) \text{ for all } \alpha. \quad (10)$$

So, the probability that a FCW exists with the forced ranking option for any α is identical to the probability that a CW exists with IC.

It is possible to extend EIAC to consider the forced ranking option in the same fashion. We assume with the *Forced Ranking IAC Condition* (FIAC) that all voting situations with a given value of *k* in (4) are equally likely to be observed. As in the case of EIAC, this does not imply that all values of *k* are equally likely to be observed.

Candidate *A* will be the FCW for voting situations with a specified *k* when (4), (7), (8) and (9) hold simultaneously. As we did above for EIAC, let $\alpha = \frac{k}{n}$ in the limit $n \to \infty$ and we obtain a representation for the probability that a FCW exists with FIAC as:

$$P^{\infty}_{FCW}(\alpha, FIAC) = \frac{\begin{bmatrix}141321\alpha^{11} - 1040976\alpha^{10} + 3707160\alpha^9 - 8776880\alpha^8 + 15362620\alpha^7 - 20263152\alpha^6 \\ +19867848\alpha^5 - 14151280\alpha^4 + 7091370\alpha^3 - 2369510\alpha^2 + 474474\alpha - 43134\end{bmatrix}}{46592(\alpha-1)^{11}}, \quad (11)$$
$$\text{for } 0 \le \alpha \le \frac{1}{2}$$
$$= \frac{37897\alpha^5 + 16403\alpha^4 - 16902\alpha^3 + 7878\alpha^2 - 1747\alpha + 151}{46592\alpha^5}, \text{ for } \frac{1}{2} \le \alpha \le 1.$$

The representations in (10) and (11) are used respectively to compute values for $P^{\infty}_{FCW}(\alpha, FIC)$ and $P^{\infty}_{FCW}(\alpha, FIAC)$ for each $\alpha = 0(.05)1$. The results are shown graphically in Figure 5. These results show that $P_{FCW}^{\infty}(\alpha, FIAC)$ does not change monotonically as α increases, but the degree of dependence that is introduced by FIAC results in a consistent increase in $P_{FCW}^{\infty}(\alpha, FIAC)$ compared to the case of complete independence with $P_{FCW}^{\infty}(\alpha, FIC)$.

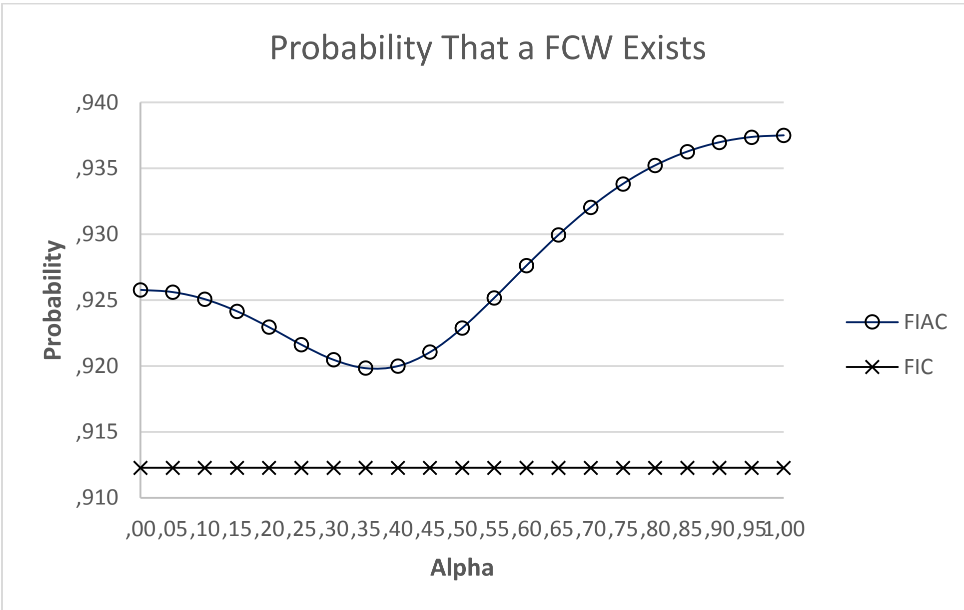

Figure 5. Computed values for $P_{FCW}^{\infty}(\alpha, FIC)$ and $P_{FCW}^{\infty}(\alpha, FIAC)$.

The probability that a FCW exists is found to be relatively large in Figure 5, so it is definitely of interest to determine the Condorcet Efficiency of various voting rules when the forced ranking option is used.

Condorcet Efficiency with Forced Rankings

The limiting Condorcet Efficiency as $n \to \infty$ that a WSR will elect the FCW, given that a CW exists in the original voting situation with EIC was considered in Gehrlein and Valognes (2001). This probability is denoted as $CE_{FCW}^{\infty}(WSR, \alpha, FIC)$, and it was shown that the basic result of the representation in equation (10) still holds in this case and

$$CE_{FCW}^{\infty}(WSR, \alpha, FIC) = CE_{CW}^{\infty}(WSR, 1, EIC), \text{ for all } \lambda \text{ and } \alpha. \quad (12)$$

So, the probability that a WSR elects the FCW with FIC is equal to the Condorcet Efficiency of that WSR with IC, for all λ and α.

Simplified limiting Condorcet Efficiency representations for the special cases of PR, NPR and BR with FIC that are obtained in earlier studies are given in Gehrlein and Lepelley (2011) with:

$$CE_{FCW}^{\infty}(PR, \alpha, FIC) = CE_{FCW}^{\infty}(NPR, \alpha, FIC) = \quad (13)$$

$$\frac{\left[\begin{array}{c}\frac{1}{4}+\frac{3}{4\pi}\left\{Sin^{-1}\left(\sqrt{\frac{2}{3}}\right)+Sin^{-1}\left(\sqrt{\frac{1}{6}}\right)+\frac{1}{2}Sin^{-1}\left(\frac{1}{3}\right)\right\}+\\ \frac{3}{4\pi^2}\left\{\left(Sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)^2-\frac{1}{4}\left(Sin^{-1}\left(\frac{1}{3}\right)\right)^2+\frac{3}{2}\int_0^{\frac{1}{3}}\frac{Sin^{-1}\left(\frac{x}{1+2x}\right)}{\sqrt{1-x^2}}dx\right\}\end{array}\right]}{P_{CW}^{\infty}(1,EIC)}\approx .7572$$

$$CE_{FCW}^{\infty}(BR,\alpha,FIC)=\frac{\left[\frac{3}{2}-\frac{3}{2\pi}\left\{Cos^{-1}\left(\sqrt{\frac{8}{9}}\right)+Cos^{-1}\left(\sqrt{\frac{2}{9}}\right)\right\}\right]}{P_{CW}^{\infty}(1,EIC)}\approx .9012 \tag{14}$$

