Equilibrium Analysis in Singleton Congestion Games

Samir Sbabou*, Hatem Smaoui† and Abderrahmane Ziad‡

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Abstract. *In this paper, we study the class of congestion games with player-specific payoff functions introduced by Milchtaich in 1996, for which, he proved that they always possess a Nash equilibrium in pure strategies. More specifically, we examine the case of two resources and we propose a simple method describing all Nash equilibria in this kind of congestion games. Additionally, we give a new and short proof establishing the existence of a Nash equilibrium in this type of games without invoking the potential function or the finite improvement property.*

*Corresponding author. Center of Research in Economics and Management, University of Caen, 19 Rue Claude Bloch, 14032 Caen, France. E-mail: samir.sbabou@gmail.com, tel.:+33 2 31 56 66 29.

†Center of Economics and Management of the Indian Ocean, University of La Réunion, 15 Avenue René Cassin, 97715 Saint Denis Cedex 9, BP 7115. E-mail: hsmaoui@univ-reunion.fr

‡Center of Research in Economics and Management, University of Caen, 19 Rue Claude Bloch, 14032 Caen, France. E-mail: abderrahmane.ziad@unicaen.fr

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1 Introduction

The study of congestion games is currently a popular area of research (economics, computer science, applied mathematics...), and multiple definitions exist that allow to formulate them. Rosenthal (1973) was the first to analyze congestion games. In this kind of games, a set of players compete for a set of resources, and the payoff of each resource depends only on the number of players using it. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. A key game-theoretic property of these games is that they always have at least one pure strategy Nash equilibrium. This result comes from the existence of a potential function (Rosenthal (1973)) which directly implies the existence of a pseudo-polynomial algorithm for calculating the Nash equilibrium (Fabrikant et al. (2004)). However, Quint and Shubik (1994), Milchtaich (1996), and Konishi et al. (1997) consider that congestion games do not admit (in general) a potential function, but are likely to admit a Nash equilibrium in pure strategies.

Milchtaich (1996) extended the original version of congestion games (those described by Rosenthal), to the family of congestion games with player-specific payoff functions : Each player is restricted to the selection of a single resource and he has his own payoff function. He outlined that the specific payoff functions are decreasing to the number of players and he proved that such games always admit at least one Nash equilibrium. Contrary to the proof proposed by Rosenthal, the result of Milchtaich does not use the concept of the potential function, but a variant of the mechanism of improvement, called the finite improvement property (FIP)[1]. In the special case of two resources, he showed that such games possess the finite improvement property and an obvious consequence of the existence of the latter one is the existence of a Nash equilibrium. Milchtaich's method implicitly contains a polynomial-time algorithm to find a Nash equilibrium of such games. However, his method has a main disadvantage : if we want to find all Nash equilibria, we have to repeat the FIP process and this, maybe to the infinity ...! Additionally, we have to note that, until now, it does not exist in the literature a mathematical formula of the potential function that allows to establish at least one Nash equilibrium in such games. Moreover, most of the studies focus on the problem of finding and computing efficiently only one Nash equilibrium.

Let us mention that Holzman and Law-Yone (1997) and Voorneveld et al. (1999) investigated the set of strong Nash equilibria[2] in monotone congestion games. It turns out that this set coincides with the set of Nash equilibria and with the set of profiles which maximizes the potential. Furthermore, Ackermann et al. (2006) showed that in congestion games with specific payoff functions, if the player chooses at least one resource and he has his own payoff function, i.e. his own utility function, then the game may not possess a Nash equilibrium.

In this paper, we are interested in congestion games with player-specific payoff functions, in the sense of Milchtaich. We use the term nonsymmetric singleton

[1]Monderer and Shapley (1996): Any sequence of strategy-profiles in which each strategy-profile differs from the preceding one in only one coordinate and the unique deviator in each step strictly increases his utility (such a sequence is called an improvement path).

[2]A strong Nash equilibrium is a profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies.

congestion games to point up the fact that players' strategies are singleton and the payoff functions are decreasing and at the same time specific to each player. We examine the special case of two resources and we essentially direct our research towards the question of identifying all Nash equilibria.

