

Apportionment and Referendum Paradox in Three-state Federations

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Abstract

In federal unions, how many mandates (or seats) should be given to each state in a two-tier voting system, given that the majority rule is used at each level? We propose to choose an apportionment of the seats among the states that minimizes the probability of the so-called *Referendum Paradox*, i.e. the probability of electing the candidate who receives less than 50% of the votes in a two candidate competition over the whole union. We consider here the three-state federation case and show that allocating seats proportionally to the the population of each state is an optimal solution to the apportionment problem under the May's model (May 1948).

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1. Introduction

In federal unions, the electoral systems are based on elections in jurisdictions (districts, states, regions) and the political decisions are taken by the assembly of representatives elected from these jurisdictions. In designing such electoral systems, a crucial question is the following: how many mandates should be given to the representative of each jurisdictions, given that these jurisdictions are not in general of equal size? The answer is not obvious because a federal union is both an union of states and an union of citizens: the first feature leads to a “one state-one vote” principle whereas the second suggests to allocate the mandates proportionally to the population of each state (“one man-one vote”).

In this note, we use a majoritarian criterion to evaluate the different two-tier voting rules: a rule is said to be *majority efficient* if it minimizes the probability that a decision is taken with a majority of mandates at the federal level though it is supported by a minority of voters over the whole union. In other words, we wish to minimize the likelihood of the so-called *Referendum Paradox* (Nurmi 1999): this paradox occurs whenever a decision taken by representatives elected in local jurisdictions conflicts with the decision that would have been adopted if the voters had directly given their opinion through a referendum. In political science, such an event is called a case of *election inversion* (Miller 2012); Such strange political situations are not unfrequent in real elections, the best known case being the election of George W. Bush against Al. Gore in 2000.

We consider here a specific class of rules, the δ -rules, that allocates a_i mandates to state i according to the law $a_i = n_i^\delta$, where n_i denotes state i population. Though restrictive, this assumption enable us to encompass the pure federalist case ($\delta = 0$ and each state has one mandate), the square root rule case ($\delta = 1/2$), the pure proportionality case ($\delta = 1$) and even the dictatorship of the biggest state ($\delta \rightarrow \infty$). We aim to identify which parameter δ minimizes the probability of the Referendum Paradox.

The probabilistic model on which our computations rely is due to May (1948), who was the first to propose analytical results on the Referendum Paradox likelihood. In his remarkable study, May assumes that the voting results across the states are independent and, in each state, voters' preferences are generated according to what is nowadays referred to as the *Impartial Anonymous Culture* hypothesis. Our model and our probabilistic assumptions are described with more details in Section 2. In Section 3, we give a complete analytical solution to our apportionment problem for the three-state case, showing that the pure proportional rule ($\delta = 1$) is optimal. Section 4 concludes.

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2. Framework and probabilistic assumption

2.1. The model

Consider a set of three states (or regions, districts, etc.) which have to take decisions altogether in a political union. We assume that n_i voters live in state i , and $\sum_{i=1}^3 n_i = n$. The vector $\tilde{n} = (n_1, n_2, n_3)$ describes the repartition of the population among the 3 states. We will assume throughout the paper that $n_1 \geq n_2 \geq n_3 > 0$. Two parties, A and B , compete in all the states; the winner in state i is the party who obtains a majority of voters on his side (abstention is not allowed). We shall denote v_i the number of votes for A in state i . Each state is represented by a_i mandates in the union, and the winner in state i gets all the mandates. For the sake of simplicity, we set that $a_1 \geq a_2 \geq a_3 \geq 0$, with at least a_1 strictly positive. Thus, the position that is officially adopted by the union is the one which obtains a majority of mandates at the federal level. Notice that we always use throughout the paper the 50% quota for all the decisions (votes in the states, vote of the delegates and popular vote nationwide).

In our search of the apportionment rules that minimize the probability of the referendum paradox, we have decided to focus our study on the family of δ -rules. That is, we assume that the vector of mandates, $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$, is entirely characterized by the parameter δ , $\delta \in [0, \infty[$ as $a_i = n_i^\delta \forall i = 1, 2, 3$.