When attention is turned to that case of FIAC, to allow some dependence among voters' preferences, we note that previous work in Gehrlein et al (2015b) indicates that NPR performs so poorly on the basis of Condorcet Efficiency that it is not worth considering as a viable option. Due to the complexity of the representations that follow, we will therefore only consider PR and BR for the case of FIAC.

Candidate *A* is the PR winner under forced rankings when:

$$n_1+n_2+n'_7+n'_8+n'_{10}+n^*_{10} > n_3+n_5+n'_9+n'_{11}+n^*_7+n^*_{11} \quad [A\boldsymbol{P}B] \tag{15}$$

$$n_1+n_2+n'_7+n'_8+n'_{10}+n^*_{10} > n_4+n_6+n'_{12}+n^*_8+n^*_9+n^*_{12} \quad [A\boldsymbol{P}C] \tag{16}$$

So, Candidate *A* is the PR winner and the FCW for voting situation with a specified *k* when (4), (7), (8), (9), (15) and (16) hold simultaneously. The resulting representations for $CE_{FCW}^{\infty}(PR,\alpha,FIAC)$ are:

$$CE_{FCW}^{\infty}(PR,\alpha,FIAC)=\frac{\left[\begin{array}{c}1717449039\alpha^{11}-10052639568\alpha^{10}+29056015440\alpha^{9}-56810610360\alpha^{8}\\+86661289260\alpha^{7}-107443705824\alpha^{6}+104702165568\alpha^{5}-75861082440\alpha^{4}\\+38751505650\alpha^{3}-13117202670\alpha^{2}+2641332694\alpha-240121154\end{array}\right]}{6561\left[\begin{array}{c}141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}\\-20263152\alpha^{6}+19867848\alpha^{5}-14151280\alpha^{4}+7091370\alpha^{3}\\-2369510\alpha^{2}+474474\alpha-43134\end{array}\right]}, \tag{17}$$

$$\text{for } 0\le\alpha\le\frac{1}{6}$$

$$\frac{\left[\begin{array}{c}9139197526179840\alpha^{16}-98508926948474880\alpha^{15}+483133182480875520\alpha^{14}-1426523596754780160\alpha^{13}\\+2810892918348103680\alpha^{12}-3873694886768738304\alpha^{11}+3809542500639203328\alpha^{10}-2679918421234728960\alpha^{9}\\+1329403039972001280\alpha^{8}-448759173180764160\alpha^{7}+96257485421121024\alpha^{6}-11671905709900800\alpha^{5}\\+768478233781440\alpha^{4}-70085275626240\alpha^{3}+4272674412960\alpha^{2}-156364302912\alpha+2598891689\end{array}\right]}{156728328192\alpha^5\left[\begin{array}{c}141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}-20263152\alpha^{6}+19867848\alpha^{5}\\-14151280\alpha^{4}+7091370\alpha^{3}-2369510\alpha^{2}+474474\alpha-43134\end{array}\right]},$$

$$\text{for } \frac{1}{6}\le\alpha\le\frac{1}{3}$$

$$\frac{-\left[\begin{array}{c}166975641221529600\alpha^{16}-956683976916860928\alpha^{15}+2370588755505315840\alpha^{14}-3079931997392732160\alpha^{13}\\+1619319220126433280\alpha^{12}+1347053128023343104\alpha^{11}-3452417880127856640\alpha^{10}+3423621840135045120\alpha^{9}\\-2107677432261772800\alpha^{8}+878727321890795520\alpha^{7}-255152086240955904\alpha^{6}+53111576840684544\alpha^{5}\\-8433144081835200\alpha^{4}+1051254951586560\alpha^{3}-86260239407520\alpha^{2}+4112978405952\alpha-83439415465\end{array}\right]}{156728328192\alpha^5\left[\begin{array}{c}141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}-20263152\alpha^{6}+19867848\alpha^{5}\\-14151280\alpha^{4}+7091370\alpha^{3}-2369510\alpha^{2}+474474\alpha-43134\end{array}\right]},$$

for $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$

$$\frac{\begin{bmatrix} 1012787056410624\alpha^{16}-11562962853298176\alpha^{15}+60075541929000960\alpha^{14}-188069541369937920\alpha^{13} \\ +395283384068751360\alpha^{12}-587084981452849152\alpha^{11}+629470891462533120\alpha^{10}-485664604716994560\alpha^{9} \\ +258969615464102400\alpha^{8}-81679789311943680\alpha^{7}+1585728157837824\alpha^{6}+13334063953077504\alpha^{5} \\ -7639914480569280\alpha^{4}+2374718010356160\alpha^{3}-452036752198560\alpha^{2}+49506141404112\alpha-2405292877601 \end{bmatrix}}{9795520512(\alpha-1)^{11}[37897\alpha^5+16403\alpha^4-16902\alpha^3+7878\alpha^2-1747\alpha+151)]},$$

for $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$

$$\frac{\begin{bmatrix} 509218937929728\alpha^{16}-5624526498103296\alpha^{15}+28220509223976960\alpha^{14}-84806992629596160\alpha^{13} \\ +169210432925614080\alpha^{12}-234166500369285120\alpha^{11}+226605561139077120\alpha^{10}-148394014074408960\alpha^{9} \\ +57182782979013120\alpha^{8}-3290755074647040\alpha^{7}-10130651591284224\alpha^{6}+6283312009438464\alpha^{5} \\ -1846215830047680\alpha^{4}+244004196381120\alpha^{3}+11167417611360\alpha^{2}-8119166370864\alpha+786404694239 \end{bmatrix}}{9795520512(\alpha-1)^{11}[37897\alpha^5+16403\alpha^4-16902\alpha^3+7878\alpha^2-1747\alpha+151)]},$$

for $\frac{2}{3} \leq \alpha \leq \frac{5}{6}$

$$\frac{16[1121971\alpha^5+1593955\alpha^4-1309670\alpha^3+359690\alpha^2-4555\alpha-7093]}{729[37897\alpha^5+16403\alpha^4-16902\alpha^3+7878\alpha^2-1747\alpha+151]}, \text{ for } \frac{5}{6} \leq \alpha \leq 1.$$

The representation in (17) perfectly matches the known value of $CE_{FCW}^{\infty}(PR, 1, FIAC) = \frac{119}{135}$ from Gehrlein (1982).