Our aim is to prove that there exists at least one Nash equilibrium, by direct and constructive proofs, without using either the notion of the potential function or the finite improvement property, to show how to compute all equilibria and give their structure using a simple and direct formula. Note that the characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic proprieties of the game such as the price of anarchy[3] (Koutsoupias and Papadimitriou (1999)). The rest of this paper is organized as follows: Section 2 provides basic definitions and notations concerning congestion games, section 3 establishes the case of two strategies and section 4 concludes.

2 Basic Definitions and Notations

A game (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the utility function of player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i\in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies. A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a

[3]When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

best-reply strategy: For each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma^*_{-i})$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy.

In a standard congestion game, defined by Rosenthal (1973), we are given a finite set $R = \{1, \ldots, m\}$ of m resources. A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \ldots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r \in \sigma_i} d_r(n_r(\sigma))$. Rosenthal shows that every congestion game possesses at least one Nash equilibrium by considering the exact potential function $P : S \to \mathbb{N}$ with $P(\sigma) = \sum_{r=1}^{m} \sum_{j=1}^{n_r(\sigma)} d_r(j)$, $\forall \sigma \in S$ [4]. This line of work has been continued by Monderer and Shapley (1996) who proved that the existence of an exact potential function implies the finite improvement property (FIP): That is, each path of single-handed (one player) profitable deviations is finite.

In 1996 Milchtaich introduced an extension of congestion games, namely the congestion games with player-specific payoff functions. He proved that any improvement path that cannot be extended terminates by a Nash equilibrium. A game in this class is defined by a tuple $\Gamma(N, R, (d_r)_{r \in R})$, where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a nonincreasing payoff function associated with resource r. The utility of player i for a profile σ is given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$.

[4] Rosenthal's potential function shows that congestion games are potential. Monderer and Shapley (1996) proved that every potential game can be represented in a form of a congestion game. Thus, classes of potential games and congestion games coincide. Hence, congestion games are essentially the only class of games for which one can show the existence of pure equilibria with an exact potential function.

We note that these games are nonsymmetric : Players are restricted to choose only one strategy, but they each have their own utility function. Since the utility of a player derived from selecting a single resource depends only on the number of the players doing the same choice, the specific utility function of this player is simply a mapping: $u_i : R \times \{1, \dots, n\} \to \mathbb{R}$, $(r, k) \mapsto u_i(r, k)$, where u_i decreases with k, $\forall k \in N$. Hereafter, we refer to this kind of games as monotone nonsymmetric singleton congestion games or singleton congestion games for short.

Thus, having the necessary background, we continue to our main results.

3 The Case of Two Strategies

Henceforth, we develop a method which simplifies the analysis of singleton congestion games. We are interested in the case of two resources and we make use of the method introduced by Sbabou et al. (2010) concerning the use of the ordinal representation of preferences, instead of the cardinal one. Indeed, according to this method, we replace the values of the payment functions (i.e. cardinal representation) by their ranks in a preference ordering representing the specific utility function of each player.

3.1 Our Model

Let $\Gamma(N, R, (\precsim_i)_{i \in N})$ be a singleton congestion game, where N is a set of n players, $R = \{a, b\}$ a set of two resources and $\precsim_i$ a weak ordering on $R \times N$. In the ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim_i (\sigma_i, \sigma^*_{-i})$ for all σ_i in R. A congestion vector $\sigma^* = (n_{r_1}, n_{r_2})$ corresponds to a Nash equilibrium if $(r_1, n_{r_1}) \succsim_i (r_2, n_{r_2} + 1)$, for all $r_1, r_2 \in R$, with $r_1 \neq r_2$. Thus, no player can benefit from joining a group of players sharing a different resource.

For the sake of clarity, we distinguish two cases. In the first case, each player has a strict order of preferences while in the second one each order of preferences may include ties.