2.2. The Impartial Anonymous Culture assumption

There are several ways to model theoretically the a priori behavior of the citizens (see *e.g.* Gehrlein 2006). Here, we model the people's vote inside each state and we assume that their behavior is described by the same probability distribution in every state. Furthermore, we assume that the votes from states to states are always drawn independently. Thus, the probabilistic behavior of a given state at the federal level is totally driven by the behavior of its voters.

In May's model, and more generally under the Impartial Anonymous Assumption (IAC) (Gehrlein 2006), it is assumed that every repartition of the votes between A and B is equally likely. Many interpretations of this model have been given (going from Polya urns to quantum Bose-Einstein statistics), a fine one being the probabilistic interpretation (Straffin 1977, Berg 1999). The idea is the following: In state i , for a given election, a "public opinion" emerges, *i.e.* an individual probability p_i for selecting one of the issue is drawn from the uniform distribution on $[0, 1]$. Thus, the probability of picking A , may be 0.1, 0.5, 0.7 or whatever you want in $[0, 1]$, with equal probability. Of course, p_i varies from one election to the other, but in average, there is no bias in favor of one alternative.

It is possible to generalize Berg's and Straffin's reasoning: Indeed, the choice of a probability p is itself of probabilistic nature through the introduction of a probability distribution function $f(p)$. The choice of $f(p)$ is a first step for a better description of the electorate behavior. In particular, it could be determined after the study of real data. The distribution $f(p)$ is defined on $0 \leq p \leq 1$, with $f(p) \geq 0$ and $\int_0^1 f(p) dp = 1$. The probability of a given configuration of n identified voters with t votes for A and $(n - t)$ votes for B is $p^t(1 - p)^{n-t}$, and for a large number of elections it reads

$$\int_0^1 f(p) p^t (1 - p)^{n-t} dp. \quad (1)$$

For $f(p) = 1$, we get May's model (IAC in social choice terminology) and we obtain

$$\frac{1}{(n+1) \binom{n}{t}} \quad (2)$$

for the probability of a configuration with t votes for A and $n - t$ votes for B . As there are $\frac{n!}{t!(n-t)!}$ voting configurations with t A 's and $(n - t)$ B 's, we recover the fact that any result is equally likely, as in May's paper.

Under this model, each point in the parallelepiped depicted on Figure 1 is a possible voting outcome; each of them is equally likely and the total volume of the parallelepiped is $n_1 n_2 n_3$. The parallelepiped is cut into eight zones by the hyperplanes $v_1 = n_1/2$, $v_2 = n_2/2$ and $v_3 = n_3/2$.

3. Results

3.1. A limited number of cases

Without loss of generality, we assume in the 3-state case that the distribution of the population is given by the vector $\tilde{n} = (n_1, n_2, n_3)$, with $\sum_{i=1}^3 n_i = n = 1$, and $n_1 \geq n_2 \geq n_3 > 0$. Hence, as in (Saari 1992), we can interpret this voting problem geometrically. Under this model, each point in the parallelepiped depicted on Figure

1 is a possible voting outcome; each of them is equally likely and the total volume of the parallelepiped is $n_1 n_2 n_3$. The parallelepiped is cut into eight zones by the hyperplanes $v_1 = n_1/2$, $v_2 = n_2/2$ and $v_3 = n_3/2$.