Candidate *A* is the winner by BR under forced rankings when:

$$n_1 + 2n_2 + n_4 + n'_7 + 2n'_8 + n^*_8 + n'_{10} + 2n^*_{10} + n'_{12} > \quad (18)$$
$$n_3 + 2n_5 + n_6 + n^*_7 + 2n'_9 + n^*_9 + n'_{11} + 2n^*_{11} + n^*_{12} \ [A\boldsymbol{B}B]$$
$$2n_1 + n_2 + n_3 + 2n'_7 + n^*_7 + n'_8 + 2n'_{10} + n^*_{10} + n'_{11} > \quad (19)$$
$$n_4 + n_5 + 2n_6 + n^*_8 + n'_9 + 2n^*_9 + n^*_{11} + n'_{12} + 2n^*_{12} \ [A\boldsymbol{B}C]$$

So, Candidate *A* is the BR winner and the FCW for voting situation with a specified *k* when (4), (7), (8), (9), (18) and (19) hold simultaneously. The resulting representations for $CE_{FCW}^{\infty}(BR, \alpha, FIAC)$ are:

$$CE_{FCW}^{\infty}(BR, \alpha, FIAC) =$$

$$\frac{\begin{bmatrix} 16395521343\alpha^{11}-78463832000\alpha^{10}+235768276000\alpha^{9}-583236360000\alpha^{8} \\ +1082604250000\alpha^{7}-1467255608000\alpha^{6}+1451229780000\alpha^{5}-1036361040000\alpha^{4} \\ +519926550000\alpha^{3}-173852250000\alpha^{2}+34824790000\alpha-3165890000 \end{bmatrix}}{81000\begin{bmatrix} 141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}-20263152\alpha^{6} \\ +19867848\alpha^{5}-14151280\alpha^{4}+7091370\alpha^{3}-2369510\alpha^{2}+474474\alpha-43134 \end{bmatrix}}, \quad (20)$$

for $0 \leq \alpha \leq \frac{1}{6}$

$$= \frac{\begin{bmatrix} 55252366807680\alpha^{16}-395534931812352\alpha^{15}+1306746201968640\alpha^{14}-2797118834872320\alpha^{13} \\ +4455089399255040\alpha^{12}-5502120721284096\alpha^{11}+5200873487986176\alpha^{10}-3640745683111680\alpha^{9} \\ +1811149363520640\alpha^{8}-603484318831680\alpha^{7}+120733162726176\alpha^{6} \\ -10982689086096\alpha^{5}+3399800040\alpha^{4}-203624820\alpha^{3}+8398890\alpha^{2}-213477\alpha+2522 \end{bmatrix}}{279936000\alpha^5\begin{bmatrix} 141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}-20263152\alpha^{6} \\ +19867848\alpha^{5}-14151280\alpha^{4}+7091370\alpha^{3}-2369510\alpha^{2}+474474\alpha-43134 \end{bmatrix}},$$

for $\frac{1}{6} \leq \alpha \leq \frac{1}{3}$

$$
= \frac{-\begin{bmatrix} 797933643412320\alpha^{16}-5468002424667648\alpha^{15}+17323874646831360\alpha^{14}-33576403816327680\alpha^{13} \\ +44428341815544960\alpha^{12}-42487596792635904\alpha^{11}+30424729805293824\alpha^{10}-16773485640088320\alpha^{9} \\ +7320199806279360\alpha^{8}-2597482364368320\alpha^{7}+756225960313824\alpha^{6}-174915110273904\alpha^{5} \\ +29902221399960\alpha^{4}-3530921175180\alpha^{3}+288718801110\alpha^{2}-14611946523\alpha+344937478 \end{bmatrix}}{279936000\alpha^{5}\begin{bmatrix} 141321\alpha^{11}-1040976\alpha^{10}+3707160\alpha^{9}-8776880\alpha^{8}+15362620\alpha^{7}-20263152\alpha^{6} \\ +19867848\alpha^{5}-14151280\alpha^{4}+7091370\alpha^{3}-2369510\alpha^{2}+474474\alpha-43134 \end{bmatrix}}, \quad \text{for } \frac{1}{3} \le \alpha \le \frac{1}{2}
$$

$$
\frac{\begin{bmatrix} 14417970606720\alpha^{16}-637576766097408\alpha^{15}+5204942664514560\alpha^{14}-21162499944929280\alpha^{13} \\ +53716942830620160\alpha^{12}-93895199244048384\alpha^{11}+119023465785928704\alpha^{10}-112655373288606720\alpha^{9} \\ +80887137556642560\alpha^{8}-44333273888766720\alpha^{7}+18499022742904704\alpha^{6}-5796326340552384\alpha^{5} \\ +1322051982320160\alpha^{4}-206420902919280\alpha^{3}+19516276465560\alpha^{2}-824530702908\alpha-3000084037 \end{bmatrix}}{1119744000(\alpha-1)^{11}[37897\alpha^{5}+16403\alpha^{4}-16902\alpha^{3}+7878\alpha^{2}-1747\alpha+151)]}, \quad \text{for } \frac{1}{2} \le \alpha \le \frac{2}{3}
$$