Case 1: Strict order of preferences

To develop our approach, we need the following notation: For all players $i \in N$, we note $(a, 0) \succ_i (b, n+1)$ (or $0 \cdot a \succ_i (n+1) \cdot b$ by adopting a simplified notation) when $(a, 1) \prec_i (b, n)$. Similarly, we note $(b, 0) \succ_i (a, n+1)$ (or $0 \cdot b \succ_i (n+1) \cdot a$) when $(b, 1) \prec_i (a, n)$.
Thus, we define the following integers:

$$p_i = \max \{p \in \{0, 1, \dots, n\} : (a, p) \succ_i (b, n+1-p)\}$$

$$q_i = \max \{q \in \{0, 1, \dots, n\} : (b, q) \succ_i (a, n+1-q)\}$$

The integer p_i denotes the maximum size of a group choosing the alternative a in a given strategy profile, in which the player i can belong. Beyond this size, the player i will choose the resource b. Indeed, by definition we have $p_i \cdot a \succ_i (n+1-p_i) \cdot b$ and $(p_i + 1) \cdot a \prec_i (n - p_i) \cdot b$. The integer q_i is interpreted in the same way; we replace a by b. We mention that $p_i + q_i = n$, $\forall i \in N$.
Using the list of integers p_i and q_i, $\forall i \in N$, we define two other integers that will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game:

$$n(a) = \max \{p \in \{0, 1, \dots, n\} : \mid \{i \in N : p_i \geq p\} \mid \geq p\}$$

$$n(b) = \max \{q \in \{0, 1, \dots, n\} :| \{i \in N : q_i \geq q\} |\geq q\}$$

We point out that $n(a)$ (resp. $n(b)$) represents the maximum size of a group of players that can choose the resource a (resp. b) without any member of this group having interest in deviating from his strategy. In this case, we necessarily have $n(a) + n(b) = n$, with the corresponding congestion vector being $v = (n(a), n(b))$.

In order to describe all Nash equilibria, we introduce the three following sets that allow us to identify the alternatives that correspond to each player:

$$A(G) = \{i \in N : p_i > n(a)\}, \quad B(G) = \{i \in N : p_i < n(a)\},$$

$$\text{and } \ C(G) = \{i \in N : p_i = n(a)\}$$

Here, N is the disjoint union of these three sets, each of which may be empty and $|C(G)| \succeq na - |A(G)|$.

Case 2: Order of preferences with ties

This time we denote:

$$p_i = \max \{p \in \{0, 1, \dots, n\} : (a, p) \succsim_i (b, n + 1 - p)\}$$

$$q_i = \max \{q \in \{0, 1, \dots, n\} : (b, q) \succsim_i (a, n + 1 - q)\}$$

where p_i and q_i, $\forall i \in N$, have the same meaning as in the above case. However, we do not necessarily have $p_i + q_i = n$ because of the possible presence of ties. Hence, $p_i + q_i \geq n$, for all $i \in N$. It is therefore possible to have $p_i + q_i > n$ for some players i. This point is important because in this case there exists the possibility to have more than one congestion vector corresponding to a Nash equilibrium.

Using the list of integers p_i and q_i, we define $n(a)$ and $n(b)$ as in Case 1. The latter ones will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game. In this case we have the inequality $n(a)+n(b) \geq n$ and the corresponding congestion vector is $v = (\alpha, \beta)$, where $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$.

In order to describe all Nash equilibria, we introduce the three following sets that allow us to identify the alternatives that correspond to each player:

$$A(G,v) = \{i \in N : p_i \geq \alpha \text{ and } q_i < \beta\}, \quad B(G,v) = \{i \in N : p_i < \alpha \text{ and } q_i \geq \beta\},$$

$$\text{and } C(G,v) = \{i \in N : p_i \geq \alpha \text{ and } q_i \geq \beta\}$$

N is the disjoint union of these three sets and that each of these sets may be empty. We do not examine the case in which $p_i < \alpha$ and $q_i < \beta$, as $p_i + q_i \geq n$ and $\alpha + \beta = n$.

3.2 The result

Our purpose is to improve the study of singleton congestion games by providing a general method to describe all Nash equilibria and to establish a comprehensive list of all of them. We investigate the special case of two resources and our approach is such that we are not making use of the potential function or the finite improvement property invoked by Rosenthal and Milchtaich respectively.

Theorem 1. *Let $R = \{a, b\}$ and $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game where all preference orderings are strict.*

1. *G admits at least one Nash equilibrium. All equilibria correspond to the same congestion vector :* $v = (n(a), n(b))$.