Similarly, we assume that the distribution of the mandates is given by $\tilde{a} = (a_1, a_2, a_3)$, with $a_1 \geq a_2 \geq a_3 \geq 0$, and $a_1 > 0$. We consider δ -rules only, $\tilde{a} = (n_1^\delta, n_2^\delta, n_3^\delta)$. But, for a weighted majority game, it is well known (see *e.g.* Leech 2002) that any vector $\tilde{a} = (a_1, a_2, a_3)$ can be identified with one of these possible cases:

- Case 1. $\tilde{a}^1 = (1, 1, 1)$. All the states have the same power. \tilde{a} is equivalent to \tilde{a}^1 if and only if $n_1^\delta < n_2^\delta + n_3^\delta$.
- Case 2. $\tilde{a}^2 = (2, 1, 1)$. \tilde{a} is equivalent to \tilde{a}^2 if and only if $n_1^\delta = n_2^\delta + n_3^\delta$. In case the opinion of states 2 and 3 conflicts with the choice of state 1, a tie breaking rule could be implemented.
- Case 3. $\tilde{a}^3 = (1, 0, 0)$ and state 1 is a dictator. \tilde{a} is equivalent to \tilde{a}^3 if and only if $n_1^\delta > n_2^\delta + n_3^\delta$.

In the three state case, there also exists a fourth possible game, $\tilde{a}^4 = (2, 2, 0)$, where state 3 is a dummy player. However, as $n_3 > 0$, no majority game with weights $(n_1^\delta, n_2^\delta, n_3^\delta)$ can be represented by such a game.

3.2. Case 1, $\tilde{a}^1 = (1, 1, 1)$

We focus first on $\tilde{a}^1 = (1, 1, 1)$, the most interesting case.

Proposition 1. *Let $P(\tilde{n}, \tilde{a}^1)$ be the likelihood of the referendum paradox for three states of large population under IAC (May's model) for the distribution \tilde{n} when each state gets one mandate. If $n_1 < \frac{1}{2}$:*

$$P(\tilde{n}, \tilde{a}^1) = \frac{n_1^3 + n_2^3 + n_3^3 - (n_1 - n_2)^3 - (n_1 - n_3)^3 - (n_2 - n_3)^3}{24n_1 n_2 n_3} \quad (3)$$

If $n_1 > \frac{1}{2}$:

$$P(\tilde{n}, \tilde{a}^1) = \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3 + 6n_2 n_3 (n_1 - n_2 - n_3)}{24n_1 n_2 n_3} \quad (4)$$

Proof: **Subcase $n_1 < \frac{1}{2}$:** We depict on Figure 2 the four cases where A get the support of at least two states. These regions are cut by the hyperplane $v_1 + v_2 + v_3 = n/2$. This defines three regions (labeled I, II and III) where the paradox occurs. Let V_t be the volume of region t . We can easily derive that:

$$V_I = \frac{1}{6} \left[\left(\frac{n_1}{2} \right)^3 - \left(\frac{n_1 - n_3}{2} \right)^3 - \left(\frac{n_1 - n_2}{2} \right)^3 \right] \quad (5)$$

$$V_{II} = \frac{1}{6} \left[\left(\frac{n_2}{2} \right)^3 - \left(\frac{n_2 - n_3}{2} \right)^3 \right] \quad (6)$$

$$V_{III} = \frac{1}{6} \left(\frac{n_3}{2} \right)^3 \quad (7)$$

As the cases where B wins at least two states are symmetric, we can deduce that:

$$P(\tilde{n}, \tilde{a}) = \frac{2(V_I + V_{II} + V_{III})}{n_1 n_2 n_3} \quad (8)$$

$$= \frac{n_1^3 + n_2^3 + n_3^3 - (n_1 - n_2)^3 - (n_1 - n_3)^3 - (n_2 - n_3)^3}{24n_1 n_2 n_3} \quad (9)$$

Subcase $n_1 > \frac{1}{2}$: In Figure 3, we observe that the volumes are the same as in the previous subcase, except for V_I :

$$V_I = \frac{1}{6} \left[\left(\frac{n_2 + n_3}{2} \right)^3 - \left(\frac{n_2}{2} \right)^3 - \left(\frac{n_3}{2} \right)^3 \right] + \frac{n_2 n_3 (n_1 - n_2 - n_3)}{8} \quad (10)$$

As the cases where B wins at least two states are symmetric, we can deduce that:

$$P(\tilde{n}, \tilde{a}) = \frac{2(V_I + V_{II} + V_{III})}{n_1 n_2 n_3} \quad (11)$$

$$= \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3 + 6n_2 n_3 (n_1 - n_2 - n_3)}{24n_1 n_2 n_3} \quad (12)$$

3.3. Case 2, $\tilde{a}^2 = (2, 1, 1)$

This very specific case only occurs when $n_1^\delta = n_2^\delta + n_3^\delta$. Moreover, we have to decide how to interpret a 2:2 deadlock, when state 1 votes for A , while states 2 and 3 endorse B .