$$
\frac{\begin{bmatrix} 33702901614720\alpha^{16}-343711150737408\alpha^{15}+1531622472514560\alpha^{14}-3791776903649280\alpha^{13} \\ +5185580649500160\alpha^{12}-1982741888784384\alpha^{11}-6871949940151296\alpha^{10}+16468130020193280\alpha^{9} \\ -20105888245597440\alpha^{8}+16334276872193280\alpha^{7}-9429912787591296\alpha^{6}+3942281316407616\alpha^{5} \\ -1189890468879840\alpha^{4}+253358701560720\alpha^{3}-35974365454440\alpha^{2}+3000674545092\alpha-107857684037 \end{bmatrix}}{1119744000(\alpha-1)^{11}[37897\alpha^{5}+16403\alpha^{4}-16902\alpha^{3}+7878\alpha^{2}-1747\alpha+151)]}, \quad \text{for } \frac{2}{3} \le \alpha \le \frac{5}{6}
$$

$$
\frac{[4066959041\alpha^{5}+247988795\alpha^{4}+464214410\alpha^{3}-810966410\alpha^{2}+393099205\alpha-63183041]}{108000[37897\alpha^{5}+16403\alpha^{4}-16902\alpha^{3}+7878\alpha^{2}-1747\alpha+151]}, \text{ for } \frac{5}{6} \le \alpha \le 1.
$$

The result in (20) perfectly matches the known value of $CE_{FCW}^{\infty}(BR,1,FIAC) = \frac{41}{45}$ from Gehrlein and Lepelley (2001).

The representations from (13), (14), (17) and (20) are used to obtain numerical values of $CE_{FCW}^{\infty}(PR,\alpha,FIC)$, $CE_{FCW}^{\infty}(BR,\alpha,FIC)$, $CE_{FCW}^{\infty}(PR,\alpha,FIAC)$ and $CE_{FCW}^{\infty}(BR,\alpha,FIAC)$ for each $\alpha = 0(.05)1$ and the results are presented graphically in Figure 6.

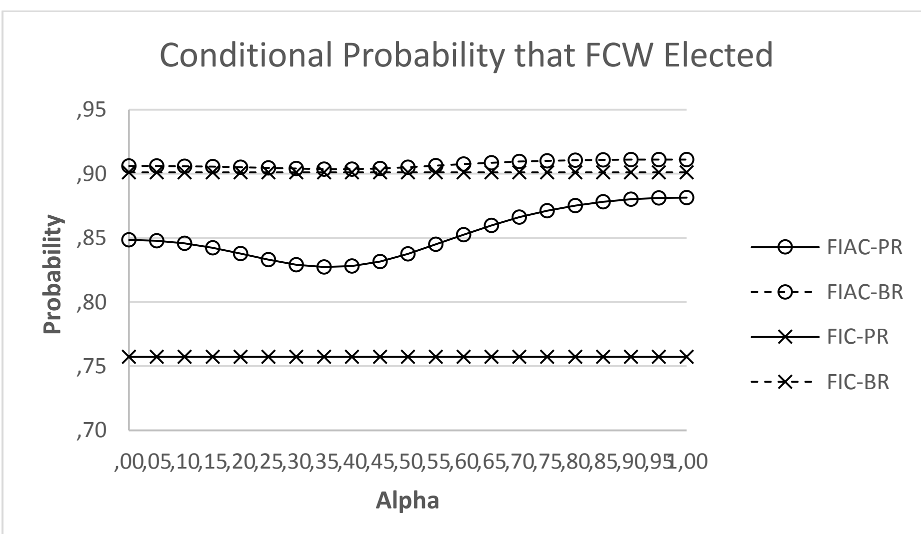

Figure 6. Conditional probability that FCW is elected by PR and BR for FIC and FIAC.

The obvious conclusion from Figure 6 is that the introduction of some dependence among voters' preferences has a dramatic impact on the Condorcet Efficiency of PR with regard to electing the FCW. A similar increase in Condorcet Efficiency also exists with BR, but the results are not as significant. However, this follows from the fact that the efficiency of BR was so high relative to PR with FIC to begin with.

The results from Figure 6 initially sound quite promising for the policy of using the forced ranking option if there is some dependence among voters' preferences, but there is another important factor that must be considered. Gehrlein (2010) considered the problem that the FCW could be a different candidate than the CW, and that Condorcet Efficiency should be based on the probability that a voting rule elects the original CW and it should not be based on the artificial FCW that is created only because voters are forced arbitrarily to rank candidates when some indifference between candidates exists. This is a particularly critical issue if there is a significant probability that the CW and FCW are not the same candidate.

Let $P_{MACW}^{\infty}(\alpha, EIC)$ denote the limiting conditional probability that there is mutual agreement between the CW and FCW with EIC, given that a CW exists for a specified α. A representation for this conditional probability is obtained in Gehrlein (2010) as:

$$P_{MACW}^{\infty}(\alpha, EIC) = \frac{3\left[\begin{array}{c} \frac{1}{16}+\frac{1}{4\pi}\left\{Sin^{-1}\left(\frac{1}{\alpha+2}\right)+Sin^{-1}\left(\sqrt{\frac{\alpha+2}{3}}\right)+Sin^{-1}\left(\sqrt{\frac{1}{3(\alpha+2)}}\right)\right\} \\ +\frac{1}{4\pi^2}\left\{\left(Sin^{-1}\left(\frac{1}{\alpha+2}\right)\right)^2+\left(Sin^{-1}\left(\sqrt{\frac{\alpha+2}{3}}\right)\right)^2-\left(Sin^{-1}\left(\sqrt{\frac{1}{3(\alpha+2)}}\right)\right)^2\right\} \\ -\frac{(1-\alpha)}{4\pi^2}\int_0^1 \frac{Cos^{-1}\left\{\frac{(\alpha+3)\{(\alpha-1)x-3(\alpha+1)(\alpha+3-x)\}+g(\alpha,x)}{g(\alpha,x)}\right\}}{\sqrt{\{(\alpha-1)x+3(\alpha+3)\}\{3(\alpha+1)-(\alpha-1)x\}}}dx \end{array}\right]}{P_{CW}^{\infty}(\alpha,EIC)}, \quad (21)$$

with $g(\alpha, x) = (\alpha + 2)\{3(\alpha + 3)(\alpha + 1) + (\alpha - 1)(x - 2)x\}$.

We use (21) to obtain values of $P_{MACW}^{\infty}(\alpha, EIC)$ for each $\alpha = 0(.05)1$ and the results are displayed graphically in Figure 7.