2. *Each Nash equilibrium of* G, $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$, *is characterized by a unique subset* D *(possibly empty) of* $C(G)$, *of cardinal* $n(a) - |A(G)|$, *such that: For all* $i \in N$, $\sigma_i^* = a$ *if* $i \in A(G) \cup D$ *and* $\sigma_i^* = b$ *if* $i \in B(G) \cup (C \setminus D)$.

3. *The game admits exactly* $C_{|C(G)|}^{n(a)-|A(G)|}$ *Nash equilibria. In particular, if* $n(a) = |A(G)|$ *the game admits a single Nash equilibrium.*

Example 1. Let $N = \{1, 2, 3, 4, 5, 6\}$ be a number of players and $R = \{a, b\}$ two alternatives. Suppose that the players' preferences are given by the following strict orderings:

$$\text{Player}_1: \quad 6b \prec 5b \prec 4b \prec 6a \prec 5a \prec 3b \prec 4a \prec 3a \prec 2b \prec 2a \prec b \prec a$$

$$\text{Player}_2: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 3b \prec 2a \prec 2b \prec a \prec b$$

$$\text{Player}_3: \quad 6b \prec 6a \prec 5a \prec 4a \prec 3a \prec 2a \prec 5b \prec 4b \prec 3b \prec 2b \prec a \prec b$$

$$\text{Player}_4: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec b \prec a$$

$$\text{Player}_5: \quad 6a \prec 6b \prec 5b \prec 4b \prec 5a \prec 4a \prec 3a \prec 2a \prec a \prec 3b \prec 2b \prec b$$

$$\text{Player}_6: \quad 6b \prec 5b \prec 4b \prec 6a \prec 5a \prec 4a \prec 3a \prec 2a \prec 3b \prec 2b \prec a \prec b$$

We have omitted the indices of players in the order of preferences. For each player i, we search the integer p_i which is the greatest p such that $p \cdot a \succ_i (n+1-p) \cdot b$

and $(n-p)\cdot b \succ_i (p+1)\cdot a$.

$$p_1 = 4: \quad 4a \succ_1 3b \quad \text{and} \quad 2b \succ_1 5a$$

$$p_2 = 3: \quad 3a \succ_2 4b \quad \text{and} \quad 3b \succ_2 4a$$

$$p_3 = 1: \quad a \succ_3 6b \quad \text{and} \quad 5b \succ_3 2a$$

$$p_4 = 3: \quad 3a \succ_4 4b \quad \text{and} \quad 3b \succ_4 4a$$

$$p_5 = 3: \quad 3a \succ_5 4b \quad \text{and} \quad 3b \succ_5 4a$$

$$p_6 = 3: \quad 3a \succ_5 4b \quad \text{and} \quad 3b \succ_5 4a$$

So, we can verify that $n(a) = 3$ and $n(b) = 3$. The only congestion vector corresponding to a Nash equilibrium is the vector $(3a, 3b)$. Furthermore, we have $A(G) = \{1\}$, $B(G) = \{3\}$ and $C(G) = \{2, 4, 5, 6\}$. By theorem 1, we know that there are exactly $C_4^2 = 6$ different Nash equilibria. All these equilibria are common: $\sigma^* = (a)$ if $i \in A(G)$ and $\sigma^* = (b)$ if $i \in B(G)$. Each of these equilibria is characterized by a subset D of $C(G)$ with $|D| = 2$ and $\sigma_i^* = a$ if $i \in D$. The list of the Nash equilibria of this game is:

$$(a,a,b,a,b,b),\ (a,a,b,b,a,b),\ (a,a,b,b,b,a),$$
$$(a,b,b,a,b,a),\ (a,b,b,a,a,b),\ (a,b,b,b,a,a).$$

Remark 1. *With the above example, we show that, contrary to the approaches already proposed in the literature, our method gives directly the exact number of Nash equilibria, before even knowing which they are. This fact allows us to avoid a repeating procedure to find all Nash equilibria, like the one given by Milchtaich. His method provides only one equilibrium at a time and if we want to find all of them, we must repeat the process of the finite improvement property, which may never finish to be repeated ...!*

Theorem 2. *Let $R = \{a, b\}$ and $G(N, R, (\precsim)_{i\in N})$ be a singleton congestion game where the order of preferences my include ties.*

1. *Each congestion vector* $v = (\alpha, \beta)$ *such that* $\alpha \leq n(a)$, $\beta \leq n(b)$ *and* $\alpha + \beta = n$, *corresponds to (at least) one Nash equilibrium of* G.