In order to derive probabilities, we will assume that, in case of 2:2 deadlock, the election is decided by tossing a fair coin, which means that only half of these situations are considered as paradoxical, depending whether or not the popular winner wins the draw.

Proposition 2. *Let $P(\tilde{n}, \tilde{a}^2)$ be the likelihood of the referendum paradox for three states of large population under IAC for the distribution \tilde{n} when $\tilde{a} = \tilde{a}^2$. Then:*

$$P(\tilde{n}, \tilde{a}^2) = P(\tilde{n}, \tilde{a}^1) - \frac{n_3^3}{12n_1n_2n_3} + \frac{1}{8} \quad (13)$$

Proof: The only difference with Case 1 comes from the regions III displayed on Figures 2 and 3. The whole region is a tossup in terms of mandates, and it accounts to 1/16 to the magnitude of the paradox, as there is a 50% chance that the draw picks the wrong candidate. Thus, to obtain formula (13), we discard expression (7) and add 1/16 instead. As 1/16 is always superior to V_{III} , the new probability is always superior to $P(\tilde{n}, \tilde{a}^1)$.

3.4. Case 3, $\tilde{a}^3 = (1, 0, 0)$

Proposition 3. *Let $P(\tilde{n}, \tilde{a}^3)$ be the likelihood of the referendum paradox for three states of large population under IAC for the distribution \tilde{n} when state 1 is a dictator. If $n_1 < \frac{1}{2}$:*

$$P(\tilde{n}, \tilde{a}^3) = \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3 - (n_2 + n_3 - n_1)^3}{24n_1n_2n_3} \quad (14)$$

If $n_1 > \frac{1}{2}$:

$$P(\tilde{n}, \tilde{a}^3) = \frac{(n_2 + n_3)^3 - (n_2 - n_3)^3}{24n_1n_2n_3} \quad (15)$$

Proof: As state 1 is a dictator, we just have to identify the regions for which $v_1 < n/2$ while $v_1 + v_2 + v_3 > n/2$. Figure 4 displays the paradoxical regions for the subcase $n_1 < \frac{1}{2}$ and easy computations lead to equation (14). The same process applies for subcase $n_1 > \frac{1}{2}$.

Notice that all the relations given in Propositions 1 to 3 have been checked by using Barvinok's algorithm that computes the number of integer points in a polytope (see Lepelley *et al.* 2008).

3.5. Comparisons

By comparing the values given by the formulas derived for the three possible apportionment cases, we are able to find the minimal value of the referendum paradox for each \tilde{n} . The corresponding minimal values of the paradox are displayed on Table 1. First, our findings are consistent with the equal population case ($\tilde{n} = (1/3, 1/3, 1/3)$) results (May 1948; Feix *et al.* 2004). It is also obvious from the proofs presented above that the minimal values can never be obtained with \tilde{a}^2 .

In Table 1 we observe that the dictatorship of state 1 is the optimal solution for IAC as soon as it gathers more than half of the population; otherwise, equal representation is the optimal solution to the minimization problem. Indeed local minima are observed when there is a perfect correspondence between the vectors of populations and the mandate vector, that is in $\tilde{n} = \tilde{a}^3 = (1, 0, 0)$ and $\tilde{n} = \tilde{a}^1 = (1/3, 1/3, 1/3)$. For a given value of n_3 , the maximal value of the paradox is obtained at $n_1 = 0.5$; the same remark holds with n_2 . Thus, the distance of n_1 to 0.5 is the main factor that explains the magnitude of the paradox.