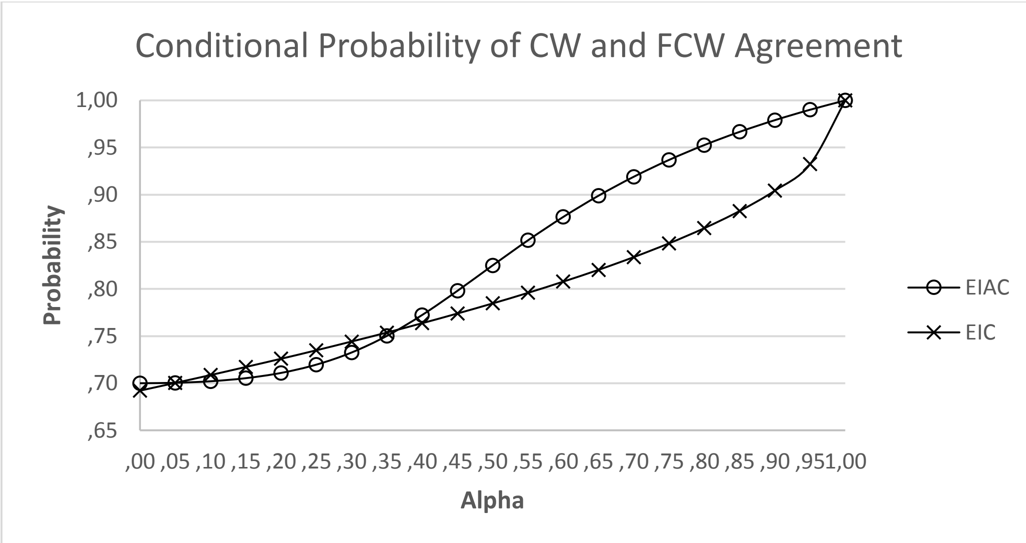

Figure 7. Computed values of $P_{MACW}^{\infty}(\alpha, EIC)$ and $P_{MACW}^{\infty}(\alpha, EIAC)$.

It is clear from Figure 7 that the conditional probability that the CW and FCW are in agreement with EIC is quite low for smaller values of α. The obvious question of interest is concerned with whether or not the imposition of some degree of dependence among voters' preferences will correct this very undesirable situation.

Candidate *A* is both the CW and the FCW for voting situation with a specified *k* when (1), (2), (4), (7), (8) and (9) hold simultaneously. The resulting representations for $P_{MACW}^{\infty}(\alpha, EIAC)$ are:

$$P_{MACW}^{\infty}(\alpha, EIAC) = \frac{-\begin{bmatrix} 115679796\alpha^{11} - 261334080\alpha^{10} - 824679840\alpha^{9} + 4551662080\alpha^{8} \\ -9900858240\alpha^{7} + 13693994496\alpha^{6} - 13397367984\alpha^{5} + 9433710000\alpha^{4} \\ -4680239850\alpha^{3} + 1553328920\alpha^{2} - 309990681\alpha + 28180971 \end{bmatrix}}{29952(1-\alpha)^{6}[1573\alpha^{5} - 6480\alpha^{4} + 12960\alpha^{3} - 13280\alpha^{2} + 6720\alpha - 1344]}, \quad (22)$$

for $0 \leq \alpha \leq \frac{1}{4}$

$$\frac{-\begin{bmatrix} 7945451219968\alpha^{16} - 44318949228544\alpha^{15} + 111554603950080\alpha^{14} - 170196730511360\alpha^{13} \\ + 179186555125760\alpha^{12} - 141260014092288\alpha^{11} + 88138201575424\alpha^{10} - 44488879452160\alpha^{9} \\ + 17748890565120\alpha^{8} - 5228982753280\alpha^{7} + 1015743688960\alpha^{6} - 106950095616\alpha^{5} \\ + 4370213120\alpha^{4} - 326762240\alpha^{3} + 16467600\alpha^{2} - 494608\alpha + 6541 \end{bmatrix}}{479232(1-\alpha)^{6}\begin{bmatrix} 822064\alpha^{10} - 1688320\alpha^{9} + 1082880\alpha^{8} - 53760\alpha^{7} - 107520\alpha^{6} - 64512\alpha^{5} \\ + 53760\alpha^{4} - 7680\alpha^{3} + 720\alpha^{2} - 40\alpha + 1 \end{bmatrix}}, \text{ for } \frac{1}{4} \leq \alpha \leq \frac{1}{3}$$

$$\frac{-\begin{bmatrix} 2534531287808\alpha^{16} - 16246754097152\alpha^{15} + 47206294748160\alpha^{14} - 80764505006080\alpha^{13} \\ + 88231154851840\alpha^{12} - 61482227509248\alpha^{11} + 23896613092352\alpha^{10} - 646702284800\alpha^{9} \\ - 4930212418560\alpha^{8} + 2971242926080\alpha^{7} - 928316316928\alpha^{6} + 186293662464\alpha^{5} \\ - 30863356160\alpha^{4} + 4872922880\alpha^{3} - 489939600\alpha^{2} + 28990480\alpha - 773773 \end{bmatrix}}{479232(1-\alpha)^{6}\begin{bmatrix} 890357\alpha^{10} - 3626090\alpha^{9} + 6298245\alpha^{8} - 5982360\alpha^{7} + 3322410\alpha^{6} \\ - 1098972\alpha^{5} + 235410\alpha^{4} - 40920\alpha^{3} + 4545\alpha^{2} - 290\alpha + 8 \end{bmatrix}}, \text{ for } \frac{1}{3} \leq \alpha \leq \frac{1}{2}$$

$$\frac{[374882485\alpha^{5} + 24559999\alpha^{4} - 98195306\alpha^{3} + 45889394\alpha^{2} - 7881451\alpha + 400559]}{9360[33073\alpha^{5} + 11539\alpha^{4} - 15626\alpha^{3} + 9794\alpha^{2} - 2791\alpha + 299]}, \text{ for } \frac{1}{2} \leq \alpha \leq 1.$$

The representation in (22) perfectly matches the obvious result that $P_{MACW}^{\infty}(1, EIAC) = 1$, and computed values of $P_{MACW}^{\infty}(\alpha, EIAC)$ are shown in Figure 7 for each $\alpha = 0(.05)1$. The results in Figure 7 predictably show that the introduction of a degree of dependence among voters' preferences does indeed increase with EIAC for $.35 \leq \alpha < 1$. However, the very surprising result is that this does not happen for smaller values of α, so that the possibility that PR and BR elect the FCW when it is not the original CW cannot be ignored when a large proportion of voters have dichotomous preferences.