2. *Each of the Nash equilibrium of* G *corresponding to the vector* v, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, *is characterized by a unique subset* D *(possibly empty)* $C(G, v)$, *of cardinal* $\alpha - |A(G, v)|$, *to ensure that: For all* $i \in N$, $\sigma_i^* = a$ *if* $i \in A(G, v) \cup D$ *and* $\sigma_i^* = b$ *if* $i \in B(G, v) \cup (C(G, v) \setminus D)$.

Example 2. Let $N = \{1, 2, 3, 4, 5\}$ be a number of players and $R = \{a, b\}$ two alternatives. Suppose that the players' preferences are given by the following weak orderings:

$$\text{Player}_1 : \quad 5a \prec 5b \prec 4b \prec 4a \prec 3b \sim 3a \sim 2a \prec 2b \sim a \prec b$$

$$\text{Player}_2 : \quad 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b$$

$$\text{Player}_3 : \quad 5a \prec 5b \prec 4b \prec 4a \sim 3b \sim 3a \sim 2b \prec 2a \prec a \prec b$$

$$\text{Player}_4 : \quad 5b \prec 4b \prec 5a \prec 4a \sim 3b \sim 3a \sim 2b \prec 2a \prec b \prec a$$

$$\text{Player}_5 : \quad 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b$$

It is easy to see that:
$p_1 = 3$, $q_1 = 3$, $p_2 = 5$, $q_2 = 5$, $p_3 = 4$, $q_3 = 3$, $p_4 = 4$, $q_4 = 3$, $p_5 = 5$, $q_5 = 5$.
Hence, $n(a) = 4$ and $n(b) = 3$. By theorem 2, the possible congestion vectors are: $v_1 = (4a, b)$, $v_2 = (3a, 2b)$, $v_3 = (2a, 3b)$.
Since $v_1 = (4a, b)$, we have $A(G, v_1) = \emptyset$, $B(G, v_1) = \{1\}$ and $C(G, v_1) = \{2, 3, 4, 5\}$. Thus, there exists a unique equilibrium corresponding to v_1, which

is the profile (b, a, a, a, a).

Similarly, as $v_2 = (3a, 2b)$ we get $A(G, v_2) = \emptyset$, $B(G, v_2) = \emptyset$ and $C(G, v_3) = \{1, 2, 3, 4, 5\}$. The Nash equilibria corresponding to v_2 are:

$$(b,b,a,a,a),\ (b,a,b,a,a),\ (b,a,a,b,a),\ (b,a,a,a,b),\ (a,b,a,a,b),$$
$$(a,a,b,a,b),\ (a,a,a,b,b),\ (a,b,a,b,a),\ (a,b,b,a,a),\ (a,a,b,b,a).$$

Finally, for $v_3 = (2a, 3b)$ we have $A(G, v_3) = \emptyset$, $B(G, v_3) = \emptyset$ and $C(G, v_3) = \{1, 2, 3, 4, 5\}$. The Nash equilibria corresponding to v_3 are:

$$(b,b,b,a,a),\ (b,b,a,b,a),\ (b,b,a,a,b),\ (b,a,a,b,b),\ (a,b,b,b,a),$$
$$(a,a,b,b,b),\ (b,a,b,b,a),\ (b,a,b,a,b),\ (a,b,b,a,b),\ (a,b,a,b,b).$$

Remark 2. *If players' preferences are represented by a strict order, there is a single congestion vector; otherwise we can find at least one congestion vector.*

Proof of Theorem 1.