At last, by comparing the formulas for $P(\tilde{n}, \tilde{a}^1)$ with $P(\tilde{n}, \tilde{a}^3)$ we can verify that $\delta = 1$ is always the optimal δ -rule. Recall that $n_1 = 1 - n_2 - n_3$. If $n_1 \leq 1/2$,

$$P(\tilde{n}, \tilde{a}^1) - P(\tilde{n}, \tilde{a}^3) = \frac{(2n_2 - 1 + 2n_3)(4n_2^2 - 4n_2 + 5n_2n_3 + 1 - 4n_3 + 4n_3^2)}{12n_1n_2n_3} \quad (16)$$

The roots for this equation are :

$$n_2^* = \frac{1}{2} - n_3, \quad n_2^{**} = \frac{1}{2} - \frac{5}{8}n_3 + \frac{1}{8}\sqrt{24n_3 - 39n_3^2}, \quad n_2^{***} = \frac{1}{2} - \frac{5}{8}n_3 - \frac{1}{8}\sqrt{24n_3 - 39n_3^2} \quad (17)$$

The domain where $n_1 \geq n_2 \geq n_3 > 0$ and $n_1 + n_2 + n_3 = 1$ is the triangle in dot lines depicted on Figure 5. The dashed line isolates situations for which $n_1 < 0.5$ (above the line). It also corresponds to the equality $n_2^* = 0$. As the plain curve describes the points such as $n_2^{**} = 0$ and $n_2^{***} = 0$, it is now obvious that equation (16) is negative when $n_1 < 0.5$, meaning that $\tilde{a}^1 = (1, 1, 1)$ is optimal. Incidentally, it corresponds to $\delta = 1$. When $n_1 \geq 1/2$, we get:

$$P(\tilde{n}, \tilde{a}^1) - P(\tilde{n}, \tilde{a}^3) = \frac{n_2 n_3 (n_1 - n_2 - n_3)}{4n_1 n_2 n_3} \quad (18)$$

This value is positive as soon as $n_1 > \frac{1}{2}$, which means that $P(\tilde{n}, \tilde{a}^3)$ always gives the optimal value. But then, $\tilde{a}^3 = (1, 0, 0)$ is also implemented with $\delta = 1$. Thus we obtain:

Proposition 4. *For three-state federations, the proportional representation ($\delta = 1$) is a δ -rule that always minimizes the likelihood of the referendum paradox under IAC.*

4. Conclusion

We studied in this note the apportionment problem in two-tier voting systems, focussing on the δ -rules, which allocate a number of mandates proportional to n_i^δ to state i . Using the May's model (which is equivalent to the IAC assumption within each state while the results across the states remain independent), we obtained a very clear answer: with exact formulas, we proved that taking $\delta = 1$ is optimal for the three-state case. This result provides a further and original argument in favour of the proportional rule.

An obvious question is then: What happens when we consider more than three states? The Monte Carlo simulations we have implemented (and described in a forthcoming paper) tend to demonstrate that the optimal feature of the proportionnal rule is a general result: for a number of states up to 50, this rule is majority efficient, *i.e.* minimizes the probability of the Referendum Paradox.

At last, we shall pinpoint that our results are dependant upon the model we use. The Impartial Culture Assumption (IC) (Gehrlein 2006) assumes that each voter will independently vote for A or B with probability $\frac{1}{2}$. In a companion paper, Lepelley *et al.* (2014) prove that under this model, for three states, $\delta = 0.42$ is the optimal apportionment method that maximize the majority efficiency. Also, using data from French local elections, Lahrach and Merlin (2012) showed that empirically, the square rule root could be the optimal solution.

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Table 1: The minimal values for the referendum paradox for three states under IAC.