This possibility is examined further with the consideration of representations for the Condorcet Efficiency $CE_{CW}^{\infty}(VR, \alpha, FIC)$ of voting rules $VR \in \{PR, BR\}$ relative to their ability to elect the original CW under the forced ranking option. Computed values of $CE_{CW}^{\infty}(VR, \alpha, FIC)$ for PR and BR are obtained from a representation that is given in Gehrlein and Lepelley (2011) for the general case of all possible WSR's and these results are displayed graphically in Figure 8. The original representation is not presented here since it is very complex, and values can only be obtained with numerical integration. These results indicate that there is a relatively low likelihood of only about 67% that the CW will be elected by PR when all voters' preferences are dichotomous with $\alpha = 0$, given that a CW exists. There is better performance with BR in the same scenario with a likelihood of only about 75% for the outcome, but this is not an encouraging result overall. The impact that introducing a degree of dependence into voters' preferences with EIAC will have is therefore definitely of interest if the forced ranking option is to be considered as a viable option.

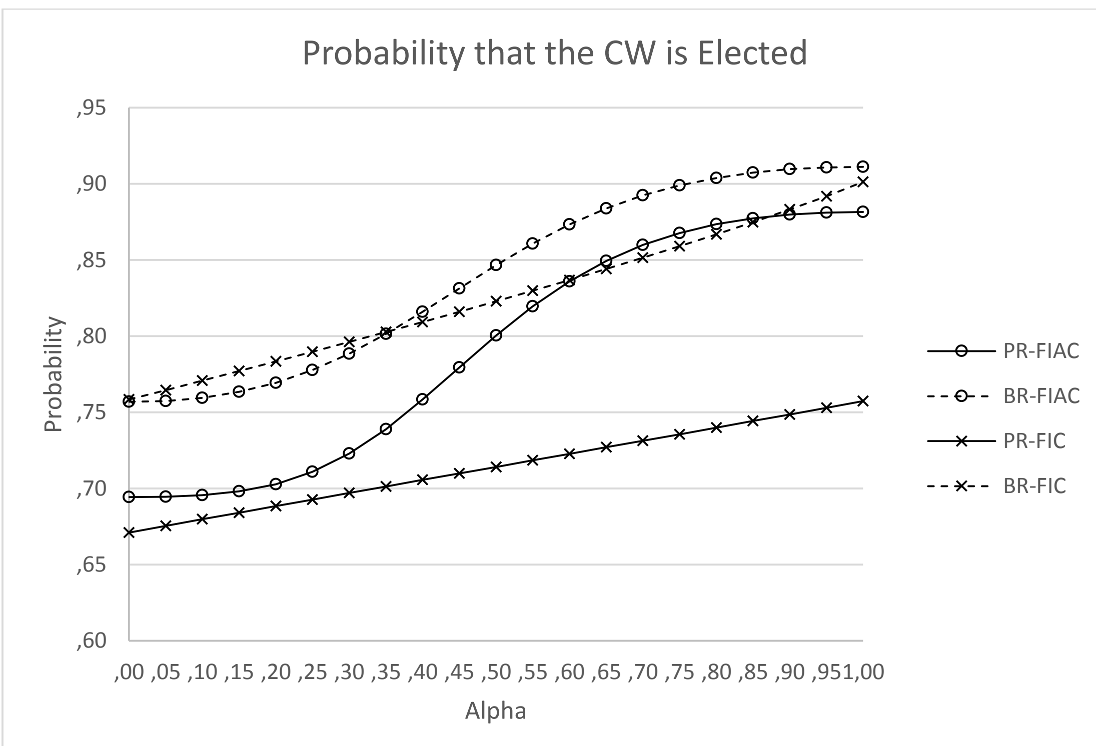

Figure 8. Computed Values of $CE_{CW}^{\infty}(VR, \alpha, FIC)$ and $CE_{CW}^{\infty}(VR, \alpha, FIAC)$ for PR and BR.

Representations are obtained for $CE_{CW}^{\infty}(VR, \alpha, FIAC)$ with PR and BR by using the techniques that were employed above, and they are predictably very complex. The integer coefficients in the polynomials in these representations become extremely large, and in order to cover the range of $0 \leq \alpha \leq 1$ it requires individual representations that correspond to 15 different segments of α values for PR and over 17 segments for BR. As a result, the formal representations that were obtained are not presented here, but they are available for any interested reader to observe in a posted online appendix (*to be noted here*). Computed values that are obtained from these representations are displayed graphically in Figure 8.

The Condorcet Efficiency results for PR improve dramatically with larger values of α with FIAC relative to FIC. The BR efficiencies do not improve as much but they always maintain superior performance relative to PR. However, the associated efficiency values for voting situations with completely dichotomous preferences at $\alpha = 0$ increase by only a very small amount for both PR and BR. This likelihood increases to only about 69% for PR. So, the possibility that the wrong candidate might be elected with the forced ranking option therefore remains as a significant issue that must be considered as being problematic, even with the introduction of some dependence among voters' preferences.

Approval Voting and Extended Scoring Rules

The forced ranking option clearly does not present a viable alternative when voters have dichotomous preferences, so it is necessary to consider how our voting rules might be modified to accommodate this scenario. There are two approaches that can be taken for this option, and the first one is to consider the use of Approval Voting (AV) which allows voters to cast a ballot for as many candidates as they want to vote for, so that not all voters will necessarily cast the same number of votes. Each voter will cast a vote for each candidate that they consider to be at least as good as all other candidates with AV, so that all voters would vote for one candidate if they have complete preferences like those in Figure 1. When voters have dichotomous preferences like those in Figure 2, they vote for each candidate in their more preferred subset. Candidate A is the AV winner with this definition whenever:

$$n_1 + n_2 + n_8 + n_{10} > n_3 + n_5 + n_9 + n_{11} \quad [A\boldsymbol{A}B] \tag{23}$$
$$n_1 + n_2 + n_7 + n_{10} > n_4 + n_6 + n_9 + n_{12} \quad [A\boldsymbol{A}C] \tag{24}$$

Brams and Fishburn (1983) showed that AV has many positive qualities when voters have dichotomous preferences, but Gehrlein and Lepelley (1998) showed that AV did not outperform either PR or NPR on the basis of Condorcet Efficiency for any α with EIC as $n \to \infty$.