1) By definition of $n(a)$, there are at least $n(a)$ players $i \in N$ such that $p_i \geq n(a)$. Therefore, we choose $n(a)$ players satisfying this condition including all players for whom $p_i > n(a)$. Denote by A the set of these players. For all players who are in $B = N \backslash A$, we must have $p_i \leq n(a)$ and therefore $q_i \geq n(b)$. It is easy, returning to the definition of p_i and q_i, to verify that the profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ defined by $\sigma_i^* = a$ if $i \in A$ and $\sigma_i^* = b$ if $i \in B$ is a Nash equilibrium. Let σ^* be a Nash equilibrium of G and let (α, β) be the congestion vector associated with σ^*. Suppose that $\alpha > n(a)$. As σ^* is a Nash equilibrium, there exist α players such that $p_i \geq \alpha$, which contradicts the maximality of $n(a)$. We must therefore have $\alpha \leq n(a)$. Similarly, we show that $\beta \leq n(b)$. As $\alpha + \beta = n$ and $n(a) + n(b) = n$, we necessarily have $\alpha = n(a)$ and $\beta = n(b)$.

2) Let D be a subset (possibly empty) of $C(G)$, of cardinal $n(a) - |A(G)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile defined by: For all $i \in N$, $\sigma_i^* = a$

if $i \in A(G) \cup D$ and $\sigma_i^* = b$ if $i \in B(G) \cup (C(G) \setminus D)$. The profile σ^* is a Nash equilibrium. Indeed, let $i \in A(G) \cup D$. By definition of $A(G)$ and D, we have $p_i \geq n(a)$. By definition of p_i and the assumption of monotonicity, we get: $n(a) \cdot a \succsim_i (n(b)+1) \cdot b$. Similarly, we show that for all i in $B(G) \cup (C(G) \setminus D$, $n(b) \cdot b \succsim_i (n(a)+1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G. It is known from (1) that the congestion vector associated with σ^* is $(n(a), n(b))$. We must have $\sigma_i^* = a$ if $i \in A(G)$ and $\sigma_i^* = b$ if $i \in B(G)$. We just have to consider $D = \{i \in N : \sigma_i^* = a \text{ and } i \notin A(G)\}$.

3) The result is obtained by a simple calculation from (2). □

Proof of Theorem 2.

It suffices to prove (2), because (1) is obtained as a consequence of (2). Let $v = (\alpha, \beta)$ be a congestion vector such that $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$. Let D be a subset (possibly empty) of $C(G, v)$, of cardinal $\alpha - |A(G, v)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a strategy profile such that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G, v) \cup D$ and $\sigma_i^* = b$ if $i \in B(G, v) \cup (C(G, v) \setminus D)$. σ^* is a Nash equilibrium. Indeed, let $i \in A(G, v) \cup D$. By definition of $A(G, v)$ and of D, we have $p_i \geq \alpha$. By definition of p_i and by the assumption of monotonicity, we obtain: $\alpha \cdot a \succsim_i (\beta + 1) \cdot b$. Similarly, we show that for all i in $B(G, v) \cup (C(G, v) \setminus D$, $\beta \cdot b \succsim_i (\alpha + 1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G and let $v = (\alpha, \beta)$ be the congestion vector associated with this equilibrium. We have $\alpha \leq n(a)$, otherwise there exist α players i with $p_i \geq \alpha > n(a)$. This is impossible by definition of $n(a)$. Similarly, we show that $\beta \leq n(b)$. By definition of a congestion vector, we also have $\alpha + \beta = n$. As σ^* is a Nash equilibrium, for any $i \in N$, we must have: $\sigma_i^* = a$ if $i \in A(G, v)$ and $\sigma_i^* = b$ if $i \in B(G, v)$. We just need to consider $D = \{i \in N : \sigma_i^* = a \text{ and } i \notin A(G, v)\}$ and to note that the case $p_i < \alpha$ and $q_i < \beta$ is not possible. □

4 Concluding remarks

Contrary to the studies done in the past, which provided only one Nash equilibrium in a specific class of games, in this paper, we have presented a method for describing the general structure of all Nash equilibria and identifying all of them in nonsymmetric singleton congestion games. Our approach is valid for the case of two resources and we consider that is complete. The ordinal representation of preferences allowed us to simplify the analysis of such games and to easily find a method for describing all Nash equilibria without using either the potential function or the finite improvement property invoked by Rosenthal and Milchtaich. It is also important to underline that our analysis can be used to obtain optimal Nash equilibria. As a future work, it would be interesting to extend our approach to the general case ($|R \geq 3|$) of nonsymmetric congestion games with player-specific payoff functions.

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