$n_2 \downarrow n_3 \rightarrow$	0^+	0.05	0.10	0.15	0.20	0.25	0.30	0.333
0^+	0^+	--	--	--	--	--	--	--
0.05	0.0132	0.0185	--	--	--	--	--	--
0.10	0.0278	0.0319	0.0417	--	--	--	--	--
0.15	0.0411	0.0486	0.0574	0.0714	--	--	--	--
0.20	0.0625	0.0681	0.0774	0.0913	0.1111	--	--	--
0.25	0.0833	0.0904	0.1013	0.1167	0.1379	0.1666	--	--
0.30	0.1072	0.1165	0.1296	0.1477	0.1722	<u>0.1509</u>	<u>0.1242</u>	--
0.333	0.1250	0.1361	0.1514	0.1722	<u>0.1646</u>	<u>0.1431</u>	<u>0.1284</u>	<u>0.1250</u>
0.35	0.1346	0.1468	0.1634	<u>0.1857</u>	<u>0.1614</u>	<u>0.1405</u>	<u>0.1276</u>	--
0.40	0.1666	0.1827	<u>0.2042</u>	<u>0.1786</u>	<u>0.1563</u>	--	--	--
0.45	0.2045	<u>0.2259</u>	<u>0.2006</u>	--	--	--	--	--
0.50^-	0.25^-	--	--	--	--	--	--	--

In bold: probabilities derived from $P(\tilde{n}, \tilde{a}^3)$.

Underlined: probabilities derived from $P(\tilde{n}, \tilde{a}^1)$.

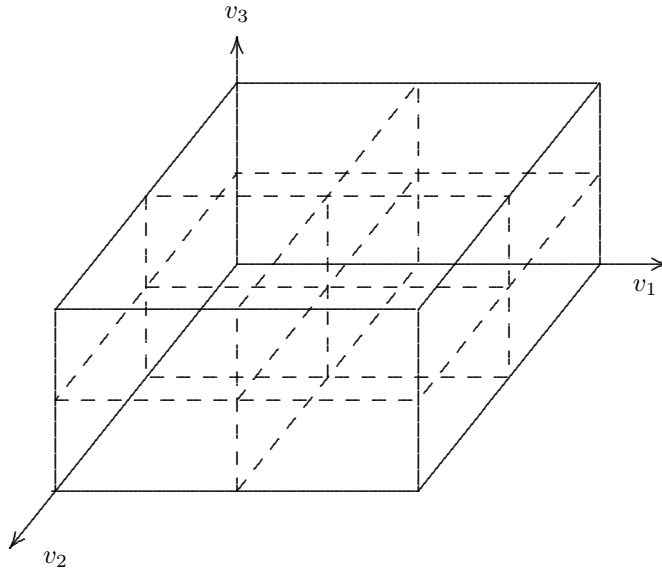


Figure 1: An example of the space of voting profiles under IAC, with $n_1 = 4$, $n_2 = 3$ and $n_3 = 2$

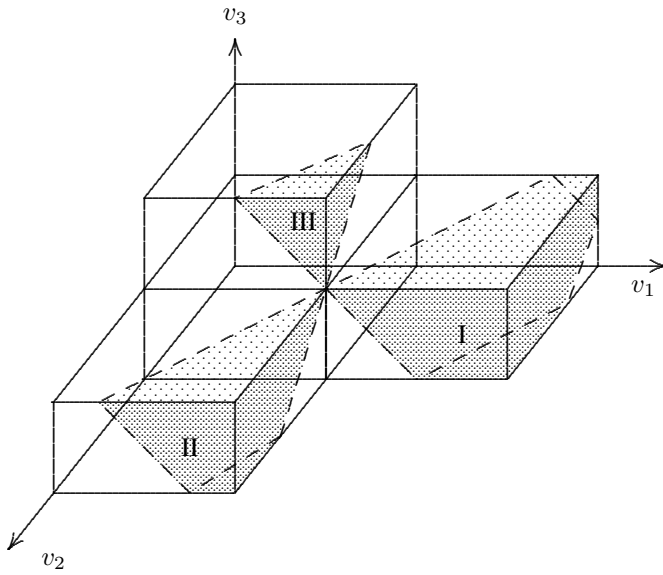


Figure 2: The volumes for the referendum paradox under IAC, with $n_1 = \frac{4}{9}$, $n_2 = \frac{3}{9}$ and $n_3 = \frac{2}{9}$

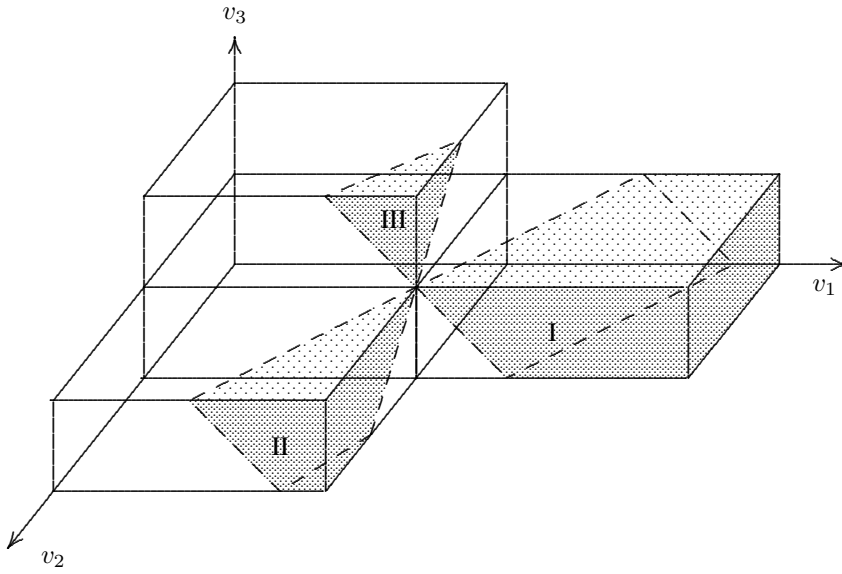


Figure 3: The volumes for the referendum paradox under IAC, with $n_1 = \frac{6}{11}$, $n_2 = \frac{3}{11}$ and $n_3 = \frac{2}{11}$

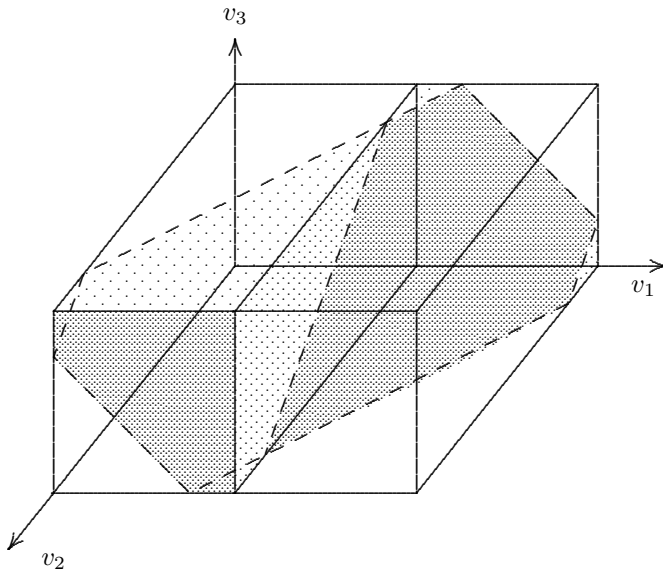


Figure 4: The volume for the referendum paradox under IAC, when state 1 is a dictator, for $n_1 < \frac{1}{2}$.

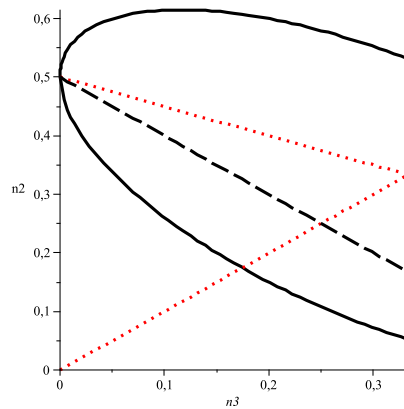


Figure 5: The boundaries between the majority game and dictatorship for the minimization of the referendum paradox. : The optimality of the proportional rule under IAC.