The second possibility is to actually modify the mechanism by which WSR's are employed to accommodate dichotomous preferences, while keeping the number of votes constant for each of the voters. An *Extended Scoring Rule* (ESR) is an adaptation of a WSR to account for the possibility of the presence of partial indifference in voter preference rankings, and has been used in a number of earlier studies including: Gehrlein and Valognes (2001), Diss et al (2010) and Gehrlein and Lepelley (2015). The notion behind the implementation of an ESR is that each voter should retain a total of $1 + \lambda$ points to be distributed to candidates. The basic definition of a WSR is used to distribute these points for voters with complete preference rankings. For voters with partial indifference with two candidates in their more preferred subset, the top ranked candidates are both given the average of $(1 + \lambda)/2$ points and the candidate in the less preferred subset is given zero points. For voters with partial indifference with two candidates in the less preferred subset, the most preferred candidate receives on point and the two bottom ranked candidates each receive an average $\lambda/2$ points.

Results from Diss et al (2010) find that the introduction of a degree of indifference into voters' preferences gives a definite advantage to AV over both Extended PR and Extended NPR on the basis of Condorcet Efficiency with the EIC model as $n \to \infty$. Gehrlein and Lepelley (2015) verified the same conclusion when some degree of dependence is inserted into voters' preferences with the EIAC model. However, Extended BR is found to dominate AV in all cases of EIC and EIAC, except for the scenario in which all voters have dichotomous preferences, where AV and Extended BR both elect the CW with certainty.

Conclusion

The most direct solution to implementing a WSR in an election in which some voters have dichotomous preferences would be to require voters to arbitrarily break indifference ties and report complete preference rankings on the candidates. This was previously found to be a very poor option for use when voters have completely independent preferences. The current study shows that while the introduction of a degree of dependence among voters' preferences typically helps to reduce the impact of such negative outcomes for elections, it still results in a significant possibility of very bad outcomes from using the forced ranking option. The forced ranking option should not be considered as a viable consideration. This is particularly true since other options like AV and ESR's that modify WSR's to accommodate dichotomous preferences have been found to be much more effective.

Appendix: Counting Lattice Points in Parametric Polytopes

Mathematical Framework

The total number of possible voting situations that result in some specified election outcome can be modeled as the number of integral solutions of a system of linear inequalities that are dependent on the total number n of all voters, and on the number k of them having complete preference rankings on the candidates. More precisely, we are dealing with a *parametric polytope* P, i.e. a family of polytopes $P_{n,k} \subset \mathbb{Q}^d$ of the form

$$P_{n,k} = \left\{ x \in \mathbb{Q}^d : Fx \leq G \binom{n}{k} + c \right\}$$

for certain d- and 2-column matrices F and G and a suitable vector c. The enumerator $\mathcal{E}_P$, defined by

$$\mathcal{E}_P(n,k) = \#\left(P_{n,k} \cap \mathbb{Z}^d\right),$$

counts the number of integral points contained in P depending on n and k. That is, it counts the number of different voting situations with the specified election outcome, given n voters with k of them having complete preference rankings.

By a mathematical theory going back to the work of Ehrhart (1962) we can express $\mathcal{E}_P(n,k)$ by a closed formula. Its use in voting theory is for instance described in Lepelley et al (2008). For the multi-parameter setting, it was proven by Clauss and Loechner (1998), that the parameter space ($\mathbb{Z}^2$ in our case) can be decomposed into finitely many polyhedra, called *chambers*, such that $\mathcal{E}_P$ is a *quasipolynomial* on each of them. The important component for us is the *leading term* LT_P of $\mathcal{E}_P$ which is known to be a polynomial on each chamber.

Let P and S denote parametric polytopes, with P contained in S. Then, the expected relative frequency of voting situations being in P, among voting situations in S is expressed by

$$Prob(n,k) = \frac{\mathcal{E}_P(n,k)}{\mathcal{E}_S(n,k)}.$$

Since the formulas for $Prob(n,k)$ are too cumbersome for practical purposes, we restrict our attention to the probability for n and k tending to infinity. More precisely, let $\alpha = k/n$ be the probability that a random voter has a complete preference ranking on the candidates, and let LT_P and LT_S be the leading terms (which are homogeneous polynomials) of $\mathcal{E}_P$ and $\mathcal{E}_S$, respectively, on some fixed chamber. Then we have

$$Prob(n,\alpha n) \xrightarrow{n\to\infty} \frac{\mathrm{LT}_P(1,\alpha)}{\mathrm{LT}_S(1,\alpha)},$$

given that $\mathcal{E}_P$ and $\mathcal{E}_S$ have the same total degree, and α is such that $\binom{n}{\alpha n}$ lies in the fixed chamber mentioned before, for all sufficiently large values of n. In the examples we are concerned with, these constraints are always satisfied, so in order to determine the limiting probability it suffices to calculate the polynomials $\mathcal{E}_P$ and $\mathcal{E}_S$. Note that each chamber gives a segment of α-values for which a closed probability formula is obtained.

Practical Computations

We use the software package `barvinok` to compute the quasipolynomials $\mathcal{E}_P$ and $\mathcal{E}_S$ on all chambers. Afterwards, we filter out the chambers, we are interested in (i.e. those containing $(n, \alpha n)^t$ for sufficiently large n, where $0 \leq \alpha \leq 1$), and cut off the leading terms of the corresponding quasipolynomials to obtain the polynomials LT_P and LT_S on those chambers.

In the following we demonstrate an example calculation of the limiting probability $P_{CW}^{\infty}(\alpha, EIAC)$ that a CW exists, given that $k = \alpha n$ of n voters have complete preferences on the candidates, where $0 < \alpha < \frac{1}{4}$. Figure A.1 shows the input file which encodes the parametric polytope P that describes all voting situations where Candidate *A* is the CW [see Verdoolaege (2016) concerning the syntax]. The first two equalities, as denoted by a 0 in the first column in Figure A.1, specify that there are exactly k voters having complete rankings (distributed among $n_1, \dots, n_6$) and $n - k$ voters with partial indifferences (distributed among $n_7, \dots, n_{12}$). The next two inequalities, as specified by a 1 in the first column, then describe those voting situations where *AMB* and *AMC* respectively. Finally, there are 12 inequalities specifying that all n_i are nonnegative.

```
16 16

#  n1  n2  n3  n4  n5  n6  n7  n8  n9  n10  n11  n12  k  n cst
0   1   1   1   1   1   1   0   0   0   0    0    0  -1  0  0
0   0   0   0   0   0   0   1   1   1   1    1    1   1 -1  0
1   1   1  -1   1  -1  -1   0   1  -1   1   -1    0   0  0 -1
1   1   1   1  -1  -1  -1   1   0  -1   1    0   -1   0  0 -1
1   1   0   0   0   0   0   0   0   0   0    0    0   0  0  0
                                    ⋮
1   0   0   0   0   0   0   0   0   0   0    0    1   0  0  0

E 0
P 2
k n
```

Figure A.1 Parametric Polytope *P* describing the voting situations where Candidate *A* is a CW.

Applying the program `barvinok_enumerate_e` to this file yields the complete list of all chambers and corresponding quasipolynomials describing the enumerator $\mathcal{E}_P$. They are too lengthy to print them here, but we can handle their leading terms. On the chamber $\{(n, k)^t \in \mathbb{Z}^2 : n \geq 4k + 2, k \geq 1\}$ for example the leading term of the quasipolynomial $\mathcal{E}_P$ is given by

$$\mathrm{LT}_P(n, k) = \frac{-1573}{58060800} k^{10} + \frac{1}{8960} n k^9 - \frac{1}{4480} n^2 k^8 + \frac{83}{362880} n^3 k^7 - \frac{1}{8640} n^4 k^6 + \frac{1}{43200} n^5 k^5.$$

Since each of the three candidates is a CW in the same number of voting situations, $3 \cdot \mathcal{E}_P$ gives the total number of voting situations where a CW exists, so the leading term of the enumerator is $3 \cdot \mathrm{LT}_P$ for the corresponding chamber. A representation of the parametric polytope S corresponding to all possible voting situations for which *k* voters have complete preference rankings is obtained from Figure A.1 by removing the first and second inequalities. On the chamber $\{(n,k)^t \in \mathbb{Z}^2 : 0 \le k \le n\}$, which clearly contains the previous chamber, the enumerator $\mathcal{E}_S$ is given by the polynomial

$$\mathcal{E}_S(n,k) = \binom{k+5}{5} \cdot \binom{n-k+5}{5}.$$

Its leading term is

$$\mathrm{LT}_S(n,k) = \frac{1}{14400}(n-k)^5 k^5.$$

Putting these results together, we obtain

$$P_{CW}^{\infty}(\alpha, EIAC) = \lim_{n\to\infty} \frac{3\cdot\mathcal{E}_P(n,\alpha n)}{\mathcal{E}_S(n,\alpha n)} = \frac{3\cdot\mathrm{LT}_P(1,\alpha)}{\mathrm{LT}_S(1,\alpha)}$$

$$= \frac{-1573\alpha^5+6480\alpha^4-12960\alpha^3+13280\alpha^2-6720\alpha+1344}{1344(1-\alpha)^5}$$

for all $0 < \alpha < \frac{1}{4}$. The probability representations concerning the remaining segments of α values are obtained in exactly the same way.

Exploiting Symmetries

If a parametric polytope $P_{n,k}$ has symmetries, i.e. if there exist permutations of the variables $x_1, \dots, x_d$ under which it stays unchanged, one may hope for some computational gain by exploiting the structure coming with these symmetries. One such approach in the one parameter case was successfully applied to voting problems in Schürmann (2013). A generalization to the setting of more than one parameter is principally possible: If certain sums of m variables, say $n_{i_1} + \dots + n_{i_m}$, occur in all relevant inequalties and equations together, we can substitute them by a single new variable, say N. In order to obtain the same counting results with $m-1$ variables less, we have to adapt the way of counting though. For a fixed nonnegative integer N we have to count $\binom{N+m}{m}$ voting situations (which is a polynomial in N of degree m), as it gives the number of possibilities to choose different nonnegative integers $n_{i_1}, \dots, n_{i_m}$ with $N = n_{i_1} + \dots + n_{i_m}$. With several of such substitutions we get a product of polynomials in the new variables, which we have to use as *weights* in a new dimension reduced *weighted counting problem*.

We applied this approach to our voting examples using the `barvinok` package, which is currently — to the best of our knowledge — the only available software package capable of weighted counting computations with more than one parameter (and with weights being quasi-polynomials). There are three different methods implemented that we could use [see Verdoolaege, Bruynooghe (2008)].

In the examples of this paper we computed all affine symmetries of the parametric polytopes, using the software Sympol [see Rehn, Schürmann (2010)]. In each case these symmetries turned out to come from permutations of coordinates, many of them allowing a reformulation into a weighted counting problem as described above. In some of the computations this has a substantial beneficial effect. In one of the cases (FCW) we were able to take advantage of a symmetry group of order 9216 giving a reduction of computation time to roughly 25%. However, in another case (BR-CW) there was no symmetry we could take advantage of. Unfortunately, this was the computation taking by far the longest to finish (more than half a year on a machine with two Xeon X5650 Six Core 2.66 GHz processors).

We finally note that we think, in the future it should be possible to take more advantage of available symmetries. Recent theoretical advances as in Baldoni et al (2014) seem to show that in particular the computation of leading terms should be easier, once a counting problem can be reduced to a lower dimensional weighted counting problem. However, at the moment we are still missing efficient software to take advantage of such reformulations in practical multi-parameter computations.